Journal of Convex Analysis
Volume 17 (2010), No. 2, 381–403

Properties of the Convex Cone of Vectors with Autocorrelated Components

Marc Fuentes
INRIA Nancy Grand Est, 615 Rue du Jardin Botanique, 54602 Villers les Nancy, France
marc.fuentes@inria.fr

Received: October 20, 2008
Revised manuscript received: March 3, 2009

This paper reviews some properties of the set of vectors with autocorrelated components. This set appears in some signal processing problems, in particular filter synthesis or statistical estimation. It turns out to be a closed convex cone enjoying several representations and various geometrical properties. The aim of this paper is to gather different aspects of the geometry of this cone. We adopt a convex analysis point of view to present known and new results.

Keywords: Convex analysis, autocorrelation, nonnegative trigonometric polynomials

1. Introduction

In the first theoretical papers [27, 28] about moment problems, and later in the applications-oriented papers [1, 6, 10, 13, 16, 24, 31], some authors introduced the notion of *nonnegative trigonometric polynomials*, which are special rational functions defined on the unit circle of the complex plane. For one-dimensional real signals, it reduces simply to *finite autocorrelation sequences* also called sometimes *positive real sequences* [8, 15, 36]. At the same time, a similar theory for nonnegative polynomials (polynomials defined on $\mathbb{R}$, or in the multivariate case, on $\mathbb{R}^n$) was developed; as a result, some useful applications in global optimization for polynomial functions appeared in the last decade [29, 33]. The study of the multivariate case in trigonometric context is more recent [12, 14, 25, 30, 34] and also promises worthwhile applications to signal processing.

Let us start with a simple example from signal processing where autocorrelation appears: for instance, assume we want to design a lowpass filter, which has the following frequency response

$$H(\omega) = \sum_{k=0}^{n} h_k e^{ik\omega}$$

where $\omega \in [0, \pi]$; we want to minimize the stopband energy of the filter as in [14, p. 138]

$$E_s = \frac{1}{\pi} \int_{\omega_s}^{\pi} |H(\omega)|^2 \; d\omega$$

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

where $[\omega_s, \pi]$ is the stopband. We have also the two affine constraints

$$\begin{aligned} |H(\omega) - 1| &\leqslant \gamma_p, \quad \forall \omega \in [0, \omega_p], \\ |H(\omega)| &\leqslant \gamma_s, \quad \forall \omega \in [\omega_s, \pi], \end{aligned}$$

where $[0, \omega_p]$ is the passband and γ_s, γ_p are error bounds. This kind of constraint is called "spectral mask constraint" [16] in signal processing. With a linear transformation based on Tchebycheff Polynomials [1], one could consider that in the two previous constraints ω lies in $[0, \pi]$. But, due to the nature of both the constraints and the criterion, the problem still remains nonlinear and nonconvex with respect to the coefficients $h_0, \cdots, h_k$. To solve this problem, one could consider the variables x_k where

$$X(\omega) = x_0 + 2\sum_{k=0}^{n} x_k \cos k\omega = |H(\omega)|^2,$$

for which both the constraints and the objective become linear. The last problem which we are facing is now that the vector x does not belong to the entire space $\mathbb{R}^{n+1}$, but to a subset $\mathcal{C}_{n+1}$ which is a convex cone of $\mathbb{R}^{n+1}$. We will study thoroughly that cone in the sequel, giving a review about some of its properties. Numerical results will not be presented here; concerning that, the reader could worthly have a look at [1, 24], but also at the 3rd chapter of [17], which is more detailed and didactic.

The outline of the paper is the following: Section 2 is devoted to recall the several definitions of the cone $\mathcal{C}_{n+1}$; the next section details geometric properties of the cone; most of this part is new: we give an inner and an outer approximation of a compact base of the cone, we describe the normal directions of the two facets of $\mathcal{C}_{n+1}$, and prove the acuteness of the cone. The last part concerns mainly questions related to polarity.

2. Definitions and basic properties

Before starting, we make some notations precise, which will be of a constant use in the sequel.

2.1. Correlations

For finite discrete signals there are two definitions for the correlation: if we consider signals as elements of the group $(\mathbb{Z}, +)$ or elements of $(\mathbb{Z}/n\mathbb{Z}, +)$; in the first case, we will define *acyclic* correlation, whereas in the second case we will use *cyclic* or *circular* correlation.

Assume we want to compute the correlation function of two signals of $(\mathbb{Z}, +)$ but with supports included in $\{0, \cdots, n\}$; hence, the correlation function has also a finite support, which is exactly $\{0, \cdots, n\}$ and it is possible to view the acyclic correlation as a bilinear function from $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$.

Definition 2.1 (Acyclic correlation). Let $x, y \in \mathbb{R}^{n+1}$; the acyclic correlation of x and y, $\mathtt{corr}_a(x, y)$, is the vector of $\mathbb{R}^{n+1}$ with components

$$\mathtt{corr}_a(x, y)_k = \sum_{i=0}^{n-k} x_i y_{i+k} \quad \text{for } k \in \{0, \cdots, n\}.$$

We deduce directly that $\texttt{corr}_a(\cdot,\cdot)$ is bilinear, nonsymmetric, and $\texttt{corr}_a(x,y)_0 = \langle x, y\rangle$ (usual scalar product of x and y).

The circular correlation of two vectors of $\mathbb{R}^{n+1}$ corresponds to $(n+1)$-periodic signals, and presents several interesting properties in common with the continuous case.

Definition 2.2 (Circular correlation). Let $x, y \in \mathbb{R}^{n+1}$; the cyclic (or circular) correlation of x and y, denoted $\texttt{corr}_c(x,y)$, is the vector of $\mathbb{R}^{n+1}$ with components

$$\texttt{corr}_c(x,y)_k = \sum_{i=0}^{n} x_i y_{\{(i+k) \bmod (n+1)\}} \quad \text{for } k \in \{0,\cdots,n\},$$

where a mod b is the remainder of the Euclidean division of a by b.

2.2. Autocorrelated components

There are multiple ways to define the cone of vectors with autocorrelated components. Our first definition is simply based on the acyclic correlation.

Definition 2.3. In the Euclidean space $\mathbb{R}^{n+1}$, we define $\mathcal{C}_{n+1}$ as

$$\mathcal{C}_{n+1} = \{\texttt{corr}_a(y,y) \mid y \in \mathbb{R}^{n+1}\}.$$

Using this definition, it is possible like [1] to reformulate it in a more geometrical point of view; let

$$\begin{array}{rccc} \mathcal{A}: & \mathcal{M}_{n+1}(\mathbb{R}) & \to & \mathbb{R}^{n+1} \\ & Q & \mapsto & \begin{pmatrix} \langle\langle A^{(0)}, Q\rangle\rangle \\ \vdots \\ \langle\langle A^{(n)}, Q\rangle\rangle \end{pmatrix}, \end{array}$$

with $\langle\langle A, B\rangle\rangle = \mathrm{Tr}(A^\top B)$ stands here for the *Frobenius* scalar product, and the $A^{(k)}$ matrices are defined as symmetric parts of right-shift operators, i.e.

$$A^{(k)} = \frac{1}{2}\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix},$$

where only the upper and lower k^{th} sub-diagonals are nonzero. The study of eigenvalues and eigenvectors for these matrices can be found in [18]. Using the linear operator $\mathcal{A}$,

$$\mathcal{C}_{n+1} = \mathcal{A}(\{yy^\top \mid y \in \mathbb{R}^{n+1}\}),$$

i.e. $\mathcal{C}_{n+1}$ is the image by $\mathcal{A}$ of *dyadic matrices* (positive semidefinite matrices of rank $\leqslant 1$). One notes that we obtain a construction of $\mathcal{C}_{n+1}$ as an image of a certain subset of the Euclidean space; we therefore point out this definition as a *definition by generators*. With this formulation, it appears that $\mathcal{C}_{n+1}$ is a cone, due to the bilinear term $yy^\top$ in the previous formulation.

Along this definition, we can also give another definition for this cone using the nonnegativity of a certain trigonometric polynomial. As this stage, we firstly consider the two definitions as defining two differents subsets of $\mathbb{R}^{n+1}$, and we prove later they are equivalent.

2.3. Nonnegative trigonometric polynomials

As mentioned in the previous papers [27, 28], there is a deep relation between moment problems and nonnegativity of polynomials. Moments are a mean to parametrize measures, which can appear when studying polar subsets of nonnegative polynomials, particularly in the multivariate case [29, 30]. To emphasis this relation in our particular case, let us define the set of even trigonometric nonnegative polynomials.

Definition 2.4. The set of coefficients of even nonnegative polynomials on the complex unit circle $\mathbb{T} = \{z \in \mathbb{C} \,||z| = 1\}$ is denoted

$$\mathcal{P}^+_{n+1}(\mathbb{T}) = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=-n}^{n} x_{|k|} z^k \geqslant 0 \,,\ \forall z \in \mathbb{T} \right\}.$$

Taking $z = e^{i\omega}$ we can also define $\mathcal{P}^+_{n+1}(\mathbb{T})$ as

$$\mathcal{P}^+_{n+1}(\mathbb{T}) = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x_0 + 2\sum_{k=1}^{n} x_k \cos k\omega \geqslant 0 \,,\ \forall \omega \in [0,\pi] \right\}.$$

Introducing the notation

$$v(\omega) = (1, 2\cos\omega, \cdots, 2\cos n\omega), \tag{1}$$

we derive the following description

$$\mathcal{P}^+_{n+1}(\mathbb{T}) = \{x \in \mathbb{R}^{n+1} \mid \langle x, v(\omega)\rangle \geqslant 0 \,,\ \forall \omega \in [0,\pi]\} = \bigcap_{\omega\in[0,\pi]} \mathcal{H}^+_{v(\omega)}$$

where $\mathcal{H}^+_a = \{x \in \mathbb{R}^{n+1} \mid \langle x, a\rangle \geqslant 0\}$ is the positive half-space delimited by the hyperplane $\ker a^\top$.

As a direct consequence, we observe that $\mathcal{P}^+_{n+1}(\mathbb{T})$ is a convex closed subset of $\mathbb{R}^{n+1}$, as an intersection of convex closed elementary sets, whence $\mathcal{P}^+_{n+1}(\mathbb{T})$ is characterized by a set of constraints that each element must verify. We therefore designate this formulation in the sequel, as a "formulation by constraints".

2.4. Equivalence of the two definitions

The two subsets $\mathcal{C}_{n+1}$ and $\mathcal{P}^+_{n+1}(\mathbb{T})$ seem to be *a priori* very different. But, it turns out that they define the same subset of $\mathbb{R}^{n+1}$.

Proposition 2.5. *The Definitions* 2.3 *and* 2.4 *describe the same closed convex subset of* $\mathbb{R}^{n+1}$*, in other words:*

$$\mathcal{C}_{n+1} = \mathcal{P}^+_{n+1}(\mathbb{T}).$$

The reader could view the proof in [3, p. 212]; roughly speaking, it suffices to expand the modulus of a polynomial of the form $Y = \sum_{l=0}^{n} y_l e^{il\omega}$, rewrite it as a single sum and then apply the following Riesz-Fejer theorem.

Theorem 2.6 (Riesz-Fjer). *An even trigonometric polynomial,*

$$X(\omega) = X_0 + 2\sum_{k=1}^{n} x_k \cos k\omega,$$

with $r_0, \cdots, r_n$ *real numbers, admits the following factorization*

$$\left|\sum_{l=0}^{n} y_l e^{il\omega}\right|^2$$

where $y_0, \cdots, y_n$ *are real numbers if, and only if,* $X(\omega) \geqslant 0$ *for all* $\omega \in [-\pi, \pi]$ *(or, equivalently, for all* $\omega \in [0, \pi]$*).*

For a proof of this theorem, see [28, p. 60] or more didactic [3, p. 213]. In the sequel, we therefore keep only the first notation $\mathcal{C}_{n+1}$ following [1].

2.5. First properties

The equivalence between the two definitions transports all the good properties of the second definition to the first one. The convexity of that cone and other properties of the following proposition were first proved by Krein as presented in [28]; we recall here due to their simplicity.

Proposition 2.7. $\mathcal{C}_{n+1}$ *is a pointed closed* convex *cone with nonempty interior. Consequently, we can define a partial order on* $\mathbb{R}^{n+1}$ *denoted by* $\preceq_{\mathcal{C}_{n+1}}$*:*

$$x \preceq_{\mathcal{C}_{n+1}} y \quad \textit{if and only if } y - x \in \mathcal{C}_{n+1}.$$

Proof. • $\mathcal{C}_{n+1}$ is a cone due to the bilinear term $yy^\top$ in the first definition.

• If we look at the Definition 2.4, we have

$$\mathcal{C}_{n+1} = \bigcap_{\omega \in [0,\pi]} \mathcal{H}^+_{v(\omega)},$$

and it is clear that $\mathcal{C}_{n+1}$ is the intersection of an infinite family of closed convex sets (the half-spaces $\mathcal{H}^+_{v(\omega)}$). Therefore, $\mathcal{C}_{n+1}$ is convex and closed.

• $\mathcal{C}_{n+1}$ has a nonempty interior, since $e_0 = (1, 0, \cdots, 0)$ verifies strictly the linear constraints from Definition 2.4, and so there exists an open ball centered at e_0 and strictly included in $\mathcal{C}_{n+1}$.

$\mathcal{C}_{n+1}$ is *pointed*, i.e. $\mathcal{C}_{n+1} \cap (-\mathcal{C}_{n+1}) = \{0\}$. Indeed, let $x \in \mathcal{C}_{n+1} \cap (-\mathcal{C}_{n+1})$; then there exists $y \in \mathbb{R}^{n+1}$ such that $x_0 = \|y\|^2$, but $-x \in \mathcal{C}_{n+1}$, so it exists also $w \in \mathbb{R}^{n+1}$ such that $-x_0 = \|w\|^2$. Consequently $x_0 = 0$ and $x = 0$.

Here, we present the convexity along the three other properties to show that the framework of conic programming[3, p. 45] is well-suited for $\mathcal{C}_{n+1}$. □

2.6. Rank Relaxation and Trace Parametrization

In the first definition, we saw that $\mathcal{C}_{n+1}$ is the image of dyadic matrices by the linear mapping $\mathcal{A}$. What happens if we relax the rank constraint (rank $\leqslant 1$) and consider the whole set of semi-definite matrices? We obtain surprisingly the same subset, and this formulation is called the *trace parametrization* by some authors [13].

Proposition 2.8. *Relaxing the rank constraint in Definition* 2.3*, we obtain the same subset of* $\mathbb{R}^{n+1}$*; therefore*

$$\mathcal{C}_{n+1} = \mathcal{A}(\mathcal{S}^+_{n+1}(\mathbb{R})). \tag{2}$$

Proof. We include here, for completeness, a slighty modified version of the proof presented in [1]. Since a dyadic matrix $yy^\top$ is trivially semidefinite positive, we have

$$\mathcal{C}_{n+1} \subset \mathcal{A}(\mathcal{S}^+_{n+1}(\mathbb{R}));$$

but the other inclusion is also true: indeed, if $x = \mathcal{A}(Y)$ with $Y \succeq 0$. Let us look at

$$z_\omega = (1, e^{i\omega}, \cdots, e^{in\omega}) \in \mathbb{C}^{n+1}\,; \tag{3}$$

if we compute

$$\langle Y z_\omega, z^*_\omega \rangle = \sum_{0 \leqslant k,l \leqslant n} y_{kl} e^{i(k-l)\omega} = \sum_{p=-n}^{n} \left(\sum_{k-l=p} y_{kl} \right) e^{ip\omega},$$

we observe that the term between parentheses is just

$$\mathcal{A}(Y)_{|p|} = \sum_{l-k=p} y_{kl},$$

and using the fact that Y is symmetric and semi-definite, we deduce

$$\langle Y z_\omega, z^*_\omega \rangle = \mathcal{A}(Y)_0 + 2 \sum_{p=0}^{n} A(Y)_p \cos p\omega \geqslant 0.$$

□

In the multivariate case, which is more complex (see [14, 30, 29]), the three previous sets are *distinct*, and we have therefore strict inclusions

$$\{\mathtt{corr}(y, y) \mid y \in \mathbb{R}^{n+1}\} \subsetneq \mathcal{A}(\mathcal{S}^+_{n+1}(\mathbb{R})) \subsetneq \mathcal{P}^+_{n+1}(\mathbb{T}).$$

The first set corresponds to "squares" or autocorrelated tensors; it is no more a convex set, but in fact a real algebraic manifold. The second set corresponds to "Sums of

Squares" trigonometric polynomials with bounded degree (the degree in this case is exactly the dimension of the space); due to its formulation, it is a convex set, which can be represented by a Linear Matrix Inequality, only practically useful in low dimension. The third set, which is convex, consists in even nonnegative trigonometric polynomials and can be approached using Sums-of-Squares relaxations.

3. Geometry of $\mathcal{C}_{n+1}$

3.1. A compact base for $\mathcal{C}_{n+1}$

Since $\mathcal{C}_{n+1}$ is pointed (cf. Proposition 2.7), we can find an affine subspace of $\mathbb{R}^{n+1}$ such that its intersection with $\mathcal{C}_{n+1}$ reduces to a compact set. The convex conical hull of such a compact set is $\mathcal{C}_{n+1}$ and we therefore refer to it as a compact base of $\mathcal{C}_{n+1}$. In our work, we have chosen the affine subspace $\{x \in \mathbb{R}^{n+1} \mid x_0 = 1\}$ which can be viewed as an image of the unit sphere $\mathbb{S}_n$ of $\mathbb{R}^{n+1}$ under a quadratic mapping.

Lemma 3.1. *Let* Θ *be the mapping defined by*

$$\Theta : \left\{ \begin{array}{rcl} \mathbb{R}^{n+1} & \rightarrow & \mathcal{C}_{n+1} \\ x & \mapsto & \mathcal{A}(xx^\top); \end{array} \right.$$

then the set $\mathcal{U}_n := \Theta(\mathbb{S}_n)$ *is a compact base for* $\mathcal{C}_{n+1}$.

Proof. The Θ mapping is quadratic, homogeneous of degre 2, i.e. for all $y \in \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, $\Theta(ty) = t^2\Theta(y)$. Let $x \in \mathcal{C}_{n+1}$, there exists $y \in \mathbb{R}^{n+1}$ such that $x = \Theta(y)$, and then $x_0 = \Theta(y)_0 = \|y\|^2$. If $x_0 = 0$, then $y = 0$ and so $x = 0$; hence x can be written as $0 \cdot z$ where z is any vector of $\mathcal{U}_n$. Otherwise

$$y = \|y\| \cdot \tilde{y} \quad \text{with } \tilde{y} = \frac{y}{\|y\|} \in \mathbb{S}_n,$$

and $x = \|y\|^2 \cdot \Theta(\tilde{y}) = x_0 \cdot u$, where $u = \Theta(\tilde{y}) \in \mathcal{U}_n$. □

Using the definition of $\mathcal{U}_n$, it yields that for all $x \in \mathcal{U}_n$, $x_0 = ||s||^2 = 1$, and so $\mathcal{U}_n$ is entirely contained in $\{x \in \mathbb{R}^{n+1} \mid x_0 = 1\}$, and it can be considered as compact set of $\mathbb{R}^n$; in the sequel, $\mathcal{U}_n$ will be considered as a subset of $\mathbb{R}^n$, or as a "flat" subset in $\mathbb{R}^{n+1}$, depending on the context.

To get a better intuition about $\mathcal{U}_n$, we propose a polyhedral approximation of it. For example, if we want to find the smallest parallelepiped $P_{\alpha,\beta} = \prod_{i=1}^n [\alpha_i, \beta_i]$ containing $\mathcal{U}_n$, we have to calculate for each component i in $\{1, \cdots, n\}$

$$\max_{x \in \mathbb{S}_n} \Theta_i(x) \quad \text{and} \quad \min_{x \in \mathbb{S}_n} \Theta_i(x).$$

But,

$$\max_{x \in \mathbb{S}_n} \Theta_i(x) = \max_{x \in \mathbb{S}_n} \langle\langle A^{(i)}, xx^\top \rangle\rangle = \max_{x \in \mathbb{S}_n} \langle A^{(i)}x, x \rangle = \lambda_{\max}(A^{(i)}),$$

the last inequality coming from the Rayleigh-Courant variational formulations for eigenvalues. We have a similar formulation *mutatis mutandis* for the minimum. To find the optimal $P_{\alpha,\beta}$, it suffices to know the largest and the smallest eigenvalues of each $A^{(i)}$. The eigenvalues of $A^{(i)}$ matrices are known with analytical formulae from [18, 20]. In the first reference, the associated eigenvectors are also given but in a form which is not very intuitive; Here, we give a more concise expression. Let denote by T^p the orthogonal matrix of the Discrete Sinus Transform

$$T^p_{kl} = \sqrt{\frac{2}{p+1}} \sin\left(\frac{kl\pi}{p+1}\right) \quad \text{for } k, l = 1, \cdots, p,$$

and c^p the vectors with components

$$c^p_k = \cos\left(\frac{k\pi}{p+1}\right) \quad \text{for } k = 1, \cdots, p;$$

then

$$A^{(i)} = V \operatorname{Diag}\left(\begin{bmatrix} c^p \otimes I_{m_1} \\ c^{p+1} \otimes I_{m_2} \end{bmatrix}\right) V^\top, \tag{4}$$

where p, m_1, m_2 are defined from $n+1$ and i by

$$p = \left\lfloor \frac{n+1}{i} \right\rfloor, \qquad m_1 = (p+1)i - (n+1) \quad \text{and} \quad m_2 = (n+1) - ip, \tag{5}$$

in such a manner that

$$n + 1 = pm_1 + (p+1)m_2.$$

The diagonalizing matrix V is an orthogonal block matrix

$$V = \left(\begin{array}{c|c} T^p \otimes \begin{bmatrix} 0_{m_2,m_1} \\ I_{m_1} \end{bmatrix} & T^{p+1}_{1:p,1:p+1} \otimes \begin{bmatrix} I_{m_2} \\ 0_{m_1,m_2} \end{bmatrix} \\ \hline 0_{m_2,pm_1} & T^{p+1}_{p+1,1:p+1} \otimes I_{m_2} \end{array}\right), \tag{6}$$

where $\otimes$ stands for the Kronecker product, and the : is the extraction operator defined as in [19]; for instance, $T^{p+1}_{1:p,1:p+1}$ corresponds to the submatrix with the first p lines and first $p+1$ columns extracted from the matrix T^{p+1}. The expressions (4) and (6) may seem a little bit complicated, but one has to remember that we want to “diagonalize” the acyclic correlation: each $A^{(i)}$ is needed to compute the i^{th} component. Acyclic correlation is well defined for signals with support on the infinite group $(\mathbb{Z}, +)$, but here we are doing computations on a finite space which is better represented by the $(\mathbb{Z}/(n+1)\mathbb{Z}, +)$ group. Therefore, one could view this diagonalization as an intermediate and more intricate case between two simpler cases: the first one is the diagonalization of circulant matrices by the Fourier matrix (cf. [19, p. 202]) for finite signals; the second one is the diagonalization of Laurent matrices by means of Fourier series for signals with infinite support (cf. [7]). We can therefore use this information to obtain an inner and an outer approximation of $\mathcal{U}_n$.

- The outer approximation is given by

$$\mathcal{O}_n = \prod_{i=1}^{n} \left[\lambda_{\min}(A^{(i)}), \lambda_{\max}(A^{(i)})\right].$$

But, as showed in [18], the spectrum of $A^{(i)}$ is symmetric with respect to the origin so $\lambda_{\min}(A^{(i)}) = -\lambda_{\max}(A^{(i)})$, and for

$$\left\lfloor \frac{n+1}{2} \right\rfloor + 1 \leqslant i \leqslant n, \text{ we have } \lambda_{\max}(A^{(i)}) = 1/2.$$

Letting $N = \lfloor \frac{n+1}{2} \rfloor$, the outer approximation reduces to

$$\mathcal{O}_n = \prod_{i=1}^{N} \Bigg[-\cos\left(\frac{\pi}{\lfloor \frac{n+1}{i} \rfloor + 2 - [i \mid (n+1)]} \right), \cos\left(\frac{\pi}{\lfloor \frac{n+1}{i} \rfloor + 2 - [i \mid (n+1)]} \right) \Bigg] \left[-\frac{1}{2}, \frac{1}{2} \right]^N,$$

where $[P(x)]$ stands for the Iverson symbol used in Computer Science[22] for a predicate P depending on a variable x. For instance, here we have

$$[i \mid (n+1)] = \begin{cases} 1 & \text{if } i \text{ divides } n+1 \\ 0 & \text{otherwise.} \end{cases}$$

- We obtain the inner approximation by means of eigenvectors: we select an eigenvector associated with the largest eigenvalue: if i is not a divisor of $n+1$, we take

$$v^i = \begin{bmatrix} T^{p+1}_{1:p,1} \otimes e^i_0 \\ T^{p+1}_{p+1,1} \otimes e^{m_2}_0 \end{bmatrix},$$

with $e^i_0 = (1, 0, \cdots, 0) \in \mathbb{R}^i$ and p, m_1 and m_2 being defined as previously; when i divides $n+1$, v^i is simply

$$v^i = T^p_{1:p,1} \otimes e^i_0.$$

Let us compute the values of the Θ application on the vectors v^i.

Proposition 3.2. *Let μ^p be the vector of $\mathbb{R}^p$ with components*

$$\mu^p_t = \frac{1}{2} \left((p+1-t) \cos\left(\frac{t\pi}{p+1} \right) + \frac{\sin\left(\frac{t\pi}{p+1} \right)}{\tan\left(\frac{\pi}{p+1} \right)} \right) \quad \textit{for } t = 1, \cdots, p; \tag{7}$$

then the extreme point of $\mathcal{U}_n$ with the maximal i^{th} component is

$$\gamma^i = \Theta(v^i) = \begin{cases} \mu^p \otimes e^i_0 & \textit{if } i \textit{ divides } n+1, \\ \begin{bmatrix} \mu^{p+1}_{1:p} \otimes e^i_0 \\ \mu^{p+1}_{p+1} \otimes e^{m_2}_0 \end{bmatrix} & \textit{otherwise.} \end{cases}$$

Proof. To alleviate the notations, we do not write the normalizing factor $\sqrt{\frac{2}{p+2}}$ in the sequel. Noting firstly in the case where i does not divide $n+1$

$$v^i = \sum_{t=1}^{p+1} \sin\left(\frac{t\pi}{p+2} \right) e^{n+1}_{i(t-1)},$$

where e_p^{n+1} is the $(p+1)^{\text{th}}$ vector of the canonical basis of $\mathbb{R}^{n+1}$, then

$$\Theta(v^i)_j = \sum_{k=0}^{n-j} v_k^i v_{k-j}^i = \sum_{k=0}^{n-j} \sum_{t,s=1}^{p+1} \sin\left(\frac{t\pi}{p+2}\right) \sin\left(\frac{s\pi}{p+2}\right) [k = i(t-1)][k+j = i(s-1)].$$

We used again the Iverson symbol here to make these computations easier. Thus, $[k = i(t-1)][k+j = i(s-1)]$ equals 1 if and only if j is a multiple of i (eliminating k in the previous expression yields $j = i(s-t)$); letting $u = s-t$, we obtain a single sum parametrized by t

$$\Theta(v^i)_{iu} = \sum_{t=1}^{p+1} \sin\left(\frac{t\pi}{p+2}\right) \sin\left(\frac{(u+t)\pi}{p+2}\right), \tag{8}$$

for components multiple of i, and

$$\Theta(v^i)_j = 0,$$

if i does not divide j. The computation of the closed form (7) for this sum is not difficult but a little bit tedious to be presented in detail here; the proof in the case where i divides $n+1$ is almost the same, it is just necessary to observe that

$$v^i = \sum_{t=1}^{p} \sin\left(\frac{t\pi}{p+1}\right) e_{i(t-1)}^{n+1}.$$

□

For the point with minimal i^{th} component, we use the

Corollary 3.3. *The extreme point of $\mathcal{U}_n$ with the minimal i^{th} component is*

$$\bar{\gamma}^i = \begin{cases} (\varepsilon_p \circ \mu^p) \otimes e_0^i & \text{if } i \text{ divides } n+1, \\ \begin{bmatrix} \left(\varepsilon_p \circ \mu_{1:p}^{p+1}\right) \otimes e_0^i \\ (-1)^p \mu_{p+1}^{p+1} \otimes e_0^{m_2} \end{bmatrix} & \text{otherwise,} \end{cases}$$

where $\varepsilon_p = (1, -1, 1, \cdots) \in \mathbb{R}^p$ and $\circ$ is the Hadamard product.

Proof. We present the proof for the case where i does not divide $n+1$, the other case being very similar and easier to prove: then

$$\lambda_{\min}(A^{(i)}) = -\lambda_{\max}(A^{(i)}) = \cos\left(\frac{(p+1)\pi}{p+2}\right),$$

and for all $t = 1, \cdots, p+1$

$$(-1)^t \sin\left(\frac{t\pi}{p+2}\right) = \sin\left(\frac{t(p+1)\pi}{p+2}\right).$$

So we find that

$$\bar{v}^i = \left[\begin{array}{c} \left(\varepsilon_p \circ T^{p+1}_{1:p,1}\right) \otimes e^i_0 \\ (-1)^p T^{p+1}_{p+1,1} \otimes e^{m_2}_0 \end{array} \right],$$

is an eigenvector associated with the smallest eigenvalue of $A^{(i)}$. Re-examinating the proof of the previous proposition, one sees that a factor $(-1)^u$ appears in front of the sum (8) and gives directly the result. $\square$

Now, we can define the inner approximation of $\mathcal{U}_n$ simply by

$$\mathcal{I}_n = \text{conv}\left(\{\gamma^i, \mid i = 1, \cdots, n\} \cup \{\bar{\gamma}^i, \mid i = 1, \cdots, n\}\right).$$

For example, we give an illustration in the case where $n = 2$: we can compute easily $\mathcal{U}_2$, as the evolute of the curve $\mathscr{C} = \{(2\cos\omega, 2\cos 2\omega) \mid \omega \in [0, \pi]\}$.

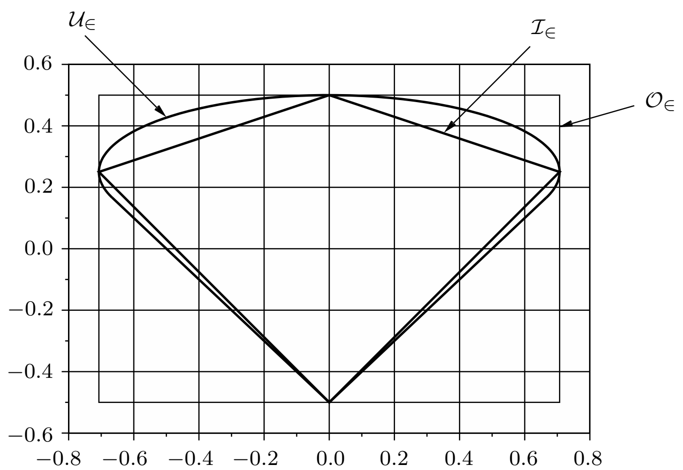

Figure 3.1: "Sandwiching" $\mathcal{U}_n$ for $n = 2$

It would be interesting to use these polyhedral approximations to determine bounds on the *modulus*

$$\rho(\mathcal{C}_{n+1}) = \frac{\text{vol}(\mathcal{C}_{n+1} \cap \mathbb{B}_{n+1})}{2\text{vol}(\mathbb{B}_{n+1})}$$

of $\mathcal{C}_{n+1}$ to compare with the results of [21], where the authors use circular and polyhedral approximations to estimate this *modulus*.

3.2. Nonnegativity of the dot product on $\mathcal{C}_{n+1}$

As for $(\mathbb{R}^n_+, \langle\cdot,\cdot\rangle)$ and $(\mathcal{S}^+_n(\mathbb{R}), \langle\langle\cdot,\cdot\rangle\rangle)$, $\mathcal{C}_{n+1}$ is *acute* for $\langle\cdot,\cdot\rangle$, i.e. the scalar product of two elements of the cone is nonnegative, or equivalently the angle between any pair of vectors is less than $\frac{\pi}{2}$. To show that, we need of a basic result connecting circular and acyclic correlations. It is very similar to results concerning circular and acyclic convolutions (cf. [32, p. 331])

Lemma 3.4 (Zero Padding). *Let $n \in \mathbb{N}^*$, $x, y \in \mathbb{R}^{n+1}$; we denote $\widetilde{x} = (x, 0, \cdots, 0)$ and $\widetilde{y} = (y, 0, \cdots, 0)$ two vectors of $\mathbb{R}^{2n+1}$ completed with zeros; then*

$$\mathtt{corr}_c(\widetilde{x}, \widetilde{y})_k = \begin{cases} \mathtt{corr}_a(x, y)_k & \text{for } k = 0, \cdots, n \\ \mathtt{corr}_a(y, x)_{2n+1-k} & \text{for } k = n+1, \cdots, 2n. \end{cases}$$

For the proof, which is straightforward, one could see [17, p. 15]. Now, we can show that

Proposition 3.5. *$\mathcal{C}_{n+1}$ is an* acute *cone,* i.e.

$$\forall x, y \in \mathcal{C}_{n+1} \quad \langle x, y \rangle \geqslant 0.$$

Proof. Let $(x, y) \in \mathcal{C}_{n+1}^2$; there exists $(z, t) \in (\mathbb{R}^{n+1})^2$ such that $x = \mathtt{corr}_a(z, z)$ and $y = \mathtt{corr}_a(t, t)$; then, let us denote $\tilde{z}$ and $\tilde{t}$ the vectors associated with t and z zero padded up to $2n + 1$,

$$\tilde{z} = \begin{pmatrix} z \\ 0_n \end{pmatrix}, \quad \tilde{t} = \begin{pmatrix} t \\ 0_n \end{pmatrix} \quad \text{and} \quad \tilde{x} = \mathtt{corr}_c(\tilde{z}, \tilde{z}), \quad \tilde{y} = \mathtt{corr}_c(\tilde{t}, \tilde{t}).$$

If $\widetilde{X}$ and $\widetilde{Y}$ are respectively the discrete Fourier transforms of $\tilde{x}$ and $\tilde{y}$ on the group $(\mathbb{Z}/(2n+1)\mathbb{Z}, +)$, the Parseval-Plancherel theorem (cf. [35, p. 26]) says that

$$\langle \tilde{x}, \tilde{y} \rangle \overset{(P.P.)}{=} \frac{1}{2n+1} \langle \widetilde{X}, \widetilde{Y} \rangle.$$

But one can observe that, due to the correlation theorem [4]

$$\widetilde{X}_i = \overline{Z}_i Z_i = |Z_i|^2 \geqslant 0 \text{ for all } i \in \{0, \cdots, 2n\},$$

which is also true for $\widetilde{Y}$, we deduce $\langle \tilde{x}, \tilde{y} \rangle \geqslant 0$. By decomposing the dot product we have

$$\begin{aligned} \langle \tilde{x}, \tilde{y} \rangle &= \tilde{x}_0 \tilde{y}_0 + \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i + \sum_{i=n+1}^{2n} \tilde{x}_i \tilde{y}_i \\ &= x_0 y_0 + \sum_{i=1}^{n} x_i y_i + \sum_{i=n+1}^{2n} \mathtt{corr}_c(\tilde{y}, \tilde{y})_i \mathtt{corr}_c(\tilde{t}, \tilde{t})_i \\ &= x_0 y_0 + \sum_{i=1}^{n} x_i y_i + \sum_{i=n+1}^{2n} \mathtt{corr}_a(y, y)_{2n+1-i} \mathtt{corr}_a(t, t)_{2n+1-i} \\ &= x_0 y_0 + 2 \sum_{i=1}^{n} x_i y_i, \end{aligned}$$

where we have used the Zero Padding Lemma. Letting $A = x_0 y_0 = ||t||^2 ||z||^2 \geqslant 0$ and $B = \sum_{i=1}^{n} x_i y_i$, this yields

$$A + 2B \geqslant 0,$$

and we conclude with

$$\langle x, y \rangle = A + B \geqslant A - A/2 = A/2 \geqslant 0.$$

$\square$

3.3. The Matryoshka property

The cones $\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_{n+1}$, are in a sense "nested": the orthogonal projection of $\mathcal{C}_{n+1}$ on $\mathbb{R}^n$ is $\mathcal{C}_n$, or in an equivalent way,

Proposition 3.6. *Let $\Im : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be the injective linear mapping defined by $\Im(x) = (x, 0)$; then*

$$\Im(\mathcal{C}_n) = \mathcal{C}_{n+1} \cap \{x_n = 0\}.$$

The proof being trivial is not given here. For the details, cf. [17, p. 44].

3.3.1. Using a support function to define $\mathcal{C}_{n+1}$

The support function σ_E of the subset $E \subset \mathbb{R}^n$ being defined as

$$\begin{array}{rccl} \sigma_E : & \mathbb{R}^n & \rightarrow & \mathbb{R} \cap \{+\infty\} \\ & x & \mapsto & \sup\limits_{s \in E} \langle x, s \rangle, \end{array}$$

there is a one-to-one correspondence between closed convex sets and support functions (see [26, p. 211]) summarized by the following formula

$$\sigma_E = \sigma_{\mathrm{conv}}(E).$$

where conv stands for the closed convex hull of a set. Another way to view this one-to-one mapping is to say that the support function of a set "could see only its convex hull". Let

$$g(x) = \max_{\omega \in [0,\pi]} \langle -v(\omega), x \rangle,$$

where $v(\omega)$ was previously defined in (1) and we can use the max notation since $[0, \pi]$ is a compact set and $\omega \mapsto \langle v(\omega), x \rangle$ is continuous; one can remark that denoting

$$S = \{-v(\omega) \mid \omega \in [0, \pi]\}, \tag{9}$$

$g = \sigma_S$ is therefore the support function of S. Since $g(x)$ is the minimal value of the trigonometric polynomial up to a change of sign, one could check if x belongs to $\mathcal{C}_{n+1}$, according to the following proposition.

Proposition 3.7. *Belonging to $\mathcal{C}_{n+1}$ is characterized by the sign of g:*

(i) $$x \in \mathcal{C}_{n+1} \Leftrightarrow g(x) \leqslant 0$$

(ii) $$x \in \operatorname{int} \mathcal{C}_{n+1} \Leftrightarrow g(x) < 0$$

(iii) $$x \in \partial\mathcal{C}_{n+1} \Leftrightarrow g(x) = 0.$$

3.3.2. Tangent and Normal cones to $\mathcal{C}_{n+1}$ at a point of $\mathcal{C}_{n+1}$

In convex analysis, to retrieve information about the local geometry of a set, we dispose of two fundamental convex sets: the tangent and the normal cone; as defined in [26, p. 136], the tangent cone $T(C, x)$ to a convex set C at a point $x \in C$ is defined as

$$T(C, x) = \mathtt{cl}\,\{d \in \mathbb{R}^n \mid d = \alpha(y - x),\ y \in C,\ \alpha \in \mathbb{R}^+\},$$

where $\mathtt{cl}(\cdot)$ denotes the closure. In our case, since

$$\mathcal{C}_{n+1} = \{x \in \mathbb{R}^{n+1} | -\langle v(\omega), x\rangle \leqslant 0,\ \forall\omega \in [0,\pi]\},$$

we can give the simple definition

$$T(\mathcal{C}_{n+1}, x) = \{d \in \mathbb{R}^{n+1} \mid \langle s, d\rangle \leqslant 0,\ \forall s \in J(x)\},$$

where

$$J(x) = \{-v(\omega) |\ \omega \in [0,\pi],\ \langle v(\omega), x\rangle = 0\}.$$

This yields

$$T(\mathcal{C}_{n+1}, x) = \{d \in \mathbb{R}^{n+1} |\langle v(\omega), d\rangle \geqslant 0,\ \text{for all } \omega \text{ such that } \langle v(\omega), x\rangle = 0\}.$$

The second fundamental set associated with C is the normal cone at x defined by

$$N(C, x) = \{d \in E \mid \forall y \in C,\ \langle y - x, d\rangle \leqslant 0\}.$$

Using again the definition of $\mathcal{C}_{n+1}$, we can describe simply $N(C, x)$ by

$$N(\mathcal{C}_{n+1}, x) = \text{cone}\,\{-v(\omega) \mid \exists\omega \in [0,\pi],\ \langle v(\omega), x\rangle = 0\}, \tag{10}$$

where cone$(\cdot)$ denotes the convex conical hull, *i.e.*

$$\text{cone}(E) = \left\{ \sum_{i=1}^{k} \alpha_i x_i \ :\ x_i \in E,\ \alpha_i \geqslant 0,\ k \in \mathbb{N}^* \right\}.$$

Knowing $N(\mathcal{C}_{n+1}, x)$ gives us more information about the boundary of $\mathcal{C}_{n+1}$, that we carry out in the next subsection

3.3.3. The boundary of $\mathcal{C}_{n+1}$ is not polyhedral

The normal cone corresponds in convex analysis to the generalization of the set of normal directions to the tangent space for differential manifolds. One could see that if at x the tangent cone is a half-space (its boundary being therefore the tangent space), then the normal cone reduces to a half-line orthogonal to the tangent space; such x is often called a smooth or regular point. For instance, $x^1 = e_0 + \frac{1}{2}e_2$ is such a point of $\partial\mathcal{C}_{n+1}$; indeed

$$\langle x^1, v(\omega)\rangle = 1 + \cos(2\omega) \geqslant 0,$$

which vanishes for $\omega = \frac{\pi}{2}$; then $x^1 \in \partial\mathcal{C}_{n+1}$ and

$$N(\mathcal{C}_{n+1}, x^1) = \text{cone}\{(-1, 0, 2, 0, \cdots, 0)\},$$

which guarantees that x^1 is a regular point. To show that $\partial\mathcal{C}_{n+1}$ is not polyhedral, it is sufficient to prove that around x^1 the curvature of $\partial\mathcal{C}_{n+1}$ does not locally equal zero. For that, we consider the direction $d = (0, 1, 0, \cdots, 0)$; it is clear that d is orthogonal to x^1. Then, computing

$$g(x^1 + \varepsilon d) = \max_{\omega\in[0,\pi]} \{-1 - 2\varepsilon\cos\omega - \cos(2\omega)\},$$

where the maximum is reached for $\omega_0 = -\arccos(\varepsilon/2)$, we obtain

$$\nabla g(x^1 + \varepsilon d) = \begin{pmatrix} -1 \\ \varepsilon \\ 2 - \varepsilon^2 \end{pmatrix}.$$

That implies

$$\frac{\nabla g(x^1 + \varepsilon d) - \nabla g(x^1)}{\varepsilon} = \begin{pmatrix} 0 \\ 1 \\ -\varepsilon \end{pmatrix},$$

which shows that the variation of ∇g along the direction d is nonlinear. The curvature does not equal zero locally at least in a neighborhood of x^1, and x^1 cannot belong to an exposed face of dimension greater than one. The boundary of $\mathcal{C}_{n+1}$ does not present an uniform regularity like the second order cone, but smooth parts and polyhedral parts, as for the cone $\mathcal{S}_n^+(\mathbb{R})$; Fig. 3.2 represents the boundary of $\mathcal{C}_{n+1}$, obtained by translating (and rescaling) the compact $\mathcal{U}_2$ from Fig. 3.1 into the direction e_0; the point x^1 lies in the plane containing $\mathcal{U}_2$ at the top of the smooth part.

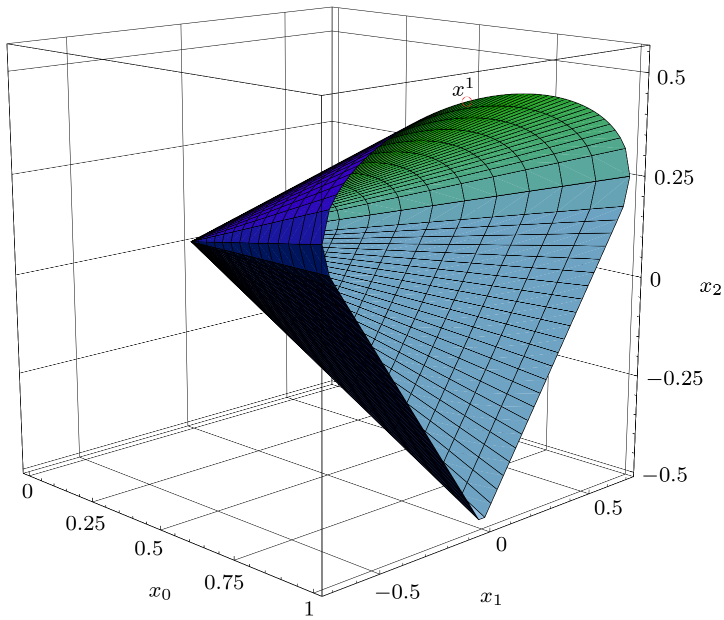

Figure 3.2: the cone $\mathcal{C}_3$

3.3.4. Faces and facets of $\mathcal{C}_{n+1}$

We know that $\mathcal{C}_{n+1}$ has not a boundary with an uniform regularity, but it is possible to get more information about the facial structure of $\mathcal{C}_{n+1}$. We need first some definitions.

Definition 3.8. Let C be a convex subset of $\mathbb{R}^n$. We call an *exposed face* each set $F \subset C$ verifying: there exists $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\begin{cases} \forall x \in C,\ \langle a, x\rangle \leqslant b \\ F = C \cap \{x \in \mathbb{R}^n \mid \langle a, x\rangle = b\}. \end{cases}$$

It means that $H_{a,b} = \{x \in \mathbb{R}^n \mid \langle a, x\rangle = b\}$ is a supporting hyperplane of C, i.e.

$$C \subset H_{a,b}^- = \{x \in \mathbb{R}^n \mid \langle a, x\rangle \leqslant b\},$$

and $H_{a,b}$ contains the affine hull of F. In the sequel, we simply speak of exposed faces. A facet is a face of dimension $n-1$. In this section, we are seeking faces or facets of $\mathcal{C}_{n+1}$. In fact, one could observe that for finding the faces of a convex cone, we can restrict our research to hyperplanes containing the origin, i.e. which can be written as $H_s = \{x \in \mathbb{R}^{n+1} \mid \langle s, x\rangle = 0\}$ due to the following lemma

Lemma 3.9. *Let K be a cone and $H_{a,b} = \{x \in \mathbb{R}^n \,|\, \langle a, x\rangle = b\}$ a supporting hyperplane of K such that $K \cap H_{a,b} \neq \emptyset$; then $b = 0$.*

For a proof, one could see [2, p. 65] The boundary and extreme rays of $\mathcal{C}_{n+1}$ are known for long time [27, 28]. The boundary is described by the vectors x such that $\langle x, v(\omega)\rangle$ has at least one zero on $[0, \pi]$. In particular, the extreme rays of $\mathcal{C}_{n+1}$ are even trigonometric polynomials which vanish exactly n times on $[0, \pi]$. One could conjecture that a face of dimension k would correspond to polynomials vanishing $n + 1 - k$ times on $[0, \pi]$. Unfortunately, to our knowledge, there is no such result in the literature. One fact related to this conjecture is that the extreme rays of the dual cone of $\mathcal{C}_{n+1}$ are composed with the vectors of the form $\lambda v(\omega)$ (see [27, p. 43]). Using this result, one computes the facets of $\mathcal{C}_{n+1}$, establishing necessary conditions about the normal directions to the facets.

Proposition 3.10. *$\mathcal{C}_{n+1}$ has two facets with outgoing normal directions $s_1 = (1, 2, \cdots, 2)$ and $s_2 = (1, -2, \cdots, 2(-1)^n)$.*

Proof. First, let us look at the necessary condition: if F is a facet, we can write F as $F = \mathcal{C}_{n+1} \cap H_s$. Since s is an outgoing normal direction to the face, and H_s a support hyperplane of $\mathcal{C}_{n+1}$, we have

$$\mathcal{C}_{n+1} \subset H_s^- = \{x \in \mathbb{R}^{n+1} \mid \langle s, x\rangle \leqslant 0\},$$

and we deduce that s belongs necessary to $\mathcal{C}_{n+1}^\circ$. Now, we show that s is necessarily an extreme ray of $\mathcal{C}_{n+1}^\circ$. Assume there exists linearly independent $s_1, s_2 \in \mathcal{C}_{n+1}^\circ$ such that

$$s = s_1 + s_2.$$

Since F has dimension n, we can find $x_1, \cdots, x_n$ linearly independent vectors and orthogonal to the normal direction s

$$0 = \langle s, x_i\rangle = \underbrace{\langle s_1, x_i\rangle}_{\leqslant 0} + \underbrace{\langle s_2, x_i\rangle}_{\leqslant 0} \quad \text{for } i = 1, \cdots, n.$$

A sum of nonpositive terms is zero if and only if each term is zero, thus

$$\langle s_1, x_i\rangle = \langle s_2, x_i\rangle = 0 \quad \text{for } i = 1, \cdots, n.$$

Therefore, s_1 and s_2 belong to the vector space Span $(\{x_1, \cdots, x_n\})^\perp$ which is of dimension one, this contradicts the linear independence of the two vectors. Using now a result from [27, p. 43], concerning the extreme rays of the dual cone of $\mathcal{C}_{n+1}$, we deduce that there exists $\omega_1 \in [0, \pi]$ such that, s is colinear to $v(\omega_1)$; therefore

$$\mathcal{A}^\star(s) = \alpha \mathcal{A}^\star(-v(\omega_1)) = -\alpha \mathrm{Re}\,(z_{\omega_1}(z^*_{\omega_1})^\top),$$

where z_ω was defined in (3) and one see that $\mathcal{A}^\star(s)$ is a matrix of rank at most 1. With a simple argument on the structure of the matrix, one proves that s_1 and s_2 are up to a multiplicative constant the only solutions for s to respect the rank constraint on $\mathcal{A}^\star(s)$ (see [9, p. 252]). Now, let us verify that s_1 and s_2 are actually directions exposing a facet. Let us start with $F_{s_1} = \mathcal{C}_{n+1} \cap H_{s_1}$ and consider the following points $(0, p_1, \cdots, p_n)$ where

$$p_i = 2e_0 - e_i \quad \text{for } i = 1, \cdots, n;$$

then according to Proposition 3.7,

$$g(p_k) = -\min_{\omega \in [0,\pi]} \langle p_k, v(\omega)\rangle = -\min_{\omega \in [0,\pi]} 2(1 - \cos k\omega) = 0,$$

therefore $p_k \in \partial\mathcal{C}_{n+1} \subset \mathcal{C}_{n+1}$. But, noticing that $\langle p_k, s_1\rangle = 0$ for all k, and the system of points $(p_k)_{k=1,\cdots,n}$ being clearly of full rank, there exist $n+1$ affinely independent points belonging to $F_{s_1} = \mathcal{C}_{n+1} \cap H_{s_1}$; hence $\dim(\mathrm{aff} F_{s_1}) = n$. For s_2, consider 0 and the system of points q_i, for $i = 1, \cdots, n$, defined by

$$q_i = 2e_0 + (-1)^i e_i,$$

and conclude in a similar way. $\square$

One could have a look at Fig. 3.2, where the two facets of $\mathcal{C}_{n+1}$ are respectively orthogonal to the directions $s_1 = (1, 2, 2)$ and $s_2 = (1, -2, 2)$; the two directions are also visible on Fig. 4.1 as edges of the unique facet of $\mathcal{C}^\circ_{n+1}$.

4. Polarity

Conic duality is the equivalent in convex analysis of the linear duality in linear analysis where roughly speaking, equalities are replaced by inequalities. As in the linear case, it is sometimes simpler to do computations in the dual space (e.g., less of parameters, etc...), and retrieve afterwards relevant information by duality. An equivalent in convex analysis of orthogonal decomposition, is the classical Moreau result

Theorem 4.1 (Moreau). *Let $F \subset E$ be a closed convex cone and $(x, x_F, x_{F^\circ}) \in E^3$; then the following conditions are equivalent:*

(i) $\quad x = x_F + x_{F^\circ}, \quad x_F \in F, \quad x_{F^\circ} \in F^\circ,$ *and* $\langle x_F, x_{F^\circ}\rangle_E = 0,$

(ii) $\quad x_F = p_F(x)$ *and* $x_{F^\circ} = p_{F^\circ}(x).$

F° stands here for the polar cone of F, i.e.

$$F^\circ = \{x \in E \mid \forall y \in F,\ \langle x, y\rangle_E \leqslant 0\}.$$

For a proof, see e.g. [26, p. 121]. F°, the negative polar of F, is a cone which enjoys several interesting properties and enables us to dualize conical constraints. For example, in some cases, computing the projection x_F of a point x on a cone F could be cheaper by computing firstly the projection x_{F° of x on F° and retrieving x_F simply by the difference $x_F = x - x_{F^\circ}$. Before determining the polar of $\mathcal{C}_{n+1}$, we need to define a subspace of $\mathcal{S}_{n+1}(\mathbb{R})$ which is involved in the computation.

4.1. Adjoint operator of $\mathcal{A}$ and symmetric Toeplitz matrices

Let us denote by $\mathcal{A}^\star$ the adjoint of $\mathcal{A}$ respect to $\langle\cdot,\cdot\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle$ defined by

$$\begin{array}{rrcl} \mathcal{A}^\star: & \mathbb{R}^{n+1} & \to & \mathcal{S}_{n+1}(\mathbb{R}) \\ & (x_0,\cdots,x_n) & \mapsto & \sum_{i=0}^{n} x_i A^{(i)}, \end{array}$$

For $x \in R^{n+1}$, $\mathcal{A}^\star(x)$ has the following structure

$$\mathcal{A}^\star(x) = \frac{1}{2}\begin{pmatrix} 2x_0 & x_1 & \cdots & x_n \\ x_1 & 2x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 \\ x_n & \cdots & x_1 & 2x_0 \end{pmatrix}.$$

The operator $\mathcal{A}^\star$ is a one-to-one mapping and realizes an isomorphism from $\mathbb{R}^{n+1}$ on its image $\mathcal{A}^\star(\mathbb{R}^{n+1})$, the set of symmetric Toeplitz matrices.

Definition 4.2. We denote by $\mathcal{T}_{n+1}(\mathbb{R})$ the subspace of $\mathcal{S}_{n+1}(\mathbb{R})$

$$\mathcal{A}^\star\left(\mathbb{R}^{n+1}\right) = \operatorname{aff}\{A^{(0)}, A^{(1)}, \cdots, A^{(n)}\}.$$

In other words,

$$\mathcal{T}_{n+1}(\mathbb{R}) = \{M \in \mathcal{S}_{n+1}(\mathbb{R}) \mid \exists x \in \mathbb{R}^{n+1} \text{ such that } M_{ij} = x_{|i-j|} \text{ for } i,j = 1,\cdots,n+1\}.$$

The set $\mathcal{T}_{n+1}(\mathbb{R})$ consists of symmetric matrices which are constant along their diagonals. This class of matrices is well-known: there are lots of fast algorithms to invert them or to compute their eigenvalues. For asymptotical results on their spectra see [7, 23]. For result in finite dimension, one could see [5, 11]. $\mathcal{T}_{n+1}(\mathbb{R})$ is of dimension of $n+1$ and the family of matrices $A^{(0)}, A^{(1)}, \cdots, A^{(n)}$ forms a basis of that space. The one-to-one mapping $\mathcal{A}^\star$ realizes a identification between $\mathcal{T}_{n+1}(\mathbb{R})$ and $\mathbb{R}^{n+1}$ for defining $\mathcal{C}^\circ_{n+1}$.

4.2. The polar cone of $\mathcal{C}_{n+1}$

4.2.1. Formulation by constraints

Proposition 4.3. *The (negative) polar cone of $\mathcal{C}_{n+1}$ is, up to the $\mathcal{A}^\star$ identification, the set of semi-definite negative Toeplitz matrices,* i.e.

$$\mathcal{C}^\circ_{n+1} = \{x \in \mathbb{R}^{n+1} \mid \mathcal{A}^\star(x) \preceq 0\}.$$

This proof was already presented in [1, 15, 27, 28] under various forms; we recall it here for completeness.

Proof. We seek $s \in \mathbb{R}^{n+1}$ such that, for all $x \in \mathcal{C}_{n+1}$, $\langle s, x\rangle \leqslant 0$; then

$$\forall y \in \mathbb{R}^{n+1} \ \langle s, \mathcal{A}(yy^\top)\rangle \leqslant 0 \Leftrightarrow \langle\langle \mathcal{A}^\star(s), yy^\top\rangle\rangle \leqslant 0 \Leftrightarrow \langle \mathcal{A}^\star(s)y, y\rangle \leqslant 0.$$

In other words, $\mathcal{A}^\star(s)$ is a semi-definite negative matrix. □

The formulation $\mathcal{A}^\star(x) \preceq 0$ guarantees that the Toeplitz form,

$$\sum_{k=0}^{n} \sum_{l=0}^{n} y_{k-l} x_k \bar{x}_l$$

is nonnegative with

$$y_k = \begin{cases} x_0 & \text{if } k = 0 \\ x_k/2 & \text{if } 1 \leqslant |k| \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

Using Bochner's theorem ([35, p. 19]), we know there exists a positive measure σ on $[0, 2\pi]$ such that

$$-\frac{1}{2} x_{|k|} = \int_0^{2\pi} e^{-ikt} d\sigma(t) \;\; \text{for all } k \in \mathbb{Z}.$$

Since x_k is a real number, it yields

$$x_k = \begin{cases} -\displaystyle\int_0^{2\pi} d\sigma & \text{if } k = 0 \\ \displaystyle\int_0^{2\pi} (-2\cos kt) d\sigma(t) & \text{for } k = 1, \cdots, n, \end{cases}$$

or, in other words, x belongs to the conical convex hull of $S = \{-v(\omega) | \omega \in [0, \pi]\}$. According to [28], this is nothing else than the formulation by generators of $\mathcal{C}_{n+1}^\circ$.

One noteworthy difference with classical cones – $(\mathbb{R}_+)^n, \mathcal{L}_n(\mathbb{R}), \mathcal{S}_n^+(\mathbb{R})$ – of convex analysis, is that $\mathcal{C}_{n+1}$ is not self-dual, i.e. $\mathcal{C}_{n+1}^\circ \neq -\mathcal{C}_{n+1}$. For that reason, a numerical approach of optimization problems involving $\mathcal{C}_{n+1}$ via its polar cone presents a reduced complexity: n parameters versus $n(n+1)/2$ under the form of the relation (2) in Proposition 2.8.

Using the formulation $\mathcal{A}^\star(x) \preceq 0$ for the polar cone, we can deduce a new expression for $N(\mathcal{C}_{n+1}, x)$: indeed, one knows that for a closed convex cone K (cf. [26, p. 137])

$$N(K, x) = \{y \in K^\circ \mid \langle x, y \rangle = 0\}.$$

Here, this gives:

$$N(\mathcal{C}_{n+1}, x) = \{y \in \mathbb{R}^{n+1} \mid \mathcal{A}^\star(y) \preceq 0, \langle x, y \rangle = 0\}.$$

4.2.2. Optimality conditions

One of the main applications of normal cones to a convex set is to write first order optimality conditions for optimization problems with constraints involving this set. Recalling from [26, p. 293] that if ones minimizes a convex function f on a convex set C, then $\bar{x}$ is a minimizer if and only if

$$\bar{x} \in C \quad \text{and} \quad 0 \in \partial f(\bar{x}) + N(C, \bar{x}).$$

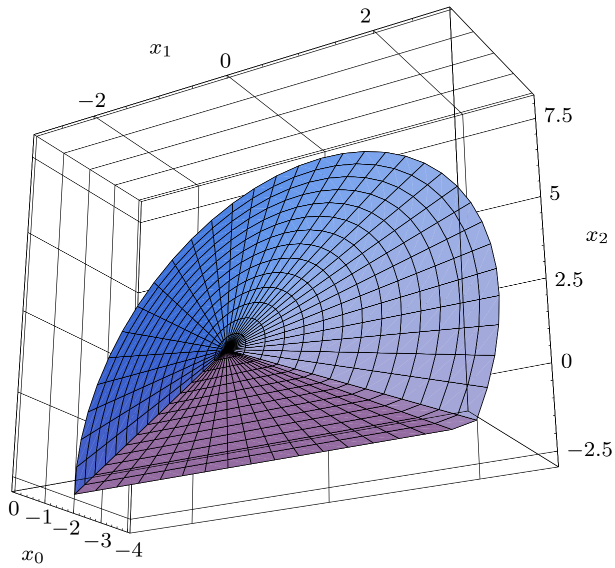

Figure 4.1: the polar cone $\mathcal{C}_3^\circ$

Here $\partial f(\bar{x})$ stands for the subdifferential of f at $\bar{x}$

$$\partial f(\bar{x}) = \{s \in E \mid \forall y \in E, \langle s, y - \bar{x}\rangle + f(\bar{x}) \leqslant f(y)\}.$$

For example, if one wants to project c on $\mathcal{C}_{n+1}$, one seeks a minimizer of $f(x) = \frac{1}{2}||c - x||_2^2$, and the optimality conditions become

$$\bar{x} \in \mathcal{C}_{n+1} \quad \text{and} \quad 0 = \bar{x} - c + s \;\; \text{with } \mathcal{A}^\star(s) \preceq 0 \text{ and } \langle s, \bar{x}\rangle = 0,$$

which can be summarized in

$$(Opt) \begin{cases} \bar{x} \in \mathcal{C}_{n+1} \\ \mathcal{A}^\star(c - \bar{x}) \preceq 0 \\ \langle \bar{x}, c - \bar{x}\rangle = 0. \end{cases} \tag{11}$$

Noting that the second condition is equivalent to $c - \bar{x} \in \mathcal{C}_{n+1}^\circ$, we have exactly the three conditions stated by Theorem 4.1 with $F = \mathcal{C}_{n+1}$.

4.2.3. A formulation by generators

We previously said that, using Bochner's theorem, one can find a formulation by generators of $\mathcal{C}_{n+1}^\circ$; nevertheless one can obtain the same result using only tools from convex analysis:

Proposition 4.4. *Let $S = \{-v(\omega) : \omega \in [0, \pi]\}$, then*

$$\mathcal{C}_{n+1}^\circ = \operatorname{cone}(S).$$

This formulation describes $\mathcal{C}_{n+1}^\circ$ as the convex conical hull of a parametric curve of $\mathbb{R}^{n+1}$. The proof of this result was given firstly by Krein et Nudelman in [28, p. 58]. One could derive (10) for $x = 0$ (the normal cone to a cone at $x = 0$ is the polar cone).

Proof. Since

$$\mathcal{C}_{n+1} = \{x \in \mathbb{R}^{n+1} | \langle x, -v(\omega)\rangle \leqslant 0 \text{ for } \omega \in [0,\pi]\},$$

by means of the generalized Farkas lemma ([26, p. 131]), if

$$\langle y, x\rangle \leqslant 0 \quad \text{for all } x \in \mathcal{C}_{n+1},$$

then

$$\begin{pmatrix} y \\ 0 \end{pmatrix} \in \text{cone}\left(\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} -v(\omega) \\ 0 \end{pmatrix} \right\}_{\omega \in [0,\pi]} \right).$$

We deduce that

$$\mathcal{C}_{n+1}^{\circ} \subset \text{cone}(S).$$

Conversely, if $y \in \text{cone}(S)$, there exists $(\alpha_0, \cdots, \alpha_n) \in (\mathbb{R}_+)^{n+1}$ such that

$$y = -\sum_{k=0}^{n} \alpha_k v(\omega_k) \quad \text{with } \omega_k \in [0,\pi];$$

and for $x \in \mathcal{C}_{n+1}$,

$$\langle y, x\rangle = -\sum_{k=0}^{n} \alpha_k \langle v(\omega_k), x\rangle \leqslant 0,$$

which implies $\text{cone}(S) \subset \mathcal{C}_{n+1}^{\circ}$. □

4.3. Conclusion

We have reviewed some results concerning the cone of vectors with autocorrelated components. This cone, which is clearly useful for applications in signal processing such as filtering or autocorrelation estimation, should deserve a better attention, in our opinion. In further support of this claim, we have studied thoroughly the geometry of $\mathcal{C}_{n+1}$ proving some results about the acuteness, the facial structure, and giving approximations of a compact base of that cone. We hope this review improve the knowledge about this cone and incite to pay more attention to this cone, or its extensions to the multivariate case; In particular, the application of the bivariate case for image processing has begun with the works [34, 14, 30]. In that case, cones which remain interesting are rather nonnegative trigonometric polynomials and sums-of-squares polynomials, which can be handled through the convex analysis paradigm thanks to linear matrix inequalities. The chapter 4 of [17] details some results and algorithms along these lines.

Acknowledgements. The author would like to acknowledge Jean-Baptiste Hiriart-Urruty for his help writing the paper, and also some other researchers from University Paul Sabatier in Toulouse, where the main part of the research work concerning this paper was done ([17]).

References

[1] B. Alkire, L. Vandenberghe: Convex optimization problems involving finite autocorrelation sequences, Math. Program., Ser. A 93 (2002) 331–359.

[2] A. Barvinok: A Course in Convexity, Graduate Studies in Mathematics 54, American Mathematical Society, Providence (2002).

[3] A. Ben-Tal, A. Nemirovski: Lectures on Modern Convex Optimization. Analysis, Algorithms, and Engineering Applications, MPS/SIAM Series on Optimization 2, SIAM, Philadelphia (2002).

[4] E. O. Brigham: The Fast Fourier Transform and its Applications, Prentice Hall, Englewood Cliffs (1988).

[5] P. Butler, A. Cantoni: Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, Linear Algebra Appl. 13 (1976) 275–288.

[6] C. I. Byrnes, A. Lindquist: A convex optimization approach to generalized moment problems, in: Control and Modeling of Complex Systems (Tokyo, 2001), K. Hashimoto et al. (ed.), Trends Math., Birkhäuser, Boston (2003) 3–21.

[7] A. Böttcher, B. Silbermann: Introduction to Large Truncated Toeplitz Matrices, Springer, New York (1999).

[8] A. Cadzow, Y. Sun: Sequences with positive semidefinite Fourier transforms, IEEE Trans. Acoust. Speech Signal Process. 34(6) (1986) 1502–1510.

[9] M. Chu, H. Golub: Inverse Eigenvalue Problems: Theory, Algorithms, and Applications, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford (2005).

[10] T. Davidson, Z. Luo, J. Sturm: Linear matrix inequality formulation of spectral mask constraints with applications to FIR filter design, IEEE Trans. Signal Process. 50(11) (2002) 2702–2715.

[11] P. Delsarte, Y. Genin: Spectral properties of finite Toeplitz matrices, in: Mathematical Theory of Networks and Systems (Beer Sheva, 1983), Lecture Notes in Control and Inform. Sci. 58, Springer, London (1984) 194–213.

[12] M. Dritschel: On factorization of trigonometric polynomials, Integral Equations Oper. Theory 49 (2004) 11–42.

[13] B. Dumitrescu: Trigonometric polynomials positive on frequency domains and applications to 2-D FIR filter design, IEEE Trans. Signal Process. 54(11) (2006) 4282–4292.

[14] B. Dumitrescu: Positive Trigonometric Polynomials and Signal Processing Applications, Springer, Dordrecht (2007).

[15] B. Dumitrescu, I. Tabus, P. Stoica: On the parametrization of positive real sequences and MA parameter estimation, IEEE Trans. Signal Process. 49 (2001) 2630–2639.

[16] L. Faybusovich: Semidefinite descriptions of cones defining spectral mask constraints, Eur. J. Oper. Res. 129(3) (2006) 1207–1221.

[17] M. Fuentes: Analyse et Optimisation de Problèmes sous Contraintes d'Autocorrélation, Université Paul Sabatier, Toulouse III (2007).

[18] M. Fuentes: Diagonalization of the symmetrized discrete i^{th} right shift operator: an elementary proof, Numer. Algorithms 44 (2007) 29–43.

[19] G. H. Golub, C. F. van Loan: Matrix Computations, 3rd Ed., Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore (1996).

[20] D. Gorsich, M. Genton, G. Strang: Eigenstructures of spatial design matrices, J. Multivariate Anal. 80 (2002) 138–165.

[21] D. Gourion, A. Seeger: Deterministic and random methods for computing volumetric moduli of convex cones, submitted (2008).

[22] R. L. Graham, D. E. Knuth, O. Patashnik: Concrete Mathematics, 2nd Ed., Addison-Wesley, Reading (1994).

[23] U. Grenander, G. Szegő: Toeplitz Forms and their Applications, 2nd Ed., Chelsea, New York (1984).

[24] Y. Hachez: Convex Optimization over Non-Negative Polynomials: Structured Algorithms and Applications, PhD Thesis, Université Catholique de Louvain (2003).

[25] Y. Hachez, H. Woerdeman: Approximating sums of squares with a single square, Linear Algebra Appl. 399 (2005) 187–201.

[26] J.-B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms. Part 1: Fundamentals, Springer, Berlin (1993).

[27] S. Karlin, W. J. Studden: Tchebycheff Systems: With Applications in Analysis and Statistics, Interscience, New York (1966).

[28] M. G. Krein, A. A. Nudelman: The Markov Moment Problem and Extremal Problems, Translations of Mathematical Monographs 50, American Mathematical Society, Providence (1977).

[29] J. B. Lasserre: Global optimization with polynomials and the problem of moments, SIAM J. Optim. 11 (2001) 796–817.

[30] A. Megretski: Positivity of trigonometric polynomials, in: Proc. 42^{nd} IEEE Conf. Decision Control (2003) 3814–3817.

[31] Y. Nesterov: Squared functional systems and optimization problems, in: High Performance Optimization, H. Frenk et al. (ed.), Kluwer, Dordrecht (2000) 405–440.

[32] A. Oppenheim, A. Willsky, I. Young: Signals and Systems, Prentice-Hall, London (1983).

[33] P. A. Parillo: Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization, PhD Thesis, California Institute of Technology, Pasadena (2000).

[34] T. Roh, B. Dumitrescu, L. Vandenberghe: Multidimensional FIR filter design via trigonometric sum-of-squares optimization, IEEE J. Sel. Topics Signal Process. 1 (2007) 641–650.

[35] W. Rudin: Fourier Analysis on Groups, Interscience Tracts in Pure and Applied Mathematics 12, Interscience, New York (1962).

[36] A. Stoica, R. Moses, P. Stoica: Enforcing positiveness on estimated spectral densities, Electron. Lett. 29 (1993) 2009–2011.