

The Monge-Kantorovich Problem for Distributions and Applications

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Dedicated to Hedy Attouch on the occasion of his 60th birthday.

Received: October 14, 2008

We study the Kantorovich-Rubinstein transshipment problem when the difference between the source and the target is not anymore a balanced measure but belongs to a suitable subspace $\mathbf{X}(\Omega)$ of first order distribution. A particular subclass $\mathbf{X}_0^\sharp(\Omega)$ of such distributions will be considered which includes the infinite sums of dipoles $\sum_k(\delta_{p_k} - \delta_{n_k})$ studied in [28, 29]. In spite of this weakened regularity, it is shown that an optimal transport density still exists among nonnegative finite measures. Some geometric properties of the Banach spaces $\mathbf{X}(\Omega)$ and $\mathbf{X}_0^\sharp(\Omega)$ can be then deduced.

Keywords: Monge-Kantorovich problem, optimal transportation, transshipment problem, flat norm, minimal connections, Jacobians

2000 Mathematics Subject Classification: 49J45, 49J20, 82C70, 90B06

1. Introduction

In the recent years, motivated by many applications, a lot of attention has been devoted by the mathematical community to mass transportation problems. They can be expressed in different equivalent formulations, that we will shortly recall below. The usual setting for these problems requires to consider source and target in the space of probability measures on a domain of \mathbb{R}^N , on a manifold, or more generally on a metric space. On the other hand, for various applications (see for instance [12, 28, 29]) it is

*The research of the second and third authors is part of the project “*Metodi variazionali nella teoria del trasporto ottimo di massa e nella teoria geometrica della misura*” of the program PRIN 2006 of the Italian Ministry of the University.

interesting to develop a theory of optimal transportation (and Wasserstein distances) for more general objects. In this paper Ω will be a convex compact subset of \mathbb{R}^N and we will focus our attention on the space of distributions of order one and zero “average”

$$\begin{aligned} \mathbf{X}_0(\Omega) := \{f \in \mathcal{D}'(\mathbb{R}^N) \mid \forall \varphi \in \mathcal{D}(\mathbb{R}^N), \\ \langle f, \varphi \rangle \leq C(\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^\infty(\Omega)}), \langle f, 1 \rangle = 0\}. \end{aligned} \quad (1)$$

Such distributions are compactly supported in Ω and the condition $\langle f, 1 \rangle = 0$ above means that whenever $\varphi \in \mathcal{D}(\mathbb{R}^N)$ is constant in Ω then $\langle f, \varphi \rangle = 0$. From (1), it is natural to endow $\mathbf{X}_0(\Omega)$ with the dual of the Lipschitz norm on smooth functions and we may identify $\mathbf{X}_0(\Omega)$ with a closed subspace of the dual of $\mathcal{C}^1(\Omega)$. Let us notice that, although it is tempting, we are not allowed to identify $\mathbf{X}_0(\Omega)$ with a subspace of the dual of $\text{Lip}(\Omega)$ since the extension of an element $f \in \mathbf{X}_0(\Omega)$ to all Lipschitz functions given by Hahn-Banach Theorem *is non unique!*.

As far as the usual setting for the Monge-Kantorovich problem is considered, one needs to work only with the subspace of measures of $\mathbf{X}_0(\Omega)$ given by

$$\mathcal{M}_0(\Omega) := \left\{ f \in \mathcal{M}(\Omega) \mid \int_{\Omega} f = 0 \right\}.$$

It is shown in [6] that the closure $\mathbf{X}_0^\sharp(\Omega)$ of $\mathcal{M}_0(\Omega)$ can be characterized as

$$\begin{aligned} \mathbf{X}_0^\sharp(\Omega) := \{f \in \mathbf{X}_0(\Omega) \mid \forall \varepsilon > 0 \exists C_\varepsilon > 0 \text{ s.t.} \\ |\langle f, \varphi \rangle| \leq C_\varepsilon \|\varphi\|_{L^\infty(\Omega)} + \varepsilon \|\nabla\varphi\|_{L^\infty(\Omega)} \forall \varphi \in \mathcal{D}(\mathbb{R}^N)\}. \end{aligned}$$

This strict subspace can be seen also as the completion of $\mathcal{M}_0(\Omega)$ with respect to the Monge-Kantorovich norm. In [6] it is proved that elements of $\mathbf{X}_0^\sharp(\Omega)$ can be represented as the distributional divergence of functions in $L^1(\Omega; \mathbb{R}^N)$ (or more in general of a suitable class of tangential vector measures). The role of this space will stand out for different reasons which will be clear through the paper. In particular by suitably extending the Monge-Kantorovich problem to all $\mathbf{X}_0(\Omega)$, we will construct a linear continuous projector from $\mathbf{X}_0(\Omega)$ onto $\mathbf{X}_0^\sharp(\Omega)$ (see Theorem 4.2).

Before considering the details of the weakened formulation of the Monge-Kantorovich mass transportation problem let us first recall the main issues about the classical version of the problem and its various formulations. Also in this short survey, for simplicity, we will limit ourselves to the case when Ω is a convex compact subset of \mathbb{R}^N .

• The most classical formulation of a mass transportation problem goes back to Monge (1781) and in modern terminology (Kantorovich, 1942) consists, given two probabilities f^+ and f^- on Ω , in finding a measure γ on $\Omega \times \Omega$ which minimizes the total transportation cost

$$\int_{\Omega \times \Omega} |x - y| d\gamma(x, y) \quad (2)$$

among all admissible transport plans γ such that $\pi_{\sharp}^1 \gamma = f^+$ and $\pi_{\sharp}^2 \gamma = f^-$. Here π_{\sharp}^1 and π_{\sharp}^2 are the usual push-forward operators associated to the projections π^1 and π^2 from $\Omega \times \Omega$ on the first and second factors respectively. Notice that the formulation above is meaningless if f^+ and f^- are distributions that are not measures.

We say that the problem is of *transshipment* type (see Kantorovich and Rubinstein [26]) when only the difference $f^+ - f^-$ is specified.

Definition 1.1. The quantity

$$\mathcal{W}^1(f^+, f^-) := \inf_{\pi_1^1 \gamma = f^+, \pi_2^2 \gamma = f^-} \int_{\Omega \times \Omega} |x - y| d\gamma(x, y) \tag{3}$$

is called Wasserstein distance of f^+, f^- . Setting $f = f^+ - f^-$ the quantity above may be redefined as:

$$\inf_{\pi_1^1 \gamma - \pi_2^2 \gamma = f} \int_{\Omega \times \Omega} |x - y| d\gamma(x, y) \tag{4}$$

and we will denote it by $\mathcal{W}^1(f)$. This last quantity is a norm in the space of measures f such that $\int df = 0$ and it is called the Kantorovich norm.

- The dual formulation of the mass transportation problem (2) introduces the Kantorovich potential u which is a solution of the maximization problem

$$\max \left\{ \int u d(f^+ - f^-) : u \in \text{Lip}_1(\Omega) \right\} = \mathcal{W}^1(f). \tag{5}$$

The value of (5) is called flat dual norm $\|f^+ - f^-\|$ in the duality $(\text{Lip}(\Omega), \text{Lip}'(\Omega))$. In the classical setting u plays a key role in proving many of the results of the Monge-Kantorovich theory.

- The mass transportation problem above can be equivalently expressed through the *Kantorovich potential* u and the transport density μ which solve the system (see [3, 21, 1])

$$\begin{cases} -\text{div}(\mu D_\mu u) = f^+ - f^- & \text{in } \Omega \\ |D_\mu u| = 1 \text{ on spt } \mu, & u \text{ is 1-Lipschitz on } \Omega \end{cases} \tag{6}$$

which consists of an elliptic PDE coupled with an eikonal equation. For general measures μ the precise sense of the PDE above has to be intended by means of a weak formulation involving the theory of Sobolev spaces with respect to a measure (see [4, 3]). The following representation formula for μ holds:

$$\mu = \int \mathcal{H}^1 \llcorner S_{x,y} d\gamma(x, y)$$

where γ is an optimal transport plan for the cost (2), \mathcal{H}^1 is the 1-dimensional Hausdorff measure and $S_{x,y}$ is the geodesic line (the segment in our Euclidean case) connecting x to y . The transport density μ appears in various applications whose models admit a Monge-Kantorovich type formulation (see for example [3, 4, 15]). Moreover the transport density was also used to prove an existence result for optimal transport maps (see [21]).

Several results on the formulations above have been obtained; in particular, it has been shown that the regularity of the measure μ (called transport density) depends on the

regularity of the data f^+ and f^- . More precisely, we summarize here below what is known on this dependence.

- When f^+ and f^- are merely nonnegative measures, the transport density μ is a non-negative measure too (see [3]). As already mentioned above, the Monge-Kantorovich PDE (6) has to be intended in the sense of Sobolev spaces with respect to a measure.
- Additional assumptions on the source terms f^+ and f^- have to be made in order to provide more regularity to the transport density μ ; more precisely, if f^+ and f^- are in $L^p(\Omega)$ with $1 \leq p \leq +\infty$ then μ is in $L^p(\Omega)$ too (see [18], [17]). Moreover, in these cases the transport density μ turns out to be unique (see [1]).
- Some recent results on the particular case of mass transportation problems that intervene in the identification of sand pile shapes (see [15], [25]) indicate that the Hölder continuity of μ has to be expected, under additional regularity on the data, while simple examples show that nothing more than Lipschitz regularity can be obtained for μ even if very strong regularity hypotheses on the data are made.
- The continuity of μ , when f is continuous, as been obtained in [23] under some additional geometric assumptions on the supports of f^+ and f^- . However the continuity of μ in the general case is an open problem.

In the present paper we will also consider the opposite question of the existence of a transport density μ when the source datum $f = f^+ - f^-$ is less regular than a measure. Indeed, we assume that f only belongs to the space $\mathbf{X}_0(\Omega)$.

As it is well explained in [14, 28, 29] in some applications f describes the location and the topological degree of singularities of a map u with values in the sphere.

Indeed if u belongs to $W^{1,N-1}(\Omega, \mathbb{R}^N)$ and is bounded ($u \in W^{1,N-1}(\Omega, \mathcal{S}^{N-1})$ is a particular case) then by the *-Hodge operator we can say that the $N - 1$ form

$$D(u) := \sum_{i=1}^N \frac{(-1)^{i-1}}{N} u_i \widehat{du}^i, \quad (\text{where } \widehat{du}^i := du_i \wedge \cdots \wedge \widehat{du}_i \wedge \cdots)$$

corresponds to an $L^1(\Omega, \mathbb{R}^N)$ vector field, and then

$$-\operatorname{div}(*D(u)) \in \mathcal{D}'(\Omega).$$

More precisely $-\operatorname{div}(*D(u))$ belongs to $\mathbf{X}_0(\Omega)$. For a smooth map u

$$-\operatorname{div}(*D(u)) = *dD(u) = *J(u)$$

then with a little abuse of words the Jacobian of a bounded map in $W^{1,N-1}(\Omega, \mathbb{R}^N)$ belongs to $\mathbf{X}_0(\Omega)$ (actually to the smaller subspace $\mathbf{X}_0^\sharp(\Omega)$ as we will see).

If $J(u)$ is a measure then the Kantorovich norm of $J(u)$ correspond to the mass of the minimal connection of $J(u)$ which plays a key role in the relaxation of the Dirichlet functional (see [14]) as well as in the Ginzburg-Landau theory [14, 31].

Another interesting issue is to establish whether an N -form (or equivalently a distribution) is a distributional Jacobian or not (see for example [16, 30]). A related question is to establish when a distribution in $\mathbf{X}_0(\Omega)$ can be approximated weakly by Jacobians

and in the negative case one may try to give a quantitative answer. We will study this question in the last section of the paper.

The next example introduces a relevant class of distributions in $\mathbf{X}_0(\Omega)$ which appear as distributional Jacobians in the theory of the Ginzburg-Landau equations and which has been studied in [11, 28, 29].

Example 1.2. Given two sequences of points $\{p_i\}, \{n_i\}$ in Ω such that $\sum_{i=0}^\infty |p_i - n_i| < \infty$ we consider the distribution

$$\langle T, u \rangle := \sum_{i=1}^\infty u(p_i) - u(n_i) \quad \forall u \in \text{Lip}(\Omega).$$

It is easy to see that $T \in \mathbf{X}_0(\Omega)$; let us show that T is also in the space $\mathbf{X}_0^\sharp(\Omega)$. Let $\varepsilon > 0$ and consider k such that $\sum_{i>k} |p_i - n_i| \leq \varepsilon$; then

$$|\langle T, u \rangle| \leq \sum_{i \leq k} |u(p_i) - u(n_i)| + \sum_{k < i} |u(p_i) - u(n_i)| \leq 2k \|u\|_\infty + \varepsilon \text{Lip}(u).$$

Notice that in this case it is not possible to define a positive and a negative part of f .

The plan of the paper is the following: we will extend the Kantorovich norm and the mass transportation problem to the space of distributions $f \in \mathbf{X}_0(\Omega)$. We show that for a wide class of source data (namely for $f \in \mathbf{X}_0^\sharp(\Omega)$) the transport density μ still remains a measure. We will then show that the space $\mathbf{X}_0(\Omega)$ may be decomposed in the direct sum of $\mathbf{X}_0^\sharp(\Omega)$ and of the space of divergences of normal measures. This decomposition is “orthogonal” in the sense of the Wasserstein norm. Some of the ideas are connected with the papers [28, 29], [6] and [27], where very interesting tools for studying the distributions of $\mathbf{X}_0(\Omega)$ have been introduced. In particular our Theorem 3.8 extends (in the most natural reformulation) Theorem 1 of [27] to the case $k = 1$ of functions in $\mathcal{C}^{0,1}(\Omega)$.

2. Preliminary results of Functional Analysis and Measure Theory

2.1. Completion of dual spaces

It looks nice to formulate the question of existence of a Kantorovich potential (see (5)) in an abstract setting. We are then considering two separable normed spaces X and Y , with $Y \hookrightarrow X$ (in our context X will be $\mathcal{C}(\Omega)$ and $Y = \text{Lip}(\Omega)$). We assume that the injection above is dense (i.e. Y is dense in X with the norm of X) and compact (i.e. bounded sequences in Y are relatively compact in X). Moreover we assume that the norm of Y is l.s.c. with respect to the convergence in X .

It is well known as James’s theorem (see for instance Remark 3 in Chapter 1 of [13]) that Y is reflexive if and only if the supremum in the dual norm of Y'

$$\|f\|_{Y'} := \sup\{\langle f, u \rangle : u \in Y, \|u\|_Y \leq 1\}$$

is attained for every $f \in Y'$. In the situation which is interesting for our purposes Y is not reflexive and we are going to consider the elements f of a space that is smaller

than the whole dual space Y' . More precisely, we denote by $Y^\#$ the space of all $f \in Y'$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ which verifies

$$|\langle f, u \rangle| \leq C_\varepsilon \|u\|_X + \varepsilon \|u\|_Y \quad \forall u \in Y.$$

We endow $Y^\#$ with the norm of Y' . The obvious inclusions are that $X' \subset Y^\# \subset Y'$. In addition to the conditions above on X and Y we assume that there exists a family $T_\delta \in \mathcal{L}(X, Y)$ such that

$$\lim_{\delta \rightarrow 0} \|T_\delta u - u\|_X = 0 \quad \text{for every } u \in Y; \quad (7)$$

$$T_\delta \text{ is bounded in } \mathcal{L}(X, Y), \text{ (i.e. } \|T_\delta u\|_Y \leq C \|u\|_X). \quad (8)$$

When X and Y are Hilbert spaces it is enough to take $T_\delta = j^t$ for all δ , where j^t is the transpose of the injection map $j : Y \hookrightarrow X$. When $X = \mathcal{C}_b(\Omega)$ or $\mathcal{C}(\Omega)$ and $Y = \text{Lip}(\Omega)$ it is enough to take $T_\delta u = \rho_\delta * u$ where ρ_δ is a family of smooth convolution kernels converging weakly* to δ_0 .

Proposition 2.1. *The space $Y^\#$ coincides with the closure of X' in the dual space Y' .*

Proof. If $\{f_n\}$ is a sequence in X' which converges to f in the dual space Y' we have

$$|\langle f, u \rangle| \leq |\langle f_n, u \rangle| + |\langle f - f_n, u \rangle| \leq C_n \|u\|_X + \varepsilon \|u\|_Y$$

where we denoted

$$C_n = \|f_n\|_{X'}, \quad \varepsilon = \|f - f_n\|_{Y'}.$$

Therefore $f \in Y^\#$. Vice versa, if $f \in Y^\#$, take $f_\delta = f \circ T_\delta$; in order to show that f is in the closure of X' in the dual space Y' it is enough to show that

$$\lim_{\delta \rightarrow 0} \langle f_\delta, u \rangle = \langle f, u \rangle \quad \forall u \in Y.$$

Fix $u \in Y$; by the definition of $Y^\#$ we have for every $\varepsilon > 0$

$$|\langle f - f_\delta, u \rangle| = |\langle f, u - T_\delta u \rangle| \leq C_\varepsilon \|u - T_\delta u\|_X + \varepsilon \|u - T_\delta u\|_Y.$$

By the assumptions (7) and (8), passing to the limit as $\delta \rightarrow 0$ in the inequality above gives

$$\limsup_{\delta \rightarrow 0} |\langle f - f_\delta, u \rangle| \leq \varepsilon \|u\|_Y (1 + C)$$

which implies our claim since $\varepsilon > 0$ was arbitrary. □

Proposition 2.2. *For every $f \in Y^\#$ the supremum in the dual norm*

$$\sup\{\langle f, u \rangle : u \in Y, \|u\|_Y \leq 1\}$$

is attained.

Proof. Let $\{u_n\}$ be a maximizing sequence for the dual norm above; since the injection $Y \hookrightarrow X$ is compact, we may assume that $u_n \rightarrow u$ in X for some $u \in Y$ with $\|u\|_Y \leq 1$. Therefore by using the fact that $f \in Y^\#$,

$$|\langle f, u_n \rangle - \langle f, u \rangle| = |\langle f, u_n - u \rangle| \leq C_\varepsilon \|u_n - u\|_X + 2\varepsilon.$$

Passing now to the limit as $n \rightarrow \infty$ and using the fact that $\varepsilon > 0$ is arbitrary, we obtain that

$$\lim_{n \rightarrow \infty} \langle f, u_n \rangle = \langle f, u \rangle$$

which shows that u is a maximizer for the dual norm. □

There is another characterization of the elements of $Y^\#$.

Proposition 2.3. *We have $f \in Y^\#$ if and only if $\langle f, u_n \rangle \rightarrow 0$ for every $u_n \rightarrow 0$ in X with $\|u_n\|_Y$ bounded.*

Proof. Take $f \in Y'$ and $u_n \rightarrow 0$ in X with $\|u_n\|_Y$ bounded. Then

$$|\langle f, u_n \rangle| \leq C_\varepsilon \|u_n\|_X + \varepsilon \|u_n\|_Y$$

which gives, at the limit as $n \rightarrow \infty$, $\langle f, u_n \rangle \rightarrow 0$.

Vice versa assume by contradiction that there exist $\varepsilon_0 > 0$ and a sequence $\{u_n\}$ such that

$$\langle f, u_n \rangle \geq n \|u_n\|_X + \varepsilon_0 \|u_n\|_Y. \tag{9}$$

With no loss of generality we can suppose $\|u_n\|_Y = 1$ so that

$$\|u_n\|_X \leq \frac{1}{n} (\langle f, u_n \rangle - \varepsilon_0) \leq \frac{K}{n}.$$

Then $u_n \rightarrow 0$ in X , which implies $\langle f, u_n \rangle \rightarrow 0$ by the hypothesis. This is a contradiction with (9), which gives $\langle f, u_n \rangle \geq \varepsilon_0$. □

2.2. Some tangential calculus for measures

Let μ be a Radon measure in \mathbb{R}^N . Following [7, 8, 9], we introduce the tangent space \mathcal{T}_μ to the measure μ which is defined μ a.e. by setting $\mathcal{T}_\mu(x) := \mathcal{N}_\mu^\perp(x)$ where (see [6] for further details related to the L^∞ -case under consideration here):

$$\begin{aligned} \mathcal{N}_\mu(x) &= \{\xi(x) : \xi \in \mathcal{N}_\mu\} \quad \text{being} \\ \mathcal{N}_\mu &= \{\xi \in (L^\infty)^\# : \exists u_n \rightarrow 0, u_n \text{ smooth, } \nabla u_n \rightharpoonup \xi \text{ weakly}^* \text{ in } L^\infty\} \end{aligned}$$

It turns out that the subspaces \mathcal{T}_μ and \mathcal{N}_μ are local in the sense that $\xi \in \mathcal{T}_\mu$ (resp. \mathcal{N}_μ) iff $\xi(x) \in \mathcal{T}_\mu(x)$ (resp. $\mathcal{N}_\mu(x)$) holds μ -a.e.

We may now define an intrinsic notion of tangential and normal vector measures in $\mathcal{M}^N(\Omega)$. It will be useful in the construction of a complement $\mathbf{X}_0^\#(\Omega)$ in the space $\mathbf{X}_0(\Omega)$.

Definition 2.4. Let $\lambda \in \mathcal{M}(\Omega)^N$. If λ can be decomposed as $\lambda = v\mu$ where μ is a positive Radon measure and $v \in (L^1_\mu)^N$ satisfies $v(x) \in \mathcal{T}_\mu(x)$ μ -a.e., then we say that λ is a tangential measure. If alternatively $v(x) \in \mathcal{N}_\mu(x)$ μ -a.e. we say that λ is a normal measure. We will denote by \mathfrak{T} the space of tangential measures and by \mathfrak{N} the space of normal measures.

Clearly we have the decomposition in direct sum

$$(\mathcal{M}(\Omega))^N = \mathfrak{T} \oplus \mathfrak{N}.$$

The following is a basic and intuitive lemma on tangent spaces which will be used in the last section of the paper.

Lemma 2.5. *Let α and μ be two nonnegative Radon measures in \mathbb{R}^N such that $\mu = \alpha + \mu_s$ where μ_s is singular with respect to α . Then*

$$T_\mu(x) \subset T_\alpha(x) \quad \alpha\text{-a.e.}$$

Proof. We will prove that $\mathcal{N}_\alpha \subset \mathcal{N}_\mu$ and then the thesis will follow from the definition of tangent space. Since $\mu = \alpha + \mu_s$ where μ_s is singular with respect to α , if $g \in (L^1_\mu)^N$ then $g \in (L^1_\alpha)^N$. Consider $\xi \in \mathcal{N}_\alpha$, then ξ is the weak-* limit in $(L^\infty_\alpha)^N$ of a sequence $\{\nabla\varphi_n\}$ with $|\varphi_n| \leq c$ and $\varphi_n \rightarrow 0$ uniformly. The sequence $\{\nabla\varphi_n\}$ is bounded in $(L^\infty_\mu)^N$ and up to subsequences converges to $\tilde{\xi}$ weakly* in $(L^\infty_\mu)^N$. Let $g \in (L^1_\mu)^N$; then on the one hand

$$\int \nabla\varphi_n \cdot g d\alpha \rightarrow \int \xi \cdot g d\alpha$$

on the other hand

$$\int \nabla\varphi_n \cdot g d\mu \rightarrow \int \tilde{\xi} \cdot g d\mu.$$

The last equality reads as

$$\int \nabla\varphi_n \cdot g d\alpha + \int \nabla\varphi_n \cdot g d\mu_s \rightarrow \int \tilde{\xi} \cdot g d\alpha + \int \tilde{\xi} \cdot g d\mu_s,$$

then $\tilde{\xi} = \xi$ α -a.e. and the conclusion follows. \square

3. The Optimal Transport Problem

We will now extend the different formulations of the optimal transport problem to a distribution $f \in \mathbf{X}_0(\Omega)$, we will compare these formulations to the classical case of measures and then prove the main existence theorems for minimizers.

3.1. Kantorovich potential and optimal transport measure

In this subsection we will see that the classical theory can be easily extended provided the distribution f belongs to the subspace $\mathbf{X}_0^\sharp(\Omega)$. To that aim we simply particularize the results of the previous section in the case $X = \mathcal{C}(\Omega)$ and $Y = \text{Lip}(\Omega)$ endowed

with their natural norms. Then Y^\sharp coincides with $\mathbf{X}_0^\sharp(\Omega)$ and by Proposition 2.2 if $f \in \mathbf{X}_0^\sharp(\Omega)$ then

$$\sup\{\langle f, u \rangle : u \in \text{Lip}(\Omega), \|u\|_{\text{Lip}(\Omega)} \leq 1\} \tag{10}$$

is attained. If f has “zero average”, that is $\langle f, 1 \rangle = 0$, we may replace the constraint $\|u\|_{\text{Lip}(\Omega)} \leq 1$ by the seminorm inequality $\|\nabla u\|_{L^\infty(\Omega)} \leq 1$ thus obtaining the flat norm of f . The subspace of the restrictions to Ω of functions in $\mathcal{C}_0^1(\mathbb{R}^N)$ is dense in $\text{Lip}(\Omega)$ for the \mathcal{C}^0 topology. Then if $f \in \mathbf{X}_0^\sharp(\Omega)$ by Proposition 2.3

$$\max_{\varphi \in \text{Lip}_1} \langle f, \varphi \rangle = \sup_{\mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1} \langle f, \varphi \rangle.$$

Definition 3.1. For every $f \in \mathbf{X}_0(\Omega)$ the Wasserstein norm of f is defined by

$$\mathcal{W}^1(f) := \sup\{\langle f, u \rangle : u \in \mathcal{C}_0^1(\mathbb{R}^N), \|\nabla u\|_\infty \leq 1\}. \tag{11}$$

By Proposition 2.3 if $f \in \mathbf{X}_0^\sharp(\Omega)$ the sup in (11) does not change if performed on $\text{Lip}_1(\Omega)$ instead than $\mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1$ and it is attained on $\text{Lip}_1(\Omega)$. We notice however that the supremum is not achieved in general for $f \in \mathbf{X}_0(\Omega) \setminus \mathbf{X}_0^\sharp(\Omega)$. It is therefore worth to characterize those elements f which belong to $\mathbf{X}_0^\sharp(\Omega)$. Two such characterizations appeared in [6]; we report here the first one while the second will be given later in this section.

Theorem 3.2 ([6]). *Let $f \in \mathbf{X}_0(\Omega)$, then $f \in \mathbf{X}_0^\sharp(\Omega)$ if and only if there exists a vector field ν in $L^1(\Omega, \mathbb{R}^N)$ such that $-\text{div } \nu = f$.*

Let us now introduce a duality argument: if we define the mapping

$$h(p) = -\sup\{\langle f, u \rangle : u \in \text{Lip}(\Omega), \|\nabla u + p\|_{L^\infty(\Omega)} \leq 1\}$$

for every continuous function p , we have that the Fenchel transform defined for all vector measure λ by

$$h^*(\lambda) = \sup_p \langle \lambda, p \rangle - h(p) = \sup_{p,u} \{\langle \lambda, p \rangle + \langle f, u \rangle : \|\nabla u + p\|_{L^\infty(\Omega)} \leq 1\},$$

is given by

$$h^*(\lambda) = \begin{cases} \int f |\lambda| & \text{if } -\text{div } \lambda = f \text{ in } \mathcal{D}' \\ +\infty & \text{otherwise.} \end{cases}$$

The duality relation $\inf_\lambda h^*(\lambda) = -h(0)$ then reads

$$\begin{aligned} & \max\{\langle f, u \rangle : \|\nabla u\|_{L^\infty(\Omega)} \leq 1\} \\ &= \min \left\{ \int |\lambda| : \lambda \in \mathcal{M}^n(\Omega), -\text{div } \lambda = f \in \mathcal{D}' \right\}. \end{aligned} \tag{12}$$

The existence and the structure of an optimal λ for the right hand side of (12) has been discussed in [6] for $f \in \mathbf{X}_0^\sharp(\Omega)$ and will be analyzed for $f \in \mathbf{X}_0(\Omega)$ in the next section. We summarize as follows.

Theorem 3.3. For every $f \in \mathbf{X}_0^\sharp(\Omega)$

(1) there exists a Kantorovich potential u which maximizes the quantity

$$\mathcal{W}^1(f) = \max\{\langle f, u \rangle : \|\nabla u\|_{L^\infty(\Omega)} \leq 1\};$$

(2) there exists an optimal measure λ which solves the problem

$$\min\{\|\lambda\| : -\operatorname{div} \lambda = f \text{ in } \mathcal{D}'\}; \quad (13)$$

(3) the two extremal values in (10) and (13) are equal; i.e.

$$\mathcal{W}^1(f) = \min\{\|\lambda\| : -\operatorname{div} \lambda = f \text{ in } \mathcal{D}'\}. \quad (14)$$

Note that, being $f \in \mathbf{X}_0^\sharp(\Omega)$, the equality $-\operatorname{div} \lambda = f$ can be equivalently considered either in \mathcal{D}' or in the duality $\langle \operatorname{Lip}'(\Omega), \operatorname{Lip}(\Omega) \rangle$. If μ denotes the total variation $|\lambda|$ of λ , we can write $\lambda = v\mu$ for a suitable vector field $v \in (L_\mu^1(\Omega))^N$. The measure μ is the transport density which appear in the Monge-Kantorovich PDE (6) and we remark that μ is still a measure even if f is only in $\mathbf{X}_0^\sharp(\Omega)$.

The next proposition (proved in [6]) characterizes the distributions in $\mathbf{X}_0^\sharp(\Omega)$ as the divergence of tangential measures in \mathfrak{T} . We report here the proof of one of the two implications.

Proposition 3.4. If $v \in (L_\mu^1)^N$ is such that $\operatorname{div}(v\mu) \in \mathbf{X}_0^\sharp(\Omega)$, then we have $v(x) \in \mathcal{T}_\mu(x)$ for μ -a.e. x . Vice versa if $v(x) \in \mathcal{T}_\mu(x)$ μ -a.e. then $\operatorname{div}(v\mu) \in \mathbf{X}_0^\sharp(\Omega)$.

Proof. We prove only the first of the two implications, for the complete proof we refer to [6]. If ξ is a normal vector field to μ , that is $\xi(x) \in \mathcal{N}_\mu(x)$ for μ -a.e. x , we have, denoting by $\{u_n\}$ the sequence corresponding to ξ

$$\int \xi \cdot v \, d\mu = \lim_{n \rightarrow \infty} \int \nabla u_n \cdot v \, d\mu = \lim_{n \rightarrow \infty} -\langle u_n, \operatorname{div}(v\mu) \rangle.$$

This last limit vanishes because $\operatorname{div}(v\mu) \in \mathbf{X}_0^\sharp(\Omega)$ by hypothesis, and $u_n \rightarrow 0$ uniformly with $\|\nabla u_n\|_{L^\infty(\Omega)}$ bounded (see Proposition 2.3). Therefore $v(x) \in \mathcal{T}_\mu(x)$ for μ -a.e. x . \square

The analysis performed in [3] can be made also in the more general case of $f \in \mathbf{X}_0^\sharp(\Omega)$; it is enough to repeat step by step what done in [3]: we obtain then the Monge-Kantorovich PDE

$$\begin{cases} -\operatorname{div}(\mu D_\mu u) = f & \text{in } \mathcal{D}' \\ u \in \operatorname{Lip}_1(\Omega) \\ |D_\mu u| = 1 & \mu\text{-a.e.} \end{cases} \quad (15)$$

In the system above the measure μ is called transport density and plays a role in the transport problem and in some of its applications.

Remark 3.5. The same conclusion holds in the more general framework of elasticity considered in [3] in which the function u is vector valued, a Dirichlet region Σ is present,

the bulk energy $\frac{1}{2}|\nabla u|^2$ is replaced by a convex p -homogeneous function $j(\nabla^{sym}u)$. For $f \in \text{Lip}_{1,\rho}(\Omega, \mathbb{R}^n)^\#$ the Monge-Kantorovich PDE (15) then takes the form

$$\begin{cases} -\operatorname{div}(\sigma\mu) = f & \text{in } \mathcal{D}'(\Omega \setminus \Sigma) \\ \sigma \in \partial j_\mu(x, e_\mu(u)) & \mu\text{-a.e. on } \Omega \\ u \in \text{Lip}_{1,\rho}(\Omega, \Sigma) \\ j_\mu(x, e_\mu(u)) = \frac{1}{p} & \mu\text{-a.e.} \\ \mu(\Sigma) = 0 \end{cases} \tag{16}$$

where we refer to [3] for the precise meaning of $j_\mu, e_\mu, \text{Lip}_{1,\rho}$. The analogy with the quoted results remains also in the scalar case where the transportation problem can be written for $f \in \mathbf{X}_0^\#(\Omega)$. Note that in this case we cannot decompose f into $f^+ - f^-$ because f is not in general a measure.

3.2. Duality, transport plan and transport densities

In order to construct the analogous of an optimal transport measure in the general case of a distribution $f \in \mathbf{X}_0(\Omega)$, we need to extend the Monge-Kantorovich duality and also to find a suitable generalization of the concept of transport plan. To that aim we now present a construction which will be shown to encompass the classical theory.

To each $\varphi \in \mathcal{C}_0^1(\mathbb{R}^N)$ we associate the function

$$D_\varphi(x, v, t) := \begin{cases} \frac{\varphi(x+tv) - \varphi(x)}{t} & \text{if } t \neq 0 \\ D\varphi(x) \cdot v & \text{if } t = 0, \end{cases}$$

which belongs to $\mathcal{C}_0(\Omega \times S^{N-1} \times [0, \infty))$. Then we can seek for a positive measure $\sigma \in \mathcal{M}(\Omega \times S^{N-1} \times [0, \infty))$ which minimizes the total variation among all the positive measures σ such that

$$\int_{\Omega \times S^{N-1} \times [0, \infty)} D_\varphi(x, v, t) d\sigma = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{C}_0^1(\mathbb{R}^N). \tag{17}$$

Then the new variational problem that we will consider reads as:

$$\min\{\|\sigma\| : \sigma \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty)), \sigma \text{ satisfies (17)}\}. \tag{18}$$

Before getting into the proof of existence in the general case of $f \in \mathbf{X}_0(\Omega)$ let us compare problem (18) with the classical Monge-Kantorovich problem so that we can understand the meaning of an optimal σ in the classical case. We will see that σ contains both optimal transport plans and optimal transport densities.

3.3. Comparisons with the classical case

Let f^+ and f^- be two finite and positive measures in Ω of equal total mass and let $f = f^+ - f^-$. Consider the map $p : \Omega \times \Omega \rightarrow \Omega \times S^{N-1} \times [0, \infty)$ defined by

$$p(x, y) := \begin{cases} \left(x, \frac{x-y}{|x-y|}, |x-y|\right) & \text{if } x \neq y \\ (x, e_1, 0) & \text{if } x = y, \end{cases}$$

where the choice of e_1 is arbitrary and it is not relevant for what follows.

Proposition 3.6. *Let f be a measure as above and let γ be a transport plan for f ; then the measure $p_{\#}(|x - y|\gamma) \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty))$ satisfies the property (17). Moreover if γ is an optimal transport plan then $p_{\#}(|x - y|\gamma)$ is optimal for problem (18). Therefore*

$$\mathcal{W}^1(f) = \inf_{\sigma \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty))} \|\sigma\| \text{ subject to } \pi_{\#}\sigma = f$$

Proof. Let $\varphi \in \mathcal{C}_0^1(\mathbb{R}^N)$ then

$$\begin{aligned} \int_{\Omega \times S^{N-1} \times [0, \infty)} D_{\varphi}(x, v, t) dp_{\#}(|x - y|\gamma) &= \int_{\Omega \times \Omega} D_{\varphi}(p(x, y))|x - y| d\gamma \\ &= \int_{\Omega \times \Omega} \varphi(x) - \varphi(y) d\gamma = \langle f, \varphi \rangle \end{aligned} \tag{19}$$

which shows the first part of the claim.

For the minimality: given a measure σ which satisfies (17) we have the inequality

$$\begin{aligned} \|\sigma\| &\geq \sup_{\varphi \in \mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1(\Omega)} \int_{\Omega \times S^{N-1} \times [0, \infty)} D_{\varphi}(x, v, t) d\sigma \\ &= \sup_{\varphi \in \mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1(\Omega)} \langle f, \varphi \rangle = \int_{\Omega} |x - y| d\gamma. \end{aligned} \tag{20}$$

On the other hand equation (19) implies that

$$\begin{aligned} \int_{\Omega} |x - y| d\gamma &= \sup_{\varphi \in \mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1(\Omega)} \int_{\Omega \times S^{N-1} \times [0, \infty)} D_{\varphi}(x, v, t) dp_{\#}(|x - y|\gamma) \\ &\leq \sup_{\psi \in \mathcal{C}_b(\Omega \times S^{N-1} \times [0, \infty))} \int_{\Omega \times S^{N-1} \times [0, \infty)} \psi(x, v, t) dp_{\#}(|x - y|\gamma) \\ &\leq \int_{\Omega} |x - y| d\gamma \end{aligned} \tag{21}$$

and the third term in the last inequality is the total variation of $p_{\#}(|x - y|\gamma)$. □

However, in the classical setting there is another way to build an optimal σ . Indeed let ν be optimal for problem (13) and consider the polar decomposition of ν as $\nu|v|$ where v is an unitary vector field. Define the map $p_{\nu} : \text{spt } \nu \rightarrow \Omega \times S^{N-1} \times [0, \infty)$ which to x associates $(x, v(x), 0)$.

Proposition 3.7. *Let f be a measure as above and let ν be optimal for problem (13). Then the measure $(p_{\nu})_{\#}\nu \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty))$ is optimal for problem (18) and it is supported on $\Omega \times S^{N-1} \times \{0\}$.*

Proof. First let us show that $\sigma = (p_{\nu})_{\#}\nu$ satisfies (17). For every $\varphi \in \mathcal{C}_0^1(\mathbb{R}^N)$

$$\int_{\Omega \times S^{N-1} \times [0, \infty)} D_{\varphi}(x, v, t) d\sigma = \int_{\Omega} D_{\varphi} d\nu = \langle \varphi, f \rangle.$$

As in (20) above the fact that σ satisfies (17) already implies the inequality

$$\sup_{\varphi \in \text{Lip}_1(\Omega)} \langle \varphi, f \rangle \leq \|\sigma\|.$$

On the other hand by definition of σ we have $\|\sigma\| = \|\nu\|$ and by the optimality of ν we obtain

$$\sup_{\varphi \in \text{Lip}_1(\Omega)} \langle \varphi, f \rangle = \|\nu\|.$$

□

3.4. Existence and structure of minimizers for problem (18)

Let us now prove the existence of an optimal σ for general $f \in \mathbf{X}_0(\Omega)$.

Theorem 3.8. *Let $f \in \mathbf{X}_0(\Omega)$. Then there exists an optimal measure σ for the transportation problem (18). Moreover the minimal value for problem (18) coincides with the supremal value for problem (10).*

Proof. Again for each $\varphi \in \mathcal{C}_0^1(\mathbb{R}^N)$ we define D_φ as above and we consider $Y := \{D_\varphi : \varphi \in \mathcal{C}_0^1(\mathbb{R}^N)\}$ equipped with the L^∞ norm.

Define the linear functional $F : Y \rightarrow \mathbb{R}$ by $F(D_\varphi) := \langle f, \varphi \rangle$ and notice that

$$|F(D_\varphi)| \leq \|f\|_{\mathbf{X}_0(\Omega)} \|\varphi\|_{\text{Lip}(\Omega)} = \|f\|_{\mathbf{X}_0(\Omega)} \|D_\varphi\|_\infty \quad \forall D_\varphi \in Y.$$

The Hahn-Banach theorem then provides an extension \tilde{F} of F to the space of bounded and continuous functions on $\Omega \times S^{N-1} \times [0, \infty)$ which preserves the norm of F , and such extension is represented by a measure $\sigma \in \mathcal{M}(\Omega \times S^{N-1} \times [0, \infty))$ which verifies

- (1) $F(D_\varphi) = \int_{\Omega \times S^{N-1} \times [0, \infty)} D_\varphi d\sigma$ for all $\varphi \in \mathcal{C}_0^1(\mathbb{R}^N)$;
- (2) $\|\sigma\| = \|\tilde{F}\| = \|F\|.$

In particular the first of the previous conditions implies that σ satisfies (17) and this gives

$$\langle f, \varphi \rangle = \int_{\Omega \times S^{N-1} \times [0, \infty)} D_\varphi d\sigma \leq \|\sigma\| \quad \forall \varphi \in \mathcal{C}_0^1(\mathbb{R}^N).$$

Then the equality

$$\|\sigma\| = \|\tilde{F}\| = \|F\|$$

implies both the equality

$$\|\sigma\| = \sup\{\langle f, \varphi \rangle : \varphi \in \mathcal{C}_0^1(\mathbb{R}^N) \cap \text{Lip}_1(\Omega)\} \tag{22}$$

and the minimality of σ . □

When f is a measure, in [3] it is shown that an optimal transportation density can be obtained through an optimal plan γ considering the total variation of the measure ν defined by the formula

$$\langle \nu, \varphi \rangle = \int \left(\int_{S_{x,y}} \varphi d\mathcal{H}^1 \right) \gamma(dx, dy) \tag{23}$$

where $S_{x,y}$ denotes the segment joining x to y (a geodesic line in the general case). We show that the same can be done when $f \in \mathbf{X}_0(\Omega)$. Decompose an optimal σ in the sum of two parts:

$$\sigma_0 := \sigma \llcorner (\Omega \times S^{N-1} \times \{0\}), \quad \sigma_+ = \sigma - \sigma_0,$$

and define the map $\pi : \Omega \times S^{N-1} \times (0, +\infty) \rightarrow \Omega \times \mathbb{R}^N$ as

$$\pi(x, v, t) = (x, x + tv).$$

Using a notation which is reminiscent of transport plans we define $\gamma_+ := \pi_{\#}\sigma_+$, and then in correspondence with σ_+ we consider the measures ν_+ defined by:

$$\langle \psi, \nu_+ \rangle := \int_{\Omega \times \Omega} \frac{1}{|x - y|} \int \frac{y - x}{|y - x|} \cdot \psi(\cdot) d\mathcal{H}^1 \llcorner [x, y] d\gamma_+(x, y). \tag{24}$$

To σ_0 instead we associate ν_0 defined by

$$\langle \psi, \nu_0 \rangle := \int_{\Omega \times S^{N-1}} \psi(x) \cdot V d\sigma_0(x, V). \tag{25}$$

Theorem 3.9. *Let ν_0 and ν_+ be defined by (25) and (24). Then*

$$-\operatorname{div}(\nu_0 + \nu_+) = f \text{ in } D'. \tag{26}$$

Moreover if σ is optimal then $\nu = \nu_0 + \nu_+$ is also optimal for (13).

Proof. For every $\varphi \in \mathcal{C}_0^1(\Omega)$ one has

$$\begin{aligned} \langle -\operatorname{div}(\nu_0 + \nu_+), \varphi \rangle &= \int_{\Omega \times \Omega} \frac{1}{|y - x|} \int_0^1 \nabla(x + t(y - x)) \cdot (y - x) dt d\tilde{\gamma}(x, y) \\ &\quad + \int_{\Omega \times S^{N-1}} \nabla\varphi(x) \cdot V d\sigma_0(x, V, 0) \\ &= \int_{\Omega \times \Omega} \frac{\varphi(y) - \varphi(x)}{|y - x|} d\sigma_+(x, y) \\ &\quad + \int_{\Omega \times S^{N-1}} \nabla\varphi(x) \cdot V d\sigma_0(x, V, 0) \\ &= \int_{\Omega \times S^{N-1} \times [0, \infty)} D_\varphi(x, v, t) d\sigma = \langle f, \varphi \rangle. \end{aligned}$$

About the minimality first observe that directly from the formula above one obtains an estimate on the total variation of ν :

$$\|\nu\| \leq \|\sigma\|. \tag{27}$$

On the other hand

$$\begin{aligned} |\nu| &= \sup_{\psi \in \mathcal{C}_0(\Omega), \|\psi\|_\infty \leq 1} \langle \nu, \psi \rangle \geq \sup_{\varphi \in \mathcal{C}^1(\Omega) \cap \operatorname{Lip}_1} \langle \nu, \nabla\varphi \rangle \\ &= \sup_{\varphi \in \mathcal{C}^1(\Omega) \cap \operatorname{Lip}_1} \langle f, \varphi \rangle = \|f\|_{\mathbf{X}_0(\Omega)} \end{aligned} \tag{28}$$

and if σ is optimal $\|f\|_{\mathbf{X}_0(\Omega)} = \|\sigma\|$ thus giving equality in (27) and the minimality of ν . □

Remark 3.10. By Theorem 3.9, it turns out that (13) and (14) are still valid for general f in $\mathbf{X}_0(\Omega)$ namely:

$$\mathcal{W}^1(f) = \min\{\|\nu\| : \nu \in \mathcal{M}(\Omega, \mathbb{R}^N), -\operatorname{div} \nu = f \text{ in } D'\}.$$

4. A decomposition of $\mathbf{X}_0(\Omega)$ and the distance to $\mathbf{X}_0^\sharp(\Omega)$

We will now apply the theory constructed so far to give an “orthogonal decomposition” of $\mathbf{X}_0(\Omega)$ and to compute the distance of a distribution $f \in \mathbf{X}_0(\Omega)$ to the space $\mathbf{X}_0^\sharp(\Omega)$ in terms of the problems introduced in the previous sections. Let us recall that, as remarked in the introduction, the space $\mathbf{X}_0^\sharp(\Omega)$ is a closed subspace of $\mathbf{X}_0(\Omega)$ and contains the weak Jacobians of maps in certain Sobolev spaces.

Let $f \in \mathbf{X}_0(\Omega)$; then by Theorem 3.9 f may be written as $f = -\operatorname{div} \nu$ for a suitable vectorial measure. Recalling Definition 2.4 we can further decompose ν as

$$\nu = \nu_T + \nu_N$$

where the measure $\nu_T \in \mathfrak{T}$ is a tangent measure and $\nu_N \in \mathfrak{N}$ is a normal measure. In other words we may write $\nu_T = v_T|\nu|$ and $\nu_N = v_N|\nu|$ where $v_T(x) \in \mathcal{T}_{|\nu|}(x)$ and $v_N(x) \in \mathcal{N}_{|\nu|}(x)$ for $|\nu|$ -a.e. x . We will use the following technical result:

Lemma 4.1. *Let α be a positive Radon measure in \mathbb{R}^N and let $\eta \in (L^1_\alpha)^N$. Then there exists a sequence $\{\varphi_n\} \subset \mathcal{C}_0^1(\mathbb{R}^N)$ such that: $\varphi_n \rightarrow 0$ uniformly, $|\nabla \varphi_n| \leq 1$ and*

$$\lim_{n \rightarrow \infty} \int \nabla \varphi_n(x) \cdot \eta(x) d\alpha = \int |\eta_N(x)| d\alpha,$$

where $\eta_N(x) \in (T_\alpha(x))^\perp$ denotes the normal component of $\eta(x)$.

Proof. Consider the integrand

$$j(x, z) = \eta(x) \cdot z + \chi_{\{|z| \leq 1\}},$$

and the functional

$$F(\varphi) = \begin{cases} \int j(x, \nabla \varphi) d\alpha & \text{if } \varphi \in \mathcal{C}_0^1(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases} \tag{29}$$

Denote by $\bar{F} : \mathcal{C}_0(\mathbb{R}^N) \rightarrow \mathbb{R}$ the relaxed functional of F with respect to the uniform convergence, then we claim that

$$\bar{F}(0) = - \int |\eta_N(x)| d\alpha. \tag{30}$$

By definition of relaxed functional (30) implies that there exists a sequence $\{\psi_n\} \subset \mathcal{C}_0^1(\mathbb{R}^N)$ such that: $\psi_n \rightarrow 0$ uniformly, $|\nabla \psi_n| \leq 1$ and $\lim_{n \rightarrow \infty} \int \nabla \psi_n(x) \cdot \eta(x) d\alpha = - \int |\eta_N(x)| d\alpha$. Then it is enough to consider $\varphi_n := -\psi_n$ to obtain the conclusion of the lemma. Let us then prove (30). By convexity $\bar{F}(0) = F^{**}(0)$ and by definition $F^{**}(0) = \sup_{g \in \mathcal{M}(\Omega)} -F^*(g) = - \inf_{g \in \mathcal{M}(\Omega)} F^*(g)$. We compute now F^* . We notice that

$F = J \circ A$ where J denotes the integral functional $J : p \in \mathcal{C}_0(\mathbb{R}^N; \mathbb{R}^N) \mapsto \int j(x, p) d\alpha$ and $A : u \in \mathcal{C}_0^1(\mathbb{R}^N) \mapsto \nabla u \in \mathcal{C}_0(\mathbb{R}^N; \mathbb{R}^N)$. As J is convex continuous at $p = 0$, by a classical duality result (see for instance [2]), we have

$$F^*(g) = \inf\{J^*(\sigma) \mid -\operatorname{div} \sigma = g\},$$

where J^* is the Fenchel conjugate of J on the dual space $\mathcal{M}(\mathbb{R}^N; \mathbb{R}^N)$. A simple computation shows that $j^*(x, w) = |w - \eta(x)|$ and by applying [10], we have

$$J^*(\sigma) = \int j^* \left(\frac{d\sigma}{d\alpha} \right) d\alpha + \int h(x, \sigma_s)$$

where σ_s represent the singular part of σ with respect to α and

$$h(x, z) = \sup \left\{ \psi(x) \cdot z \mid \int j(x, \psi(x)) d\alpha < \infty, \psi \in \mathcal{C}_0(\mathbb{R}^N; \mathbb{R}^N), |\psi| \leq 1 \right\} = |z|.$$

Therefore if we decompose all measures σ such that $-\operatorname{div} \sigma = g$ in its absolutely continuous and singular parts with respect to α so that $\sigma = w\alpha + \sigma_s$, we can write

$$F^*(g) = \inf \left\{ \int_{\mathbb{R}^N} |w - \eta| d\alpha + \int_{\operatorname{spt} \alpha} |\sigma_s| \mid -\operatorname{div}(w\alpha + \sigma_s) = g \right\}.$$

Let us choose $w = \eta_T$, $\sigma_s = 0$. Then $\bar{g} = -\operatorname{div}(\eta_T \alpha)$ and we get

$$\inf F^*(g) \leq F^*(\bar{g}) = \int |\eta_N| d\alpha,$$

and this prove the first inequality of (30). To prove the opposite inequality for a given $g = -\operatorname{div}(w\alpha + \sigma_s)$ define $m = \alpha + \sigma_s$ and set

$$q(x) = \begin{cases} w(x) & \alpha\text{-a.e.} \\ \frac{d\sigma_s}{d|\sigma_s|} & \sigma_s\text{-a.e.} \end{cases}$$

Since $g = -\operatorname{div} (q(x)m)$ is a measure, by Proposition 3.4, there holds $q(x) \in T_m(x)$ for m -a.e. x and then by Lemma 2.5 $w \in T_\alpha(x)$ for α -a.e. x . Thus

$$\int_{\mathbb{R}^N} |w - \eta| d\alpha + \int_{\operatorname{spt} \alpha} |\sigma_s| = \int_{\mathbb{R}^N} (|w - \eta_T| + |\eta_N|) d\alpha + \int_{\operatorname{spt} \alpha} |\sigma_s| \geq \int_{\mathbb{R}^N} |\eta_N| d\alpha.$$

It follows that $\inf F^* \geq \int_{\mathbb{R}^N} |\eta_N| d\alpha$ and we are led to the equality in (30). □

We are now in position to state the main theorem of this section.

Theorem 4.2. *For every $f \in \mathbf{X}_0(\Omega)$, there holds*

$$\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) = \min \left\{ \int |\nu_N| \mid \nu \in \mathcal{M}(\Omega, \mathbb{R}^N), -\operatorname{div} \nu = f \right\}.$$

Moreover there exists a unique decomposition $f = f_T + f_N$ with $f_T \in \mathbf{X}_0^\sharp(\Omega)$ and $f_N = \operatorname{div} \beta$ for some normal measure $\beta \in \mathfrak{N}$. We have in addition

$$\mathcal{W}^1(f) = \mathcal{W}^1(f_T) + \mathcal{W}^1(f_N).$$

Proof. By Theorem 3.9, there exists a measure ν such that $-\operatorname{div} \nu = f$. By the definition of $\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega))$ and recalling that elements of $\mathbf{X}_0^\sharp(\Omega)$ can be represented as divergence of tangential measures (see 3.4), we derive successively

$$\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) = \inf_{g \in \mathbf{X}_0^\sharp(\Omega)} \sup_{u \in \mathcal{C}_0^1 \cap \operatorname{Lip}_1} \langle f - g, u \rangle \tag{31}$$

$$= \inf_{g \in \mathbf{X}_0^\sharp(\Omega)} \sup_{u \in \mathcal{C}_0^1 \cap \operatorname{Lip}_1} \langle -\operatorname{div} \nu_N - \operatorname{div} \nu_T - g, u \rangle \tag{32}$$

$$= \inf_{G \in \mathfrak{T}} \sup_{u \in \mathcal{C}_0^1 \cap \operatorname{Lip}_1} \langle -\operatorname{div} \nu_N - \operatorname{div} G, u \rangle \tag{33}$$

$$= \inf_{G \in \mathfrak{T}} \sup_{u \in \mathcal{C}_0^1 \cap \operatorname{Lip}_1} \langle \nu_N + G, \nabla u \rangle \leq \int |\nu_N|. \tag{34}$$

On the other hand, by applying Lemma 4.1 to the measure $\nu_N + G$ of the last inequality, we obtain an equality. It follows in particular that for all ν such that $-\operatorname{div} \nu = f$

$$\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) = \int |\nu_N|. \tag{35}$$

The decomposition $f = f_T + f_N$ of an element $f \in \mathbf{X}_0(\Omega)$, is obtained by considering any ν such that $-\operatorname{div} \nu = f$ and $\mathcal{W}^1(f) = \int |\nu|$ (see Remark 3.10) and then by setting: $f_T := -\operatorname{div} \nu_T$ and $f_N = -\operatorname{div} \nu_N$. The uniqueness of such decomposition is straightforward since the divergence of a normal measure cannot belong to $\mathbf{X}_0^\sharp(\Omega)$ unless it vanishes. □

A second formula is related to the measures $\sigma \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty))$ such that $\pi_\sharp \sigma = f$ which are then admissible for problem (18). Indeed we introduced the natural decomposition $\sigma = \sigma_0 + \sigma_+$ and by equations (25) and (24) we associated a measure ν_0 to σ_0 and a measure ν_+ to σ_+ . By construction ν_+ is always a tangential measure while ν_0 is not necessarily so.

Theorem 4.3. *For every $f \in \mathbf{X}_0(\Omega)$ we have*

$$\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) = \inf\{\|\sigma_0\| : \sigma \in \mathcal{M}^+(\Omega \times S^{N-1} \times [0, \infty)) \text{ and } \pi_\sharp \sigma = f\}.$$

Proof. Let σ be such that $\pi_\sharp \sigma = f$ then as noticed before the measure ν_+ associated to σ by equation (24) is always tangential and then:

$$\mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) \leq \mathcal{W}^1(f, -\operatorname{div} \nu_+) \leq \|\nu_0\| \leq \|\sigma_0\|. \tag{36}$$

Let f_n be a sequence of measures in $\mathcal{M}(\Omega)$ such that $f_n = f_n^+ - f_n^-$ with $\|f_n^+\| = \|f_n^-\| < \infty$ and

$$\mathcal{W}^1(f, f_n) \leq \mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) + \varepsilon_n.$$

Let $\xi^n \in \mathcal{M}(\Omega \times S^{N-1} \times [0, \infty))$ of minimal total variation among the positive measures such that $\pi_\sharp \xi^n = f - f_n$. We decompose ξ^n as $\xi_0^n + \xi_+^n$. Then $\|\xi^n\| = \|\xi_0^n\| + \|\xi_+^n\|$ and therefore

$$\|\xi_+^n\| \leq \mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) + \varepsilon_n - \|\xi_0^n\| \leq \mathcal{W}^1(f, \mathbf{X}_0^\sharp(\Omega)) + \varepsilon_n. \tag{37}$$

Let γ_n be an optimal transport plan for f_n and consider $\sigma^n := \xi^n + p_{\sharp}(|x - y|\gamma_n)$ where p is the map introduced in Subsection 3.3. By the linearity $\pi_{\sharp}\sigma^n = f$ that is σ^n is admissible and by construction $\sigma_0^n = \xi_0^n$. Then (37) shows that σ^n is optimal up to infinitesimal constant ε_n . \square

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