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When are extreme points enough?

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Douglas Baker

In memory of Doug Baker.

We establish sufficient conditions for when the image a linear transformation on a compact, convex set in a real linear Hausdorff space is the same as the image of the linear transformation on the extreme points of that set.

We show why several of those conditions cannot be relaxed and give an application.

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This paper is an extension of the second author's thesis work before his untimely passing. He was inspired by the work of C. Akemann and J. Anderson on theorems of Lyapunov type [1] and in this paper we give three additional theorems of Lyapunov type.

The setting is as follows: let X and V be linear spaces. Let $T : X \to V$ be a linear map. Let $K \subset X$ be convex and let $E(K)$ be the set of extreme points of K. We would like to find circumstances under which the following equation holds:

$$T(K) = T(E(K)). \tag{1}$$

Theorems 21, 11 and 22 establish sufficient conditions for equation 1 to hold. In the language of [1] these are Lyapunov theorems of type 1. An interesting feature is that in our results we require minimal structure on V. Another distinction of note between our results and those in [1] is that our maps need not be continuous.

The price we pay is that we must make several restrictions on K and T. For example, our results depend on requiring K to have at most one non-singleton proper face.

Our main result is Theorem 11. We give several examples that show why the hypotheses in 11 cannot be relaxed without significant modification. Both Theorems 21 and 22 are examples of such modifications.

An application of Theorem 11 is given in Theorem 13.

We begin by recalling a well-known definition. Refer to [5] for details.

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Definition 1. Let X be a real linear space. A subset A of X is an **affine subspace** if it is a translation of a subspace, M, of X; that is, $A = \mathbf{x}_0 + M$ for some $\mathbf{x}_0 \in X$.

Notation 2. For the remainder of this paper, we use the following notation. The set X is a real linear locally convex Hausdorff space. For $K \subset X$ non-empty and convex, we denote the smallest affine subspace of X containing K by aff K. The subspace of X corresponding to aff K will be denoted by M. We define the **dimension** of K to be the dimension of M. We denote the dimension of K by $\dim K$. We define

$$E(K) := \{\mathbf{x} \in K \mid \mathbf{x} \text{ is an extreme point of } K\}. \tag{2}$$

It will often be helpful to discuss line segments. For $\mathbf{a}, \mathbf{b} \in X$ we define

$$\begin{aligned} L[\mathbf{a}, \mathbf{b}] &= \{t\mathbf{a} + (1-t)\mathbf{b} \mid t \in [0,1]\} \\ L(\mathbf{a}, \mathbf{b}) &= \{t\mathbf{a} + (1-t)\mathbf{b} \mid t \in (0,1)\} \\ L(\mathbf{a}, \mathbf{b}] &= \{t\mathbf{a} + (1-t)\mathbf{b} \mid t \in [0,1)\} \\ L[\mathbf{a}, \mathbf{b}) &= \{t\mathbf{a} + (1-t)\mathbf{b} \mid t \in (0,1]\}. \end{aligned} \tag{3}$$

The topology on X induces a topology on aff K (and the corresponding M)[6]. The topology used in the following definition is the induced topology on aff K.

Definition 3. We define the **relative boundary** of K (with respect to aff K) to be the boundary of K in aff K. The **relative interior** of K is the interior of K in aff K. We denote the relative boundary of K by ∂K and the relative interior of K by K°.

The following definition is from [1].

Definition 4. If $\mathbf{x}$ and $\mathbf{y}$ are in K and there exists $\lambda > 0$ such that

$$(1+\lambda)\mathbf{x} - \lambda\mathbf{y} \in K, \tag{4}$$

then $\mathbf{x}$ is **interior relative** to $\mathbf{y}$. If $\mathbf{x} \in K$ is interior relative to each point in K, we say that $\mathbf{x}$ is a **weak internal point** of K.

Remark 5. Notice that $\mathbf{x}$ is interior relative to itself. Indeed any $\lambda > 0$ will do.

Lemma 6. *If $\boldsymbol{x} \in K$, then $\boldsymbol{x}$ is a weak internal point of K if and only if the smallest face containing $\boldsymbol{x}$ is K.*

Proof. Let

$$F = \{\mathbf{y} \in K \mid \mathbf{x} \text{ is interior relative to } \mathbf{y}\}. \tag{5}$$

By Theorem 1.2 (1) in [1], F is a face of K; and by 1.2 (3) in [1], it is the smallest face containing $\mathbf{x}$.

If $\mathbf{x}$ is a weak internal point of K then by 1.2 (4) in [1], $K = F$.

On the other hand Theorem 1.2 (2) in [1] states that $\mathbf{x}$ is a weak internal point of F and hence if $K = F$, then $\mathbf{x}$ is a weak internal point of K. □

Proposition 7. *The collection of weak internal points of K is convex.*

Proof. Let E be the collection of weak internal points. Let $\mathbf{x}, \mathbf{y} \in E$ and $t \in (0,1)$. Let $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$. We would like to show that $\mathbf{z} \in E$. Fix $\mathbf{w} \in K$. Since $\mathbf{x}, \mathbf{y} \in E$, there exists $\alpha > 0$ such that

$$\mathbf{x} + \alpha(\mathbf{x} - \mathbf{w}),\ \mathbf{y} + \alpha(\mathbf{y} - \mathbf{w}) \in K. \tag{6}$$

Then

$$\begin{aligned} \mathbf{z} + \alpha(\mathbf{z} - \mathbf{w}) &= t\mathbf{x} + (1-t)\mathbf{y} + \alpha(t\mathbf{x} - t\mathbf{w}) + \alpha((1-t)\mathbf{y} - (1-t)\mathbf{w}) \\ &= t(\mathbf{x} + \alpha(\mathbf{x} - \mathbf{w})) + (1-t)(\mathbf{y} + \alpha(\mathbf{y} - \mathbf{w})) \in K, \end{aligned} \tag{7}$$

since K is convex. Thus, $\mathbf{z}$ is interior relative to $\mathbf{w}$. Since $\mathbf{w}$ is arbitrary, $\mathbf{z}$ is a weak internal point of K. $\square$

Proposition 8. *If $\boldsymbol{x}$ is in K°, then $\boldsymbol{x}$ is a weak internal point of K.*

Proof. Let $\mathbf{x}$ be in the relative interior of K. There exists $\mathbf{x}_0 \in X$, and M a subspace of X such that aff $K = \mathbf{x}_0 + M$. Thus, $\mathbf{x} = \mathbf{x}_0 + \mathbf{m}$ for some $\mathbf{m} \in M$ and, since $\mathbf{x} \in K^\circ$, given any $\mathbf{z} \in M$, there exists $\lambda > 0$ such that $\mathbf{x} + \lambda\mathbf{z} \in K$. Let $\mathbf{y} \in K$. Then $\mathbf{y} = \mathbf{x}_0 + \mathbf{n}$ for some $\mathbf{n} \in M$. For any $\lambda \in \mathbb{R}$, we have:

$$(1+\lambda)\mathbf{x} - \lambda\mathbf{y} = \mathbf{x} + \lambda(\mathbf{x} - \mathbf{y}) = \mathbf{x} + \lambda(\mathbf{m} - \mathbf{n}). \tag{8}$$

Notice that $\mathbf{m} - \mathbf{n} \in M$ since M is a vector space. By choosing an appropriate $\lambda > 0$, $\mathbf{x} + \lambda(\mathbf{m} - \mathbf{n}) \in K^\circ$ and hence $\mathbf{x}$ is internal to $\mathbf{y}$. Since $\mathbf{y}$ is arbitrary, $\mathbf{x}$ is a weak internal point of K. $\square$

It is natural to ask whether the converse to the above proposition holds. The answer is no in general, but yes in the finite dimensional situation.

Proposition 9. *If $\dim K < \infty$, and $\boldsymbol{x}$ is a weak internal point of K, then $\boldsymbol{x}$ is in the relative interior of K.*

Proof. Let $\dim K = n$. Then we can choose $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in X$ such that aff $K = \mathbf{x}_0 + M$, $M = \langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$, and $\mathbf{x}_0 + \mathbf{x}_k \in K$ for each $k = 1, \ldots n$. Since the topology on X is translation invariant, we may assume without loss of generality that $\mathbf{x}_0 = \mathbf{0}$. By the same reasoning, we may also assume that $\mathbf{x} = \mathbf{0}$. Since $\mathbf{0}$ is a weak internal point of K, there exists $\lambda_k > 0$ such that

$$\lambda_k(-\mathbf{x}_k) = -\lambda_k\mathbf{x}_k \in K \quad (k = 1, \ldots, n). \tag{9}$$

Let $\lambda = \min\{\lambda_1, \ldots, \lambda_n, 1\}$ and $\mathbf{y}_k = \lambda\mathbf{x}_k$ for $k = 1, \ldots, n$. Since K is convex, and λ is in the interval $(0,1]$, $\mathbf{y}_k \in K$. Since λ is in the interval $(0, \lambda_k]$, it follows that $-\mathbf{y}_k$ is also in K. Let Z be the convex hull of $\{\pm\mathbf{y}_k\}_{k=1}^n$. Then $Z \subset K$ since K is convex. Let $H(\mathbf{y}_k) = \mathbf{e}_k$ where $\{\mathbf{e}_k\}_{k=1}^n$ is the standard basis for $\mathbb{R}^n$. Extend H linearly to M. Since M is Hausdorff, we may apply Theorem 7.3 from [3] which states that M is topologically isomorphic to $\mathbb{R}^n$, with H a suitable isomorphism. Now H takes Z onto the convex hull of $\{\pm\mathbf{e}_k\}_{k=1}^n$ which contains an open neighbourhood U of the origin.

Since H is linear and M is finite dimensional, H is continuous [4]. Thus $H^{-1}(U)$ is an open neighbourhood of $\mathbf{0} \in M$, and $H^{-1}(U) \subset Z \subset K$. We conclude that $\mathbf{0} \in K^\circ$. $\square$

Proposition 10. *Proposition* 9 *does not extend to infinite dimensions.*

Proof. Let X be the real Hilbert space with countably infinite basis. Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be the standard orthonormal basis of X. Let K be the Hilbert cube: that is,

$$K = \left\{ \sum_{k=1}^{n} x_k \mathbf{e}_k \mid |x_k| \leq \frac{1}{k} \right\}. \tag{10}$$

By Tychonoff's Theorem [3], K is compact, and by construction K is convex. Note that $\mathbf{0}$ is a weak internal point. To see this, let $\mathbf{y} \in K$. Then $-\mathbf{y} \in K$. Hence, if $\lambda \in (0, 1]$,

$$(1 + \lambda)\mathbf{0} - \lambda\mathbf{y} = -\lambda\mathbf{y} \in K, \tag{11}$$

as desired.

Let $\epsilon > 0$. There exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\epsilon}{2}$. Define

$$\mathbf{z} = \frac{\epsilon}{2}\mathbf{e}_k. \tag{12}$$

Then $||\mathbf{z}|| < \epsilon$ but $\mathbf{z}$ is not in K. Hence, $\mathbf{0}$ is not in the interior of K. □

Theorem 11. *Let V be a real vector space. Suppose that K is compact and has exactly one non-singleton proper face F. Let $T : X \to V$ be a linear map. If there exists $\boldsymbol{f}_0 \in F$, $\boldsymbol{x}_0 \in K$, with $\boldsymbol{f}_0 \neq \boldsymbol{x}_0$ and $\boldsymbol{x}_0 - \boldsymbol{f}_0 \in \ker T$, then*

$$T(K) = T((E(K)).$$

Proof. We may assume that $\mathbf{f}_0 = \mathbf{0}$. Then $\mathbf{x}_0 \in \ker T$. For $\mathbf{z} \in K$, define:

$$\begin{aligned} g_{\mathbf{z}} : \mathbb{R} &\to X \\ t &\mapsto \mathbf{z} + t\mathbf{x}_0. \end{aligned} \tag{13}$$

Since $\mathbf{x}_0 \neq \mathbf{0}$, $g_{\mathbf{z}}$ is injective. Since $g_{\mathbf{z}}$ is an affine map on a finite dimensional vector space, $g_{\mathbf{z}}$ is continuous. Let $W = g_{\mathbf{z}}(\mathbb{R})$. Then W is a one dimensional affine subspace of X and hence has a unique topology [2, Ch. 1]. Thus $g_{\mathbf{z}}$ is an open map from $\mathbb{R}$ to W. Consequently, $g_{\mathbf{z}}$ is a homeomorphism onto W.

Now the inverse map of $g_{\mathbf{z}}$, $g_{\mathbf{z}}^{-1}$, is continuous. Since K is compact, $g_{\mathbf{z}}^{-1}(K \cap W)$ is compact. Since $g_{\mathbf{z}}(0) = \mathbf{z} \in K$, it follows that $0 \in g_{\mathbf{z}}^{-1}(K \cap W)$.

Suppose $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ are in K for some $a \leq b$. Since K is convex,

$$t g_{\mathbf{z}}(a) + (1 - t) g_{\mathbf{z}}(b) \in K \tag{14}$$

for $t \in [0, 1]$. Thus,

$$t\mathbf{z} + ta\mathbf{x}_0 + (1 - t)\mathbf{z} + (1 - t)b\mathbf{x}_0 = \mathbf{z} + (ta + (1 - t)b)\mathbf{x}_0 \in K, \tag{15}$$

and so $ta + (1 - t)b \in g_{\mathbf{z}}^{-1}(K \cap W)$.

Hence, $g_{\mathbf{z}}^{-1}(K \cap W)$ is convex.

We have now shown that $g_{\mathbf{z}}^{-1}(K \cap W)$ is a compact, convex, subset of $\mathbb{R}$ which contains 0. Hence

$$g_{\mathbf{z}}^{-1}(K \cap W) = [a, b] \tag{16}$$

for some $a \leq 0 \leq b$.

We next show that $g_{\mathbf{z}}(a), g_{\mathbf{z}}(b)$ are not weak internal points of K.

For $\lambda > 0$,

$$\begin{aligned} (1+\lambda)g_{\mathbf{z}}(a) - \lambda g_{\mathbf{z}}(b) &= g_{\mathbf{z}}(a) + \lambda(g_{\mathbf{z}}(a) - g_{\mathbf{z}}(b)) \\ &= \mathbf{z} + a\mathbf{x}_0 + \lambda(a-b)\mathbf{x}_0 \\ &= \mathbf{z} + (a + \lambda(a-b))\mathbf{x}_0. \end{aligned} \tag{17}$$

Since $a + \lambda(a-b) \leq 0$ for $\lambda > 0$, the vector $\mathbf{z} + (a + \lambda(a-b))\mathbf{x}_0$ is in K only if $a = b$. Hence, $g_{\mathbf{z}}(a)$ is interior to $g_{\mathbf{z}}(b)$ only if $a = b$. A similar calculation shows that $g_{\mathbf{z}}(b)$ is interior to $g_{\mathbf{z}}(a)$ if and only if $a = b$. Thus, the only way that either $g_{\mathbf{z}}(a)$ or $g_{\mathbf{z}}(b)$ can be weak internal points of K is if $a = b = 0$, and hence

$$g_{\mathbf{z}}(a) = g_{\mathbf{z}}(b) = g_{\mathbf{z}}(0) = \mathbf{z}; \tag{18}$$

that is

$$g_{\mathbf{z}}^{-1}(K \cap W) = \{0\}. \tag{19}$$

Assuming then that equation 19 holds, suppose that for some $\lambda \geq 0$,

$$(1+\lambda)\mathbf{z} - \lambda\mathbf{0} \in K. \tag{20}$$

Since K is convex,

$$t\mathbf{x}_0 + (1-t)(1+\lambda)\mathbf{z} \in K \tag{21}$$

for each $t \in [0, 1]$. Fix $t = \frac{\lambda}{1+\lambda}$. Then

$$(1-t)(1+\lambda) = \frac{1}{(1+\lambda)}(1+\lambda) = 1, \tag{22}$$

and so

$$\frac{\lambda}{1+\lambda}\mathbf{x}_0 + \mathbf{z} \in K. \tag{23}$$

Since $g_{\mathbf{z}}^{-1}(K \cap W) = \{0\}$, $\lambda = 0$. Consequently, $\mathbf{z}$ is interior to $\mathbf{0}$ if and only if $\mathbf{z} = \mathbf{0}$.

But $t\mathbf{x}_0 \in K$ for $t \in [0, 1]$ so $g_{\mathbf{0}}^{-1}(K \cap W) \neq \{0\}$. Hence $\mathbf{z} \neq \mathbf{0}$. We have a contradiction.

We conclude that under no circumstances are either $g_{\mathbf{z}}(a)$ or $g_{\mathbf{z}}(b)$ weak internal points.

Notice that

$$T(g_{\mathbf{z}}(a)) = T\mathbf{z} + tT\mathbf{x}_0 = T\mathbf{z}, \tag{24}$$

and similarly,

$$T(g_{\mathbf{z}}(b)) = T\mathbf{z}. \tag{25}$$

Therefore, to complete our proof, we need to show that at least one of $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ is in $E(K)$.

Since both vectors in question are not weak internal points, both $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ are in proper faces of K by Lemma 6.

If at least one of them is not in F, then by hypothesis that one is an extreme point of K.

Suppose that $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ are in F. There are two cases to consider.

First suppose that $g_{\mathbf{z}}(a) \neq g_{\mathbf{z}}(b)$. Then neither vector is interior to the other in F as we argued above. Thus, they must be in proper faces of F. Since proper faces of F are proper faces of K, and since F is the only non-singleton proper face of K, $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ are extreme points of F, hence of K. (See [2, Chapter 1] for proofs of these assertions.)

Next suppose that $g_{\mathbf{z}}(a) = g_{\mathbf{z}}(b) = \mathbf{z}$. Then $\mathbf{z} \neq \mathbf{0}$ and as we have already seen $\mathbf{z}$ is not interior to $\mathbf{0}$. Hence, in this situation, $\mathbf{z}$ is in a proper face in F, and therefore is an extreme point of F. Thus $\mathbf{z}$ is an extreme point of K as required. □

Corollary 12. *Let $F \subset X$, and suppose K is the closed convex hull of F. If K satisfies the conditions in Theorem* 11 *and T is a linear map that satisfies the conditions in Theorem* 11*, then*

$$T(F) \subset T(E(K)).$$

Proof. This follows immediately since $F \subset K$ and $T(K) = T(E(K))$ by Theorem 11. □

The following shows how Theorem 11 can be applied.

Theorem 13. *Suppose that K is compact and has exactly one non-singleton proper face F. Let $T : X \to \mathbb{R}^n$ be a linear map. If $\dim F = m \geq n$, then*

$$T(K) = T(E(K)).$$

Proof. It will be sufficient to find an $\mathbf{f}_0$ and an $\mathbf{x}_0$ satisfying the hypotheses in Theorem 11.

Since F is a non-singleton face, there exists $\mathbf{f}_0 \in F \backslash E(F)$. Now the set

$$F_0 = \{\mathbf{z} \in F \mid \mathbf{f}_0 \text{ is interior to } \mathbf{z}\} \tag{26}$$

is itself a face of F, hence of K. Since $\mathbf{f}_0$ is not an extreme point and F is the only proper non-singleton face of K, we conclude that $F_0 = F$. Hence, F has weak internal points. Via translation we may assume that $\mathbf{0}$ is one of those weak internal points. (That is, $\mathbf{0}$ will be our $\mathbf{f}_0$.)

Since $\mathbf{0}$ is a weak internal point, we can choose a linearly independent set, $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\} \subset F$ so that $\{-\mathbf{y}_1, \ldots, -\mathbf{y}_n\} \in F$. To see this, let $\{\mathbf{x}_1, \ldots \mathbf{x}_m\}$ be a basis for aff F with each $\mathbf{x}_k \in F$. Then $\mathbf{0}$ is interior to $\mathbf{x}_k$ so there exists $\lambda > 0$ such that

$$(1 + \lambda)0 - \lambda \mathbf{x}_k = -\lambda \mathbf{x}_k \in F. \tag{27}$$

Since F is convex $0 < \epsilon \leq \min\{1, \lambda\}$

$$\pm \epsilon \mathbf{x}_k := \pm \mathbf{y_k} \in F \tag{28}$$

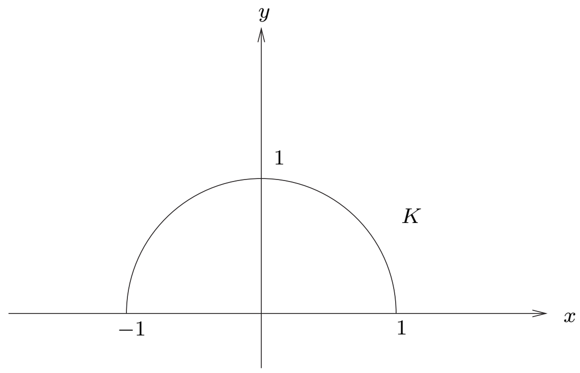

Figure 1: Illustration for Example 14.

whenever $0 < \epsilon \le \min\{1, \lambda\}$.

We restrict our attention to $\{\mathbf{y}_k\}_{k=1}^n$ and select $\mathbf{z}_{n+1} \in K \backslash F$. Define

$$Y = \langle \mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{z}_{n+1} \rangle, \tag{29}$$

and note that $\dim Y = n + 1$.

Define $H(\mathbf{y}_k) = \mathbf{e}_k$ and $H(\mathbf{z}_{n+1}) = \mathbf{e}_{n+1}$, where the $\mathbf{e}_k$'s form the standard basis for $\mathbb{R}^{n+1}$, and extend H linearly to all of Y. Notice that H is a vector space isomorphism by Theorem 7.3 in [3]. The map $T \circ H^{-1}$ is a linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. Since $n+1 > n$, $T \circ H^{-1}$ has a non-trivial kernel. Let $\mathbf{z}$ be a non-zero element in the kernel. Write $\mathbf{z} = \sum_{k=1}^{n+1} z_k \mathbf{e}_k$. Now $-\mathbf{z}$ is also in the kernel, so we may without loss of generality assume that $z_{n+1} \ge 0$. Let $U \subset \mathbb{R}^n$ be the closed convex hull of $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n, \mathbf{e}_{n+1}\}$ (we do not include $-\mathbf{e}_{n+1}$), and observe that $H^{-1}(U) \subset K$ since K is convex and contains $H^{-1}(E(U))$. By scaling if necessary, we may assume that $\mathbf{z} \in U$.

Define $\mathbf{x}_0 = H^{-1}(\mathbf{z})$. Since H^{-1} is an isomorphism, $\mathbf{x}_0 \ne \mathbf{0}$ and $\mathbf{x}_0 \in \ker T$. We now apply Theorem 11 to get the desired result. □

It is clear that the conditions laid out in Theorem 11 are sufficient but not always necessary. For example, if K is any convex, non-empty, compact subset of X and T is the zero map on X, we have $T(K) = T(E(K))$.

(It is crucial to note that $E(K) \ne \emptyset$ since we are assuming throughout that X is locally convex, K is compact, and hence the Krein-Milman theorem applies.)

We now ask if any of the conditions in Theorem 11 can be relaxed. The following examples explore several of the possibilities.

Example 14. In the proof of Theorem 11, we assumed that $\mathbf{f}_0 = \mathbf{0}$ and hence we required that there be a point, $\mathbf{x}_0 \in K \backslash \{\mathbf{0}\}$ such that $\mathbf{x}_0 \in \ker T$. We need such an $\mathbf{x}_0$ to exist.

Proof. Let $X = \mathbb{R}^2$ and let

$$K = \left\{ (x, y) \mid -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2} \right\}. \tag{30}$$

Note that K has one proper non-singleton face and $(0,0)$ is in that face (see Figure 1). Let T be the identity map on X. Then T has trivial kernel, and $(0,0)$ is not an extreme point of K. Hence, $T(K) \neq T(E(K))$. □

Example 15. Theorem 11 is false if we assume that K is closed rather than compact.

Proof. Let $X = \mathbb{R}^2$. Let

$$\begin{aligned} K_1 &= \{(x, y) \mid y \geq 0, x \leq 1\} \\ K_2 &= \{(x, y) \mid x \geq 1, y \geq x^2 - 1\} \\ K &= K_1 \cup K_2. \end{aligned} \tag{31}$$

Then K is closed, convex and has exactly one proper non-singleton face, namely

$$F = \{(x, y) \mid x \leq 1\}. \tag{32}$$

Moreover,

$$E(K) = \{(x, x^2 - 1) \mid x \geq 1\}. \tag{33}$$

For $(x, y) \in \mathbb{R}^2$ define $T(x, y) = x$. Then T is a linear map. Notice that if $(x, y) \in E(K)$ then $T(x, y) \geq 1$. However, $(0, 0) \in K$ and hence

$$T(E(K)) \neq T(K). \tag{34}$$

□

Since K is assumed to be compact, K is bounded when X is locally convex.

Example 16. If we only require K to be bounded rather than compact, the conclusion of Theorem 11 need not hold.

Proof. Let $X = \mathbb{R}^2$. Let

$$K = \left\{(x, y) \mid -1 < x < 1, 0 \leq y < \sqrt{1 - x^2}\right\}. \tag{35}$$

Then K is convex and bounded and has only one proper face which is not an extreme point. Hence, if T is the zero map on X, all of the hypotheses in Theorem 11 are satisfied, except we have replaced compact with bounded. However, the conclusion to Theorem 11 does not hold for the simple reason that K has no extreme points. □

Example 17. In Theorem 11, it is crucial that K has at most one non-singleton proper face F.

Proof. Let $X = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}$ be given by $T(x, y) = x - y$. Let

$$K = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{2x - x^2}\right\}. \tag{36}$$

Notice that K is the top left quadrant of the disk of radius one with center $(1, 0)$. Hence, K is compact, convex, and has two proper non-singleton faces. Let

$$Q = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y = \sqrt{2x - x^2}\right\}. \tag{37}$$

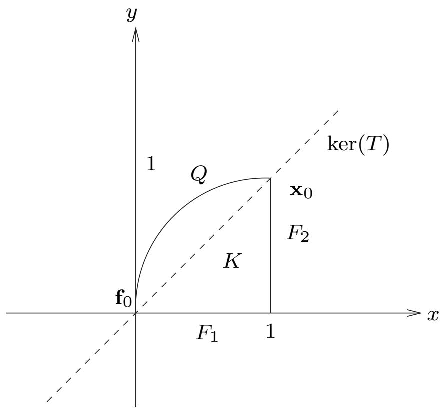

Figure 2: Illustration for Example 17.

The extreme points of K are the points in Q together with the point $(1,0)$. (See Figure 2.)

If $(x,y) \in Q$, $y \geq x$, and so $T \leq 0$ on Q. Since $T(1,0) = 1$, there is no extreme point of K whose output is $\frac{1}{2}$. On the other hand, $T\left(\frac{1}{2},0\right) = \frac{1}{2}$ and $\left(\frac{1}{2},0\right) \in K$. Further, by taking

$$\begin{aligned} F_1 &= \{(x,0) \mid 0 \leq x \leq 1\} \\ F_2 &= \{(1,y) \mid 0 \leq y \leq 1\} \\ \mathbf{f}_0 &= (0,0) \\ \mathbf{x}_0 &= (1,1), \end{aligned} \tag{38}$$

we have $\mathbf{f}_0 \in F$ and $\mathbf{x}_0 - \mathbf{f}_0 \in \ker(T)$. Hence, all the hypotheses of Theorem 11 are satisfied except for the requirement about the number of proper non-singleton faces. Thus, the theorem is false if we allow K to have two proper non-singleton faces. □

Remark 18. This example can be easily modified to show that Theorem 11 is false if we allow K to have n proper non-singleton faces for $n \in \mathbb{N}$, $n \geq 2$.

If the only proper faces of K are singletons we have to reformulate the hypotheses.

Proposition 19. *Suppose the only proper faces of K are singletons. Let V be a real vector space. Suppose $T : X \to V$ is a linear map such that there exist $\boldsymbol{f}_0 \in E(K)$ and $\boldsymbol{x}_0 \in K$ with $\boldsymbol{x}_0 - \boldsymbol{f}_0 \in \ker(T)$ and $\boldsymbol{x}_0 \neq \boldsymbol{f}_0$. If X is finite-dimensional, then*

$$T(K) = T(E(K)).$$

Proof. Since X is finite-dimensional, we may assume that $X = \mathbb{R}^n$. Further, we may assume that $\mathbf{f}_0 = \mathbf{0}$, and so $\mathbf{0}$ is an extreme point of K. Since we are really only interested in how T acts on K we may also assume that $\dim K = n$.

By assumption $\mathbf{x}_0 \in \ker(T)$. Hence, $\lambda \mathbf{x}_0 \in \ker T$ for $\lambda \in \mathbb{R}$. Let $\mathbf{x} \in K$. Let l be the line parameterized by $\mathbf{c}(\lambda) = \mathbf{x} + \lambda \mathbf{x}_0$. Notice that T is constant on l since $\mathbf{x}_0 \in \ker(T)$. Moreover, $l \cap \partial K \neq \emptyset$ since K is compact in $\mathbb{R}^n$ and therefore closed and bounded. Since K has no proper non-singleton faces and X is finite-dimensional, all points in the boundary of K must be extreme points. Hence, there exists $\lambda \in \mathbb{R}$ such that $\mathbf{z} := \mathbf{x} + \lambda \mathbf{x}_0 \in E(K)$ and $T\mathbf{z} = T\mathbf{x}$. Since $\mathbf{x}$ was arbitrary, the result follows. $\square$

Proposition 19 does generalize to infinite dimensions but the proof in the infinite dimensional case is a bit more complicated. The proof depends on the following lemma which we believe must be known. However, we have not found it in the literature.

Lemma 20. *Let K be a non-empty compact convex set in X with dimension at least two. Suppose that the only proper faces of K are singletons. Then there exist K_1 and K_2 such that the following hold:*

(*i*) $K_1 \cup K_2 = K$
(*ii*) K_1, K_2 *are compact, convex, and non-empty.*
(*iii*) K_1 *and* K_2 *have exactly one non-singleton proper face.*
(*iv*) $E(K_1) \cup E(K_2) = E(K)$.

We believe that this lemma is intuitively appealing. When constructing the proof, we essentially think of K as a 'ball' which we are cutting in half. Then K_1 and K_2 are the resulting two 'hemispheres' which 'obviously' have one proper non-trivial face each. Here are the details.

Proof. Via translation, we may assume that $\mathbf{0} \in K$. Let $\mathbf{x}_0 \in K \backslash \{\mathbf{0}\}$.

By Theorem 1.2.10 in [2] there exists a continuous linear functional $\rho : X \to \mathbb{R}$ such that $\rho(\mathbf{x}_0) > 0$. (Remember that we are assuming that X is locally convex.) By scaling, we may assume that $\rho(\mathbf{x}_0) = 1$.

We now define several (non-empty) sets. See Figure 3 for a schematic. Let:

$$
\begin{aligned}
H_0 &= \left\{ \mathbf{x} \in X \mid \rho(\mathbf{x}) = \frac{1}{2} \right\} \\
H_1 &= \left\{ \mathbf{x} \in X \mid \rho(\mathbf{x}) \leq \frac{1}{2} \right\} \\
H_2 &= \left\{ \mathbf{x} \in X \mid \rho(\mathbf{x}) \geq \frac{1}{2} \right\} \\
K_0 &= K \cap H_0 \\
K_1 &= K \cap H_1 \\
K_2 &= K \cap H_2.
\end{aligned}
\tag{39}
$$

It is immediate that $K = K_1 \cup K_2$ and so (*i*) is satisfied.

Note that

$$K_0 \subset K_1, \qquad K_2 \subset K \tag{40}$$

and that $K = K_1 \cup K_2$. Further K_0, K_1, and K_2 are intersections of closed, convex sets and hence are closed and convex. Additionally, since K is compact, K_0, K_1, and K_2 are compact. Hence (*ii*) is satisfied.

In the sequel we will repeatedly make use of the fact that if $\mathbf{c} \in K_1 \backslash K_0$ and $\mathbf{d} \in K_2 \backslash K_0$ then $L[\mathbf{c}, \mathbf{d}] \cap K_0$ is a singleton set.

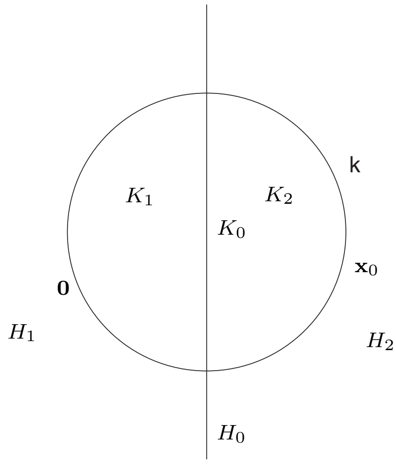

Figure 3: Schematic for Lemma 20.

We will also make use of the fact that real 2-dimensional vector spaces are isomorphic to $\mathbb{R}^2$ and hence elementary Euclidian geometry is applicable.

We now show that K_0 is a face of both K_1 and K_2. First, since $\frac{1}{2}\mathbf{x}_0 \in K_0$ it follows that K_0 is non-empty.

Let $\mathbf{u}, \mathbf{v} \in K_1$ and suppose that there exists $t \in (0,1)$ such that

$$t\mathbf{u} + (1-t)\mathbf{v} \in K_0. \tag{41}$$

Now $\rho(\mathbf{u}), \rho(\mathbf{v}) \leq \frac{1}{2}$ while

$$\rho(t\mathbf{u} + (1-t)\mathbf{v}) = \frac{1}{2}. \tag{42}$$

Hence,

$$\begin{aligned} t\rho(\mathbf{u}) + (1-t)\rho(\mathbf{v}) &= \frac{1}{2} \\ \frac{t}{2} + (1-t)\rho(\mathbf{v}) &\geq \frac{1}{2} \\ (1-t)\rho(\mathbf{v}) &\geq \frac{1-t}{2} \\ \rho(\mathbf{v}) &\geq \frac{1}{2}, \end{aligned} \tag{43}$$

whence $\rho(\mathbf{v}) = \frac{1}{2}$. Thus, $\mathbf{v} \in K_0$ and analogously so is $\mathbf{u}$. Therefore K_0 is a face of K_1. A similar argument shows that K_0 is a face of K_2.

We next note that K_0 is not a singleton set (and this is where we require K to have dimension bigger than two).

We observe that since $\rho(\mathbf{x}_0) = 1$ the only point in the span of $\mathbf{x}_0$ that is in K_0 is $\frac{1}{2}\mathbf{x}_0$.

Choose $\mathbf{y} \in K$ so that $\mathbf{y}$ is not in the span of $\mathbf{x}_0$. We will use $\mathbf{y}$ to find a point in K_0 that is not $\frac{1}{2}\mathbf{x}_0$. There are three cases.

If $\rho(\mathbf{y}) = \frac{1}{2}$ then K_0 is a non-singleton set.

If $\rho(\mathbf{y}) > \frac{1}{2}$ then $\mathbf{z} = \frac{1}{2\rho(\mathbf{y})}\rho(\mathbf{y}) \in K_0$ and $\mathbf{z}$ is not in the span of $\mathbf{x}_0$. Hence, K_0 is a non-singleton set in this instance as well.

If $\rho(\mathbf{y}) < \frac{1}{2}$ take $t = \frac{1}{2(1-\rho(\mathbf{y}))}$ and $\mathbf{z} = t\mathbf{y} + (1-t)\mathbf{x}_0$. It is straightforward to check that $t \in (0,1)$.

Notice that

$$\begin{aligned} \rho(\mathbf{z}) &= t\rho(\mathbf{y}) + (1-t)\rho(\mathbf{x}_0) \\ &= t\rho(\mathbf{y}) + 1 - t \\ &= \frac{\rho(\mathbf{y})}{2(1-\rho(\mathbf{y}))} + \frac{2(1-\rho(\mathbf{y}))-1}{2(1-\rho(\mathbf{y}))} \\ &= \frac{1-\rho(\mathbf{y})}{2(1-\rho(\mathbf{y})} \\ &= \frac{1}{2}. \end{aligned} \tag{44}$$

Since $\mathbf{z} \neq \mathbf{x}_0$ is in $L[\mathbf{y}, \mathbf{x}_0]$ and $\mathbf{y}$ is not in the span of $\mathbf{x}_0$ it follows that $\mathbf{z}$ is not in the span of $\mathbf{x}_0$. Hence K_0 is a non-singleton set in this situation as well.

So far, we have shown that K_0 is a proper non-singleton face of K_1 and K_2. We would like to show that it is unique in that regard. We will prove this for K_1. The proof for K_2 is essentially identical.

There are several cases to consider.

Case 1. We start with the case when $F \cap K_0$ is empty.

Choose $\mathbf{x} \in F$ and $\mathbf{y} \in K_0$. By assumption, $\mathbf{x} \neq \mathbf{y}$. Since F is a face, the only line segment in F with endpoint $\mathbf{y}$ that intersects $\mathbf{x}$ is $L[\mathbf{x}, \mathbf{y}]$ for otherwise $\mathbf{y}$ would be in F.

We claim that F is a face of K. Let $\mathbf{a}, \mathbf{b} \in K$ and suppose that $\mathbf{x} \in L[\mathbf{a}, \mathbf{b}]$. If both $\mathbf{a}$ and $\mathbf{b}$ are in K_1 then, since F is a face of K_1, $\mathbf{a}, \mathbf{b} \in F$. Suppose that $\mathbf{b} \in K_2$. Since $\mathbf{x} \notin K_2$ and K is convex, $\mathbf{a} \in K_1$. Hence, $L[\mathbf{a}, \mathbf{b}]$ is a line segment in K with one end point in K_1 and the other in K_2. Notice that there exists a unique $\mathbf{c} \in L[\mathbf{a}, \mathbf{b}] \cap K_0$. But $\mathbf{c} \in K_0 \subset K_1$ and hence, since F is a face of K_1 it follows that $\mathbf{c} \in F$; which contradicts the fact that F does not intersect K_0. Hence, $\mathbf{b}$ cannot be in K_2. Therefore, F is indeed a proper face of K and by assumption, must be an extreme point.

Before we proceed with the second case, we first note that since K_0 has more than one element and X is locally convex (so that the Krein-Milman theorem is applicable)

$$\emptyset \neq E(K_0) \subset K_0. \tag{45}$$

Case 2. We assume that $F \cap K_0 \neq \emptyset$ and

$$\mathbf{x} \in F \cap K_0 \Rightarrow \mathbf{x} \in E(K_0). \tag{46}$$

It immediately follows that $F \cap K_0$ is a singleton set containing the point $\mathbf{x}$, say. We will show that $F = F \cap K_0$.

Let $\mathbf{z} \in K_2 \backslash K_0$ and $\mathbf{y} \in K_1$. Since K is convex, $L[\mathbf{y}, \mathbf{z}] \subset K$ and $L[\mathbf{y}, \mathbf{z}] \cap K_0$ is a singleton set. (For unless a line segment is parallel to K_0, the linear functional ρ is injective on the line segment.) Let $\mathbf{z}_0 \in L[\mathbf{y}, \mathbf{z}] \cap K_0$. If $\mathbf{z}_0 \neq \mathbf{x}$ then (since F is a face of K_1 and $\mathbf{z}_0 \notin F$)

$$L[\mathbf{y}, \mathbf{z}] \cap F = L[\mathbf{y}, \mathbf{z}_0] \cap F = \{\mathbf{y}\} \tag{47}$$

if $\mathbf{y} \in F$. Otherwise $L[\mathbf{y}, \mathbf{z}_0] \cap F$ is empty. Therefore, $L[\mathbf{y}, \mathbf{z}]$ is of interest only if $\mathbf{z}_0 = \mathbf{x}$.

The remainder of our discussion for this case uses a series of arguments by contradiction. The big assumption we will make is that there exists $\mathbf{y} \in F \backslash K_0$. After some discussion it will become clear that this assumption is unsustainable. Proceeding under that assumption, let us now suppose that there exists $\mathbf{y}_1 \in F$ with $\mathbf{y}_1 \neq \mathbf{x}$ and

$$L[\mathbf{y}_1, \mathbf{x}] \cap L[\mathbf{y}, \mathbf{x}] = \{\mathbf{x}\}. \tag{48}$$

Then

$$L[\mathbf{y}_1, \mathbf{z}] \cap K_0 = \{\mathbf{u}\}, \tag{49}$$

say. Notice that $\mathbf{u} \neq \mathbf{x}$.

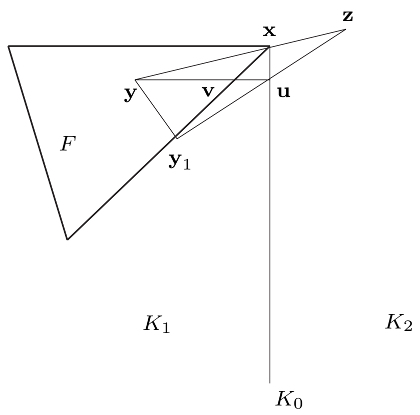

Figure 4: Illustration for Case 2 in Lemma 20.

Now $\mathbf{u}$ and $\mathbf{x}$ lie on the triangle with vertices $\mathbf{y}, \mathbf{y}_1$, and $\mathbf{z}$. (see Figure 4.) Hence $\mathbf{y}, \mathbf{u}, \mathbf{y}_1$, and $\mathbf{x}$ are coplanar. In particular $L[\mathbf{y}, \mathbf{u}] \cap L[\mathbf{y}_1, \mathbf{x}]$ is non-empty. Let $\mathbf{v}$ be the (necessarily unique) point in that intersection. Since $L[\mathbf{y}_1, \mathbf{x}] \subset F$ it follows that $\mathbf{v} \in F$. Since F is a face of K_1, it follows that $\mathbf{u} \in F$ which is a contradiction. Hence, no such $\mathbf{y}_1$ can exist.

Thus, F is a subset of the line, l, passing through $\mathbf{y}$ and $\mathbf{x}$. Notice that $\mathbf{z} \in l$. Since ρ is continuous and K is compact there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha < \frac{1}{2} < \beta \\ &\alpha \leq \rho(\mathbf{w}) \leq \beta \end{aligned} \tag{50}$$

for each $\mathbf{w} \in l \cap K$. Having taken the supremum over α and the infimum over β we may assume without loss of generality that $\rho(\mathbf{y}) = \alpha$ and $\rho(\mathbf{z}) = \beta$.

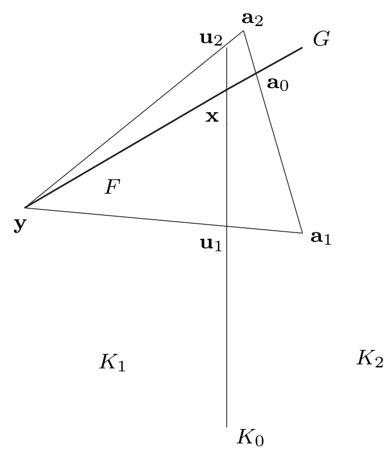

Figure 5: Illustration for Case 2b in Lemma 20.

Let $G = l \cap K$ and note that $F \subset G$. We will now argue that G is a face of K. Suppose not: Then we can find two distinct points $\mathbf{a}_1, \mathbf{a}_2 \in K$ at least one of which is not in G and $t \in (0, 1)$ such that

$$\mathbf{a}_0 = t\mathbf{a}_1 + (1 - t)\mathbf{a}_2 \in G. \tag{51}$$

There are several situations to consider.

Case 2a. Suppose first that $\mathbf{a}_1, \mathbf{a}_2 \in K_1$. Then $\mathbf{a}_0 \in K_1$ (since K_1 is convex) and so $\mathbf{a}_0 \in F$. But F is face, and so $\mathbf{a}_1, \mathbf{a}_2 \in F \subset G$, contradicting our assumption that at least one of $\mathbf{a}_1$ and $\mathbf{a}_2$ is not in G.

Case 2b. Suppose then that $\mathbf{a}_1, \mathbf{a}_2 \in K_2$ and at least one of $\mathbf{a}_1$ and $\mathbf{a}_2$ is not in $K_0 \subset K_1$. Then

$$\begin{aligned} &L[\mathbf{y}, \mathbf{a}_1] \cap K_0 = \{\mathbf{u}_1\} \\ \text{and} \quad &L[\mathbf{y}, \mathbf{a}_2] \cap K_0 = \{\mathbf{u}_2\}. \end{aligned} \tag{52}$$

Now $\mathbf{a}_0, \mathbf{u}_1$, and $\mathbf{u}_2$ are on the triangle with vertices $\mathbf{y}, \mathbf{a}_1$, and $\mathbf{a}_2$ and thus are coplanar. Therefore, the line segments $L[\mathbf{y}, \mathbf{a}_0]$ and $L[\mathbf{u}_1, \mathbf{u}_2]$ intersect. Since $L[\mathbf{u}_1, \mathbf{u}_2] \subset K_0$ it follows that they intersect at $\mathbf{x}$ which contradicts our assumption that $\mathbf{x}$ is an extreme point of K_0. (See Figure 5.)

Case 2c. The remaining situation occurs when $\mathbf{a}_1 \in K_1 \backslash K_0$ and $\mathbf{a}_2 \in K_2 \backslash K_0$. Then

$$L[\mathbf{a}_1, \mathbf{a}_2] \cap K_0 = \{\mathbf{v}\}. \tag{53}$$

Suppose that $\mathbf{v} \neq \mathbf{x}$: then we can replace $\mathbf{a}_1$ with $\mathbf{v}$ if $\mathbf{a}_0 \in K_2$ and $\mathbf{a}_2$ with $\mathbf{v}$ if $\mathbf{a}_0 \in K_1$, thereby reducing to the previous situations.

What if $\mathbf{v} = \mathbf{x}$? By our choice of $\mathbf{a}_1$ and $\mathbf{a}_2$ the line segment $L[\mathbf{a}_1, \mathbf{a}_2]$ is not a subset of G. Hence, $\mathbf{x} = \mathbf{a}_0$ as well. Moreover, the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{y}, \mathbf{x}$ and $\mathbf{z}$ are coplanar (see Figure 6). The line segments $L[\mathbf{y}, \mathbf{a}_2]$ and $L[\mathbf{a}_1, \mathbf{z}]$ intersect K_0 at $\mathbf{u}_1$ and $\mathbf{u}_2$ say; and $\mathbf{x} \in L[\mathbf{u}_1, \mathbf{u}_2]$ again contradicting the assumption that $\mathbf{x}$ is an extreme point of K_0.

Therefore G is indeed a face of K. But that too is a contradiction since G is a proper non-singleton face of K.

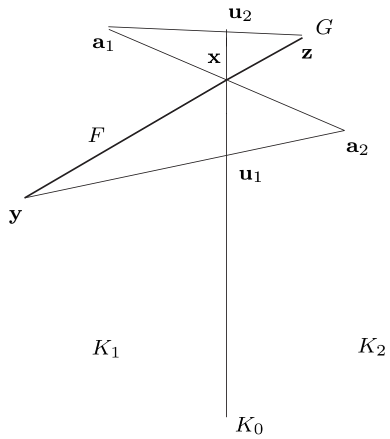

Figure 6: Illustration for Case 2c in Lemma 20.

Consequently, our assumption that there exists $\mathbf{y} \in F \backslash K_0$ has been shown to be absurd and as result, F is a singleton set and $\mathbf{x}$ is an extreme point of K_1.

Before moving on to the third case, we note that if F is a face of K_1 then $F \cap K_0$ is a face of K_0 (when the intersection is nonempty).

Case 3. We now assume that F is a face of K_1 which contains K_0 and there exists $\mathbf{y} \in F \backslash K_0$.

Define $G = F \cup K_2$. Notice that G is convex. To see this, observe that for any line segment with endpoints in G satisfies one of the following criteria:

(i) The end points are in K_2.
(ii) The end points are in F.
(iii) One end point is in $F \backslash K_0$ and the other is in $K_2 \backslash K_0$. In this case, the line segment is the union of a line segment in F and a line segment in K_2.

We will show that G is a face of K. To see this, suppose that $\mathbf{a}, \mathbf{b} \in K$, $t \in (0, 1)$, and $\mathbf{c} = t\mathbf{a} + (1 - t)\mathbf{b} \in G$.

If $\mathbf{a}, \mathbf{b} \in K_2$ then $\mathbf{a}, \mathbf{b} \in G$.

If $\mathbf{a}, \mathbf{b} \in K_1$ then $\mathbf{c} \in K_1$ since K_1 is convex. Hence, $\mathbf{c} \in F$. Since F is a face of K_1, it follows that $\mathbf{a}, \mathbf{b} \in F$ and hence $\mathbf{a}, \mathbf{b} \in G$.

If $\mathbf{a} \in K_1 \backslash K_0$ and $\mathbf{b} \in K_2$ then the location of $\mathbf{c}$ becomes important. Note that $L[\mathbf{a}, \mathbf{b}] \cap K_0 = \{\mathbf{d}\}$ for some $\mathbf{d}$. Also, it is clear that $\mathbf{b} \in G$. We need to show that $\mathbf{a} \in G$. If $\mathbf{c} \in L(\mathbf{a}, \mathbf{d})$ then from the preceding paragraph $\mathbf{a} \in G$ also. Suppose then that $\mathbf{c} \in L[\mathbf{d}, \mathbf{b})$.

If $\mathbf{a}$, $\mathbf{y}$, and $\mathbf{d}$ are colinear then either $\mathbf{y} \in L(\mathbf{a}, \mathbf{d})$ or $\mathbf{a} \in L[\mathbf{y}, \mathbf{d})$. In the first instance, we may conclude that $\mathbf{a} \in F$ (and hence G) since F is a face of K_1 and $\mathbf{y} \in K_1$. In the second instance, we may likewise conclude that $\mathbf{a} \in F$ since F is convex.

Suppose that $\mathbf{a}, \mathbf{y}$, and $\mathbf{d}$ are not colinear.

Note that $L[\mathbf{y}, \mathbf{b}] \subset G$. Since $\mathbf{y} \notin K_0$, $L[\mathbf{y}, \mathbf{b}] \cap K_0 = \{\mathbf{e}\}$ for some $\mathbf{e}$. The points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{y}$ are coplanar since they are points on the triangle with vertices $\mathbf{a}, \mathbf{b}$,

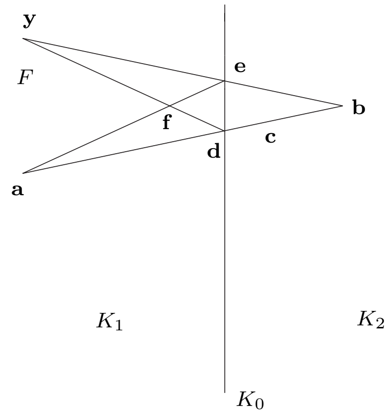

Figure 7: Illustration for Case 3 in Lemma 20.

and $\mathbf{y}$ (see Figure 7). Thus, the line segments $L[\mathbf{a}, \mathbf{e}]$ and $L[\mathbf{d}, \mathbf{y}]$ intersect at a point $\mathbf{f}$. Since $\mathbf{d}, \mathbf{y} \in F$ and F is convex, $\mathbf{f}$ is in F. Since $\mathbf{a}, \mathbf{e} \in K_1$, $\mathbf{f} \in L(\mathbf{a}, \mathbf{e}) \cap F$ and F is a face of K_1 it follows that $\mathbf{a} \in F \subset G$.

Hence, G is a non-singleton face of K. By the hypothesis in the statement of the theorem, $G = K$.

Case 4. We now assume that F is a proper face of K_1 and that $G = F \cap K_0$ is non-empty and hence a proper face of K_0. Let $\mathbf{x} \in G$. By Lemma 5 and since K has no proper non-singleton faces, $\mathbf{x}$ is either an extreme point of K or is a weak internal point of K.

Suppose that $\mathbf{x}$ is a weak internal point of K. Let $\mathbf{y} \in K_0$. By hypothesis there exists $\lambda > 0$ such that $\mathbf{z} = (1+\lambda)\mathbf{x} - \lambda\mathbf{y} \in K$. Notice that

$$\rho(\mathbf{z}) = (1+\lambda)\rho(\mathbf{x}) - \lambda\rho(\mathbf{y}) = \frac{1}{2}. \tag{54}$$

Thus $\mathbf{z} \in K_0$. Therefore $\mathbf{x}$ is interior relative to $\mathbf{y}$. Since $\mathbf{y}$ was arbitrary it follows that $\mathbf{x}$ is a weak internal point of K_0. Therefore, the smallest face of K_0 containing $\mathbf{x}$ is K_0 which contradicts the assumption that that G is a proper face of K_0. Thus, G contains only extreme points of K which are therefore extreme points of K_0. Therefore G is a singleton set and hence Case 2 is applicable. Consequently statement (iii) in the theorem is satisfied.

We now see that any point in K_0 that is not an extreme point of K is a weak-internal point of K_0. Combining this observation with the first case and the fact that our above arguments can be easily modified for K_2, we conclude that

$$E(K) = E(K_1) \cup E(K_2). \tag{55}$$

Thus (iv) is satisfied and the proof is complete. □

Theorem 21. *Suppose the only proper faces of K are singletons. Let V be a real vector space. Suppose $T : X \to V$ is a linear map such that there exist $\boldsymbol{f}_0 \in E(K)$ and $\boldsymbol{x}_0 \in K$ with $\boldsymbol{x}_0 - \boldsymbol{f}_0 \in \ker T$ and $\boldsymbol{x}_0 \neq \boldsymbol{f}_0$. Then*

$$T(K) = T(E(K)).$$

Proof. Without loss of generality $\mathbf{f}_0 = \mathbf{0}$ and hence $\mathbf{x}_0 \neq \mathbf{0}$ is in the kernel of T.

By Proposition 19 we may assume that the dimension of K is infinite (and hence at least 2). The conditions of Lemma 20 are satisfied. Let K_0, K_1, and K_2 be the sets constructed in the proof of that lemma.

Notice that K_1 and K_2 satisfy Theorem 11. (The role of $\mathbf{f}_0$ from that theorem is the unique point in $L[\mathbf{0}, \mathbf{x}_0] \cap K_0$.) Hence

$$T(K_1) = T(E(K_1)) \tag{56}$$

and

$$T(K_2) = T(E(K_2)). \tag{57}$$

Consequently,

$$T(K) = T(K_1 \cup K_2) = T(E(K_1) \cup E(K_2)) = T(E(K)). \tag{58}$$

□

Our next result is similar to Theorem 11. We relax the compactness of K but require more structure on X.

Theorem 22. *Let V be a real vector space. Suppose that X is a Banach space and that K is closed and bounded (rather than compact). Suppose that K has exactly one non-singleton proper face F. Let $T : X \to V$ be a linear map. If there exists $\boldsymbol{f}_0 \in F$, $\boldsymbol{x}_0 \in K$, with $\boldsymbol{f}_0 \neq \boldsymbol{x}_0$ and $\boldsymbol{x}_0 - \boldsymbol{f}_0 \in \ker T$, then $E(K) \neq \emptyset$. Moreover*

$$T(K) = T(E(K)).$$

Proof. The proof is essentially the same as in Theorem 11. As before, we assume without loss of generality that $\mathbf{f}_0 = \mathbf{0}$ and so $\mathbf{x}_0 \in \ker T$. For $\mathbf{z} \in K$ and $t \in \mathbb{R}$, we define $g_{\mathbf{z}}(t) = \mathbf{z} + t\mathbf{x}_0$. The map $g_{\mathbf{z}}$ is continuous and injective. Thus, $g_{\mathbf{z}}^{-1}(K)$ is closed. Further, $g_{\mathbf{z}}^{-1}(K)$ is convex. Note that

$$||g_{\mathbf{z}}(t)|| \geq |t|||\mathbf{x}_0|| - ||\mathbf{z}|| \to \infty \tag{59}$$

as $|t| \to \infty$. Since K is bounded, $g_{\mathbf{z}}(t) \notin K$, for sufficiently large $|t|$. By the Heine-Borel theorem (Theorem 5.3.1 in [6]), $g_{\mathbf{z}}^{-1}(K)$ is a compact interval, $[a, b]$ say. Proceeding as before, we note that $g_{\mathbf{z}}(a)$ and $g_{\mathbf{z}}(b)$ are not weak internal points and are therefore both in proper faces of K. Moreover, at least one of them must be an extreme point of K and so $E(K) \neq \emptyset$. To complete the proof, note that

$$T\mathbf{z} = Tg_{\mathbf{z}}(a) = Tg_{\mathbf{z}}(b) \tag{60}$$

since $\mathbf{x}_0 \in \ker T$. □

Example 23. Suppose that K is closed, convex, and bounded in a Banach space, and that K has exactly one proper non-singleton face. It does not follow that K is compact.

Proof. Let $X = l^2$ and let B be the unit ball in X. Since X is a Hilbert space, and hence is strictly convex [4, Ch. 6]. By Exercise 15 in Section 6.2 of [4], $E(B) = \partial B$. Let H be a hyperplane through the origin, with corresponding functional ρ. Let $K = \{\mathbf{x} \in B \mid \rho(\mathbf{x}) \geq 0\}$. Then $E(K) = \partial B \cap K$ and K has exactly one proper non-singleton face, $F = K \cap H$. (This can be verified separately, but we can apply Lemma 20 using the weak-$*$ topology.) Further, K is closed and bounded but not compact. $\square$

We close by mentioning a further direction of research. In our main results, the key point was that there were relatively few faces (either 0 or 1) of small positive dimension. The following conjecture was suggested to the first author by C. Akemann.

Conjecture 24. *With the notation in Theorem* 11*: Suppose that K has exactly one proper non-singleton face of dimension n and that $\dim V = n$. Can we conclude that $T(K) = T(E(K))$, possibly under additional assumptions like the continuity of T? Note that K is allowed to have proper faces of dimension larger than n.*

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