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On the Continuous Representation of Quasiconcave Functions by their Upper Level Sets

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We provide a continuous representation of quasiconcave functions by their upper level sets. A possible motivation is the extension to quasiconcave functions of a result by Ulam and Hyers, which states that every approximately convex function can be approximated by a convex function.

Keywords: Quasiconcave, upper level set

1. Introduction

Since a seminal paper of Hyers and Ulam [1], several papers have studied the problem of the stability of functional inequalities. In particular, the first issue given (and solved) by Hyers and Ulam was roughly the following: is it possible to approximate approximately convex functions by convex functions? Now, an important extension of convexity, especially for mathematical economics or game theory, is quasiconcavity. In the following, let X be a convex subset of some vector space.

Definition 1.1. A function $f : X \to \mathbb{R}$ is quasiconcave if the following property is true: $\forall (x, y) \in X \times X,\ \forall \lambda \in [0, 1],\ f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.

Equivalently, $f : X \to \mathbb{R}$ is quasiconcave if its upper level sets $S_\lambda(f) = \{x \in X,\ f(x) \geq \lambda\}$ are convex subsets of X for every $\lambda \in \mathbb{R}$. Clearly, concave mappings are quasiconcave. Besides, non-increasing or non-decreasing mappings from $\mathbb{R}$ to $\mathbb{R}$ are quasiconcave.

In order to raise the issue of the stability of the quasiconcavity property, one defines the notion of approximately quasiconcave function as follows:

Definition 1.2. For every $\epsilon > 0$, the function $f : X \to \mathbb{R}$ is $\epsilon-$quasiconcave if the following property is true:

$$\forall (x, y) \in X \times X,\ \forall \lambda \in [0, 1],\ f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\} - \epsilon.$$

A natural question, in the vein of Hyers and Ulam's stability result, is to know whether it possible to approximate an ϵ-quasiconcave function $f : X \to \mathbb{R}$ by a quasiconcave function. Such question could have applications in game theory or in mathematical

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economics, but it also has an interest for its own sake. A possible answer to this question could be to build the quasiconcave envelop $\tilde{f}$ of f as follows: first, associate to the $\epsilon-$quasiconcave function f its upper level sets $S_\lambda(f) = \{x \in X,\ f(x) \geq \lambda\}$.[1] Then convexify the upper level sets by defining their convex hulls $\operatorname{co} S_\lambda(f)$. Last, try to find a quasiconcave function $\tilde{f}$ whose upper level sets are the $\operatorname{co} S_\lambda(f)$.

A first objective of the present paper is to clarify the previous construction.[2] In particular, one could hope that the function $\tilde{f}$ is a good approximation of f if ϵ is small enough. In view of the construction of $\tilde{f}$, a question which naturally arises is the continuity of the representation of each function $f : X \to \mathbb{R}$ by its upper level sets. This is the main purpose of this paper, and this study is given in Section 2. In particular, it requires the definition of a good topology on the set of upper level sets. Then, in Section 3, one will study the continuity properties of the convex hull operator in the set of upper level sets, and will prove the stability of the quasiconcave property, as an immediate byproduct.

2. Continuous representation of a quasiconcave function by its upper level sets

The purpose of this section is to prove that the standard representation of quasiconcave functions by their upper level sets (see Crouzeix [2], Chapter 3) can be made continuous by defining adapted topologies. The representation result requires the following definition:

Definition 2.1. Let X be a vector space. A multivalued function T from $\mathbb{R}$ to X is said to be a u.l.s. (upper level sets) multivalued function if the following properties are satisfied:

i) **Monotony.** For every $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$, $\lambda \leq \mu$ implies $T_\mu \subset T_\lambda$.

ii) **Left Continuity.** For every $\lambda \in \mathbb{R}$, one has $\bigcap_{n \in \mathbb{N}} T_{\lambda - \frac{1}{n}} = T_\lambda$.

iii) **Limits at $+\infty$ and $-\infty$.** One has $\bigcup_{\lambda \in \mathbb{R}} T_\lambda = X$ and $\bigcap_{\lambda \in \mathbb{R}} T_\lambda = \emptyset$.

Throughout this paper, $\mathcal{T}$ denotes the set of all multivalued functions from $\mathbb{R}$ to X satisfying Assumptions i) and iii), and $\mathcal{T}^{\text{u.l.s.}}$ the set of all u.l.s. multivalued functions from $\mathbb{R}$ to X. Besides, $\mathcal{F}$ denotes the set of all functions from X to $\mathbb{R}$.

Remark 2.2. Condition iii) of Definition 2.1 guarantees that f takes finite values.

In the following Proposition 2.3, we define a generalized metric structure on $\mathcal{T}^{\text{u.l.s.}}$. Recall that for every set E, a function $\delta : E \times E \to \mathbb{R}^+ \cup \{+\infty\}$ is a generalized pseudo metric if these three conditions are true:

(i) for every $x \in E$, $d(x, x) = 0$.

(ii) for every $(x, y) \in E \times E$, $d(x, y) = d(y, x)$.

(iii) for every $(x, y, z) \in E^3$, one has $d(x, z) \leq d(x, y) + d(y, z)$.

[1]The representation of a function by its upper level sets is a well known process (see, for example, Crouzeix [2], Chapter 3.) In this paper, we are more interested in the continuity of such representation.

[2]Other constructions of the quasiconcave envelop could be given. But we think the construction here given, which is very natural, raises interesting questions we would like to answer in this paper.

The generalized pseudo metric d is a generalized metric if in addition, for every $(x, y) \in E^2$, $d(x, y) = 0$ implies $x = y$. In the following proposition, by convention, the infimum of an empty set is $+\infty$.

Proposition 2.3. *The function $\delta : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^+ \cup \{+\infty\}$ defined for every $(T, T') \in \mathcal{T} \times \mathcal{T}$ by*

$$\delta(T, T') = \sup_{\lambda \in \mathbb{R}} \{\max\{\inf\{a \geq 0 \mid T_\lambda \subset T'_{\lambda-a}\}, \inf\{a \geq 0 \mid T'_\lambda \subset T_{\lambda-a}\}\}\}$$

is a generalized pseudo-metric, and its restriction to $\mathcal{T}^{\text{u.l.s.}} \times \mathcal{T}^{\text{u.l.s.}}$ is a generalized metric.

Proof. Clearly, to prove that δ is a generalized pseudo metric, one only has to prove the triangular inequality. Let $(T, T', T'') \in \mathcal{T}^3$ such that $\delta(T, T') < +\infty$ and $\delta(T, T') < +\infty$ (if one of these two inequalities is not true, then the inequality $\delta(T, T'') \leq \delta(T, T') + \delta(T', T'')$ is clearly true). First, from the monotony assumption satisfied by T and T' and from the definition of δ, one has

$$T_\lambda \subset T'_{\lambda - \delta(T,T') - \epsilon}$$

for every $\lambda \in \mathbb{R}$ and for every $\epsilon > 0$. Similarly, applying the previous inclusion to T' and T'' and replacing λ by $\lambda - \delta(T, T') - \epsilon$, one obtains

$$T'_{\lambda - \delta(T,T') - \epsilon} \subset T''_{\lambda - \delta(T,T') - \delta(T',T'') - 2\epsilon}$$

for every $\lambda \in \mathbb{R}$ and for every $\epsilon > 0$. Thus, the combination of the two previous inclusions gives, for every $\lambda \in \mathbb{R}$ and every $\epsilon > 0$,

$$T_\lambda \subset T''_{\lambda - \delta(T,T') - \delta(T',T'') - 2\epsilon}$$

and similarly, by symmetry,

$$T''_\lambda \subset T_{\lambda - \delta(T,T') - \delta(T',T'') - 2\epsilon}.$$

Consequently, from the definition of $\delta(T, T'')$, for every $\epsilon > 0$ one has $\delta(T, T'') \leq \delta(T, T') + \delta(T', T'') + 2\epsilon$, which entails the triangle inequality.

Now, to prove that the restriction of δ to $\mathcal{T}^{\text{u.l.s.}} \times \mathcal{T}^{\text{u.l.s.}}$ is a generalized metric, let $(T, T') \in \mathcal{T}^{\text{u.l.s.}} \times \mathcal{T}^{\text{u.l.s.}}$ such that $\delta(T, T') = 0$. Thus, for every $\lambda \in \mathbb{R}$, one has

$$T_\lambda \subset \bigcap_{n \in \mathbb{N}} T'_{\lambda - \frac{1}{n}}.$$

But from Property ii) of Definition 2.1, this last set is equal to T'_λ; similarly, one obtains, for every $\lambda \in \mathbb{R}$, $T'_\lambda \subset T_\lambda$, and finally $T = T'$. □

Throughout this paper, $\mathcal{T}$ is equipped with the topology Top induced by the previous generalized pseudo metric δ (this topology is generated by the collection of open sets $B(x, r) = \{y \in \mathcal{T}, \delta(x, y) < r\}$ where $x \in \mathcal{T}$ and $r > 0$), and the topological space such defined will be denoted $(\mathcal{T}, \delta)$.

Since we aim at representing continuously quasiconcave functions from X to $\mathbb{R}$ by elements of $\mathcal{T}^{\text{u.l.s.}}$, we also have to define a topology on $\mathcal{F}$: for every $(f,g) \in \mathcal{F}$, define $\|f-g\|_\infty = \sup_{x\in X} \mid f(x)-g(x) \mid$. Then, let Top$'$ be the topology on $\mathcal{F}$ generated by the collection of open sets $B(f,r) = \{g \in \mathcal{F}, \|f-g\|_\infty < r\}$, where $f \in \mathcal{F}$ and $r > 0$. Remark that $\|.\|_\infty$ is not a norm, since one has $\|f\|_\infty = +\infty$ if $f \in \mathcal{F}$ is not bounded. Nevertheless, $(\mathcal{F}, \|.\|_\infty)$ will denote the set $\mathcal{F}$ equipped with the topology Top$'$.

In the following, let $\mathcal{F}^q$ be the set of all quasiconcave functions in $\mathcal{F}$, and for every $\epsilon > 0$, let $\mathcal{F}^q_\epsilon$ be the set of all $\epsilon-$quasiconcave functions in $\mathcal{F}$. Besides, let $\mathcal{T}^{\text{conv}}$ be the set of all multivalued functions of $\mathcal{T}^{\text{u.l.s.}}$ with convex (possibly empty) values, and finally let $\mathcal{T}^{\text{conv}}_\epsilon$ be defined as follows:

Definition 2.4. The set $\mathcal{T}^{\text{conv}}_\epsilon$ is the set of all multivalued functions of $\mathcal{T}^{\text{u.l.s.}}$ such that for every $\lambda \in \mathbb{R}$, one has $\operatorname{co} T_\lambda \subset T_{\lambda-\epsilon}$.

We will see in the next section the interpretation of this last condition in terms of distance, which is roughly that the multivalued function T and its convex hull $\operatorname{co} T$ are not too far. Throughout this paper, all the subsets of $\mathcal{F}$ and of $\mathcal{T}$ are equipped with the topologies induced by Top$'$ and Top.

We now state the following refinement of a result of Crouzeix ([2], Chapter 3):

Theorem 2.5.

i) *The function* $\Phi : (\mathcal{F}, \|.\|_\infty) \to (\mathcal{T}^{\text{u.l.s.}}, \delta)$ *defined by*

$$\forall f \in \mathcal{F}, \forall \lambda \in \mathbb{R},\ \Phi(f)(\lambda) = S_\lambda(f) := \{x \in X,\ f(x) \geq \lambda\}$$

is an isometric isomorphism, i.e. for every $(f,g) \in \mathcal{F} \times \mathcal{F}$*, one has* $\|f-g\|_\infty = \delta(\Phi(f), \Phi(g))$*, and* $\Psi = \Phi^{-1}$ *is defined by*

$$\forall T \in \mathcal{T}^{\text{u.l.s.}},\ \Psi(T)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in T_\lambda\}.$$

ii) Φ *is an isometric isomorphism from* $\mathcal{F}^q$ *to* $\mathcal{T}^{\text{conv}}$*, and also from* $\mathcal{F}^q_\epsilon$ *to* $\mathcal{T}^{\text{conv}}_\epsilon$*.*

Proof. The fact that Φ is bijective and the definition of Φ^{-1} is a consequence of Crouzeix ([2], Chapter 3). Besides, Statement *ii)* is left as an exercise. Thus, one only has to prove that Φ is an isometry. Let $(f,g) \in \mathcal{F} \times \mathcal{F}$, and suppose $\epsilon := \|f-g\|_\infty$ is finite (the case $\|f-g\|_\infty = +\infty$ can be treated similarly). From the definition of Φ, one clearly has $\Phi(g)(\lambda) \subset \Phi(f)(\lambda-\epsilon)$ and $\Phi(f)(\lambda) \subset \Phi(g)(\lambda-\epsilon)$, which implies $\delta(\Phi(g), \Phi(f)) \leq \epsilon$.

Now, by contradiction, suppose that there exists $a > 0$ such that $\delta(\Phi(g), \Phi(f)) < \epsilon - a$. From the definition of Ψ, for every $x \in X$ there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to $\Psi(\Phi(f))(x) = f(x)$ such that $x \in \Phi(f)(\lambda_n)$ for every integer n. From $\delta(\Phi(f), \Phi(g)) < \epsilon - a$, the condition $x \in \Phi(f)(\lambda_n)$ implies $x \in \Phi(g)(\lambda_n - \epsilon + a)$, which is equivalent by definition to $g(x) \geq \lambda_n - \epsilon + a$. Passing to the limit, one obtains $g(x) \geq f(x) - \epsilon + a$. Similarly, one proves $f(x) \geq g(x) - \epsilon + a$. Thus, $\|f(x) - g(x)\|_\infty < \epsilon$, a contradiction with the definition of ϵ. $\square$

3. Application to the stability of the quasiconcave property

3.1. The convex hull

In this section, one defines the convex hull co from the set of all multivalued function from $\mathbb{R}$ to X to itself as follows: for every multivalued function T from $\mathbb{R}$ to X and for every $\lambda \in \mathbb{R}$, $(\operatorname{co} T)_\lambda$ is the convex hull of T_λ.

We have seen in the introduction that a natural method to build the quasiconcave envelop of a function f is to take the convex hull $\operatorname{co}\Phi(f)$ of the upper level sets of f, then to define a new quasiconcave function $\Phi^{-1}(\operatorname{co}\Phi(f))$, if it is possible. A problem is that the multivalued function $\operatorname{co}\Phi(f)$ may not be in $\mathcal{T}^{\text{u.l.s.}}$, so that $\Phi^{-1}(\operatorname{co}\Phi(f))$ could be not well defined. The following examples provide a u.l.s. multivalued function T from $\mathbb{R}$ to $\mathbb{R}$ whose convex hull is not u.l.s.

Example 3.1. Let T be the multivalued function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$T_x = [x, -x] \cup ([0, 1[\cap \mathbb{Q}) \ \text{ if x} \le 0,$$

$$T_x = ([0, 1-x[\cup[x, 1[) \cap \mathbb{Q} \ \text{ if x} \in [0, 1[,$$

$$T_1 = \{0\}$$

and

$$T_x = \emptyset \ \text{ if x} > 1.$$

It is easy to check that $T \in \mathcal{T}^{\text{u.l.s.}}$. Yet, the multivalued function $\operatorname{co} T$ is defined by

$$\operatorname{co} T_x = [x, -x] \ \text{ if x} \le -1,$$

$$\operatorname{co} T_x = [x, 1[\ \text{ if x} \in]-1, 0],$$

$$\operatorname{co} T_x = [0, 1[\ \text{ if x} \in [0, 1[,$$

$$\operatorname{co} T_1 = \{0\}$$

and

$$\operatorname{co} T_x = \emptyset \ \text{ if x} > 1.$$

Clearly, $\operatorname{co} T$ does not satisfy Property ii) of Definition 2.1 at $x = 1$ (left continuity), so is not u.l.s.

Example 3.2. Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined as follows: for every $(x, y) \in \mathbb{R}^2$, $f(x, y)$ is the opposite of the Euclidean distance between (x, y) and the set

$$K = \{(0,0)\} \cup \left\{ \left(x, \frac{1}{x}\right) \in \mathbb{R}^2, x < 0 \right\}.$$

The function f is continuous, $\Phi(f)(0) = K$ and $\operatorname{co}\Phi(f)(0) = \operatorname{co} K$ is not closed. Now, for every integer $n > 0$, one has

$$\overline{\operatorname{co} K} \subset \operatorname{co}\Phi(f)\left(-\frac{1}{n}\right),$$

so that

$$\bigcap_{n\in\mathbb{N}} \operatorname{co}\Phi(f)\left(-\frac{1}{n}\right) \neq \operatorname{co}\Phi(f)(0),$$

and so the multivalued function $\operatorname{co}\Phi(f)$ is not u.l.s.

So, the first task to avoid this problem is to extend $\Psi = \Phi^{-1}$ on a larger set. We prove that Ψ is, as a matter of fact, an isometry on all the set $\mathcal{T}$:

Proposition 3.3.

$$\forall T \in \mathcal{T},\ \Psi(T)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in T_\lambda\}$$

is well defined, and Ψ is an isometry. More precisely, for every $(T, T') \in \mathcal{T} \times \mathcal{T}$, one has $\|\Psi(T) - \Psi(T')\|_\infty = \delta(T, T')$.

Proof. By definition of $\delta(T, T')$, there exist two sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, where $(\mu_n)_{n\in\mathbb{N}}$ converges to $\delta(T, T')$, and such that for every integer n, T_{λ_n} is not contains in $T'_{\lambda_n-\mu_n}$. It means that for every integer n, there exists $x_n \in X$ such that $\Psi(T)(x_n) \geq \lambda_n$ and $\Psi(T')(x_n) \leq \lambda_n - \mu_n$. Thus, $\|\Psi(T) - \Psi(T')\|_\infty \geq \mu_n$, and one obtains at the limit $\|\Psi(T) - \Psi(T')\|_\infty \geq \delta(T, T')$. Notice that this proof works if $\delta(T, T') = +\infty$. Now, by contradiction, suppose that $\delta(T, T') < \|\Psi(T) - \Psi(T')\|_\infty$. Thus, there exists $x \in X$ such that for $\epsilon > 0$ small enough, one has

$$\Psi(T)(x) - \Psi(T')(x) > \delta(T, T') + \epsilon$$

(switching T and T' is necessary). Consequently, from the definition of $\Psi(T')(x)$, one has

$$x \notin T'_{\Psi(T)(x)-\delta(T,T')-\epsilon}.$$

But from the definition of $\Psi(T)(x)$, one also has $x \in T_{\Psi(T)(x)-\frac{\epsilon}{2}}$. Finally, the definition of $\delta(T, T')$ gives

$$\delta(T, T') \geq \left(\Psi(T)(x) - \frac{\epsilon}{2}\right) - (\Psi(T)(x) - \delta(T, T') - \epsilon) = \delta(T, T') + \frac{\epsilon}{2},$$

a contradiction. □

Unfortunately, there could remain some multivalued function $T \in \mathcal{T}^{\text{u.l.s.}}$ such that $\operatorname{co} T$ is not in $\mathcal{T}$, as shown in the following example:

Example 3.4. Let T be the multivalued function from $\mathbb{R}$ to $\mathbb{R}$ defined by $T_x =]-\infty, -x] \cup [x, +\infty[$ if x ≥ 0 and $T_x = \mathbb{R}$ if x ≤ 0. It is clear that $T \in \mathcal{T}^{\text{u.l.s.}}$, but $\operatorname{co} T \notin \mathcal{T}$, because $\bigcap_{x\in\mathbb{R}} \operatorname{co} T_x = \mathbb{R}$.

So, the second task is to find a subset $\mathcal{T}'$ of $\mathcal{T}$ such that for every $T \in \mathcal{T}'$, $\operatorname{co} T \in \mathcal{T}$. This is the aim of the following definition:

Definition 3.5. Define $\mathcal{T}'$ the set of multivalued functions of $\mathcal{T}$ such that $\bigcap_{\lambda\in\mathbb{R}} \operatorname{co} T_\lambda = \emptyset$.

It is clear from the previous definition that $\mathrm{co}(\mathcal{T}') \subset \mathcal{T}$. Remark that for every bounded function $f : X \to \mathbb{R}$, $\Phi(f) \in \mathcal{T}'$, because $\Phi(f)(n) = \emptyset$ for $n \in \mathbb{N}$ large enough. The following proposition, whose proof is straightforward, relates this new set $\mathcal{T}'$ to $\mathcal{T}_\epsilon^{\mathrm{conv}}$:

Proposition 3.6. *For every $\epsilon > 0$, $\mathcal{T}_\epsilon^{\mathrm{conv}} \subset \mathcal{T}'$.*

To finish, notice that from the definition of $\mathcal{T}_\epsilon^{\mathrm{conv}}$, each point of $\mathcal{T}_\epsilon^{\mathrm{conv}}$ is almost stabilized by the convex hull operator in the following sense:

Proposition 3.7. *For every $T \in \mathcal{T}_\epsilon^{\mathrm{conv}}$, $\delta(\mathrm{co}(T), T) \leq \epsilon$.*

3.2. Stability of the quasiconcavity property

To illustrate the utility of the previous results, one proves the stability of the quasiconcave property.

Let $f \in \mathcal{F}_\epsilon^q$ an $\epsilon-$quasiconcave function from X to $\mathbb{R}$. By Theorem 2.5, $\Phi(f) \in \mathcal{T}_\epsilon^{\mathrm{conv}}$, so $\Phi(f) \in \mathcal{T}'$ from Proposition 3.6. Since $\mathrm{co}(\mathcal{T}') \subset \mathcal{T}$, one has $\mathrm{co}(\Phi(f)) \in \mathcal{T}$. Hence, from Proposition 3.3, $\tilde{f} := \Psi(\mathrm{co}(\Phi(f)))$ is well defined. Besides, since $\Psi(\Phi(f)) = f$, one has $\|\tilde{f} - f\|_\infty = \|\Psi(\mathrm{co}(\Phi(f))) - \Psi(\Phi(f))\|_\infty$. Now, from Proposition 3.3, Ψ is an isometry on $\mathcal{T}$; thus, one has $\|\Psi(\mathrm{co}(\Phi(f))) - \Psi(\Phi(f))\|_\infty = \delta(\mathrm{co}(\Phi(f)), \Phi(f)) \leq \epsilon$ from Proposition 3.7 applied to $T = \Phi(f) \in \mathcal{T}_\epsilon^{\mathrm{conv}}$.

Thus, one obtains a result similar to Hyers-Ulam's one for quasiconcave functions:

Proposition 3.8. *For every $\epsilon-$quasiconcave function $f : X \to \mathbb{R}$ there exists $\tilde{f} : X \to \mathbb{R}$ a quasiconcave function such that $\|\tilde{f} - f\|_\infty \leq \epsilon$.*

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