Journal of Convex Analysis
Volume 17 (2010), No. 3&4, 1089–1111

# Upper and lower bounds for Poincaré-type constants in BH*

**Danilo Percivale**
*Dipartimento di Ingegneria della Produzione, Università di Genova, Piazzale Kennedy, Fiera del Mare, Pad. D, 16129 Genova, Italy*
*percivale@diptem.unige.it*

*Dedicated to Hedy Attouch on the occasion of his 60th birthday.*

Received: October 27, 2008

The paper provides upper and lower bounds for the Poincaré constant in some inequalities involving BH functions and shows their application to some problems in the Calculus of Variations.

## 1. Introduction

In this paper we estimate the Poincaré constant for inequalities involving $BH$ functions, i.e. those functions whose second derivatives are bounded Radon measures on a lipschitz open set $\Omega$ of $\mathbb{R}^N$, $N \geq 1$. Indeed, in some problems in the Calculus of Variations (see [8], [9], [10], [11], [12]) it seems useful to give the best possible characterization of the constant $\lambda(\Omega)$ such that the inequality

$$\int_\Omega |v|\,dx \leq \lambda(\Omega)|\Delta v|_\Omega \tag{1}$$

holds true for every $v \in BH(\Omega)$ vanishing at the boundary. The key idea in the results presented here is quite simple and it starts, roughly speaking, from Green's identity

$$\int_\Omega z\Delta v\,dx = \int_\Omega v\Delta z\,dx. \tag{2}$$

When the test function $z \in H^1_0(\Omega)$ satisfies $\Delta z = \operatorname{sign} v$, the right hand side of (2) is exactly the $L^1$-norm of $v$, while the left hand side can be estimated by above with the sup norm of $z$ times the total variation of $\Delta v$: a careful choice of $z$ provides the optimal Poincaré constant. The rough idea is that of choosing

$$\lambda(\Omega) = \max_{\overline{\Omega}} z, \tag{3}$$

where $z \in H^1_0(\Omega)$ and $\Delta z = -1$ in $\Omega$. This choice seems to be optimal since if there exists a function v, null on the boundary, such that $\Delta v = \delta_{\overline{x}}$ with $z(\overline{x}) = \max_\Omega z$, then (2) becomes

$$\int_\Omega |v|\,dx = \lambda(\Omega) = \lambda(\Omega)|\Delta v|_\Omega. \tag{4}$$

*This research was partially supported by MIUR project 2006 "Variational Problems with Multiple Scales".

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Unfortunately such a $v$ does not belong to $BH(\Omega)$ for $N \geq 2$, but a careful manipulation of this argument gives the expected result. As far as it concernes the more delicate situations in which $|\Delta v|_\Omega$ is changed into

$$|\Delta v|_\Omega + \int_{\partial\Omega} \left|\frac{\partial v}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1} \tag{5}$$

or more generally into

$$|\Delta v|_\Omega + \int_{\Sigma} \left|\frac{\partial v}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1} \tag{6}$$

with $\Sigma \subset \partial\Omega$, the results of the paper provide an estimate from below and from above of the constant $\lambda(\Sigma, \Omega)$ such that the inequality

$$\int_\Omega |v|\, dx \leq \lambda(\Sigma,\Omega) \left( |\Delta v|_\Omega + \int_{\Sigma} \left|\frac{\partial v}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1} \right) \tag{7}$$

holds for every $v \in BH(\Omega)$ null on the boundary. In particular we show that

$$\frac{1}{2}\lambda(\Omega) \leq \lambda(\partial\Omega,\Omega) \leq \lambda(\Omega), \qquad \lambda(\emptyset,\Omega) = \lambda(\Omega) \tag{8}$$

where $\lambda(\Omega)$ is defined as in (3).

In the last section we show how from these results follows an almost sharp existence theorem for some problems in the Calculus of Variations (see [3], [4], [5], [7], [8], [9], [10], [11], [12]) and how these results lead to an optimal design problem.

## 2. Notations and results

In the following $\Omega$ will denote a Lipschitz open bounded subset of $\mathbb{R}^N$, $N \geq 1$ and $\mathbf{n}$ the outer unit vector normal to $\partial\Omega$. For every $r > 0$ $B_r(x)$ will be the ball of radius $r$ centered in $x$, $\omega_N$ the Lebesgue measure of the unit ball in $\mathbb{R}^N$, $\mathcal{L}^N$ and $\mathcal{H}^k$ the $N$-dimensional Lebesgue measure and the $k$-dimensional Hausdorff measure in $\mathbb{R}^N$ respectively.

For every $t \in \mathbb{R}$ we set $\operatorname{sign} t = t/|t|$ if $t \neq 0$ and $\operatorname{sign} 0 = 0$; if $u \in C(\overline{\Omega})$ we denote with $\operatorname{osc}_\Omega u$ the oscillation of $u$ in $\overline{\Omega}$ that is

$$\operatorname{osc}_{\overline{\Omega}} u = \max_{\overline{\Omega}} u - \min_{\overline{\Omega}} u.$$

The space of vector-valued Radon measures in $\Omega$ will be denoted with $\mathcal{M}(\Omega;\mathbb{R}^m)$ and if $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ let

$$\mu = \mu^a + \mu^j + \mu^c$$

the standard decomposition into absolutely continuous, atomic, and Cantor part. The total variation of $\mu$ in $\Omega$ will be denoted with $|\mu|_\Omega$. We then define

$$BH(\Omega) = \left\{ u \in W^{1,1}(\Omega) : D^2 u \in \mathcal{M}(\Omega;\mathbb{R}^{N^2}) \right\}, \tag{9}$$

$$BH_0(\Omega) = \{ u \in BH(\Omega) : u = 0 \text{ in } \partial\Omega \}, \tag{10}$$

$$SBH(\Omega) = \left\{ u \in BH(\Omega) : (D^2 u)^c \equiv 0 \right\} \tag{11}$$

and set $(D^2u)^a = \nabla^2 u$. $S_{\nabla u}$ will denote the singular set of $\nabla u \in BV(\Omega)$; for $\mathcal{H}^{N-1}$ almost all $x \in S_{\nabla u}$ there exist $\nu(x) \in \partial B_1$ and $\nabla u^+(x), \nabla u^-(x)$, the outer and inner trace of $\nabla u$ on $S_{\nabla u}$ in the direction of $\nu(x)$ (see [2]). It is readily seen that if $(D^2u)^j$ denotes the *jump part* of $D^2u$ then (see [2])

$$(D^2u)^j = \left[\frac{\partial u}{\partial \nu}\right] \nu \otimes \nu \, d\mathcal{H}^{N-1} \llcorner S_{\nabla u} \tag{12}$$

where

$$\left[\frac{\partial u}{\partial \nu}\right] = (\nabla u^+ - \nabla u^-) \cdot \nu. \tag{13}$$

Since $\mathrm{Tr}(\nu \otimes \nu) = \|\nu \otimes \nu\| = 1$ we get

$$\mathrm{Tr}(D^2u)^j = (\Delta u)^j = \left[\frac{\partial u}{\partial \nu}\right] \, d\mathcal{H}^{N-1} \llcorner S_{\nabla u} \tag{14}$$

and therefore

$$|(D^2u)^j|_\Omega = |(\Delta u)^j|_\Omega = \int_{S_{\nabla u}} \left|\frac{\partial v}{\partial \nu}\right| \, d\mathcal{H}^{N-1}. \tag{15}$$

Let now $\Sigma \subset \partial\Omega$ such that

$$\Sigma,\ \partial\Omega \setminus \Sigma \text{ are closed subset of } \partial\Omega \tag{16}$$

and we notice that if (16) holds then $\mathcal{H}^{N-1}(\Sigma) = 0$ if and only if $\Sigma = \emptyset$. Indeed if $\bar{x} \in \Sigma$ then by (16) there exists $\rho > 0$ such that $B_\rho(\bar{x}) \cap \partial\Omega \subset \Sigma$ and $B_\rho(\bar{x}) \cap \partial\Omega$ is athe graph of a Lipschitz function, hence $\mathcal{H}^{N-1}(\Sigma) \geq \mathcal{H}^{N-1}(B_\rho(\bar{x}) \cap \partial\Omega) > 0$. We are looking for the infimum of all $C(\Sigma, \Omega)$ such that

$$\int_\Omega |v| \, dx \leq C(\Sigma, \Omega) \left( |\Delta v|_\Omega + \int_\Sigma \left|\frac{\partial v}{\partial \mathbf{n}}\right| \, d\mathcal{H}^{N-1} \right) \tag{17}$$

holds true for every $v \in BH(\Omega)$ null on the boundary and we begin with two particular cases.

### 2.1. The cases $\Sigma = \partial\Omega$ and $\Sigma = \emptyset$.

We introduce at once the constant which has the leading role in the whole paper, namely

**Definition 2.1.** Let $u \in H_0^1(\Omega)$ such that $\Delta u = -1$ in $\Omega$. We set

$$\lambda(\Omega) = \max_{\overline{\Omega}} u. \tag{18}$$

We show now a continuity property of $\lambda(\Omega)$ with respect to $\Omega$ in the sense of the following statement.

**Proposition 2.2.** *Let $\Omega_h$ be a sequence of open Lipschitz subsets of $\Omega$ such that*

$$\max_{x\in\partial\Omega_h} d(x,\partial\Omega)\to 0, \tag{19}$$

$$\mathcal{L}^N(\Omega\setminus\Omega_h)\to 0 \tag{20}$$

*then*

$$\lambda(\Omega_h)\to\lambda(\Omega). \tag{21}$$

**Proof.** Let $u_h\in H_0^1(\Omega)$ such that $u_h\equiv 0$ in $\Omega\setminus\Omega_h$, $\Delta u_h=-1$ in $\Omega_h$ and let $w_h=u-u_h$. Then $\Delta w_h=0$ in $\Omega_h$, $w_h=u$ in $\partial\Omega_h$ and by the maximum principle

$$0\le w_h\le\max\left\{\max_{\Omega\setminus\Omega_h}u,\max_{\partial\Omega_h}u\right\}.$$

Since $u\in C(\overline{\Omega})$ and $u=0$ on $\partial\Omega$ by (19) and (20) we get

$$\max\left\{\max_{\Omega\setminus\Omega_h}u,\max_{\partial\Omega_h}u\right\}\to 0$$

hence $w_h\to 0$ uniformly in $\Omega$, that is $u_h\to u$ uniformly and therefore

$$\lambda(\Omega_h)=\max_{\overline{\Omega}}u_h\to\max_{\overline{\Omega}}u=\lambda(\Omega),$$

thus proving (21). □

**Examples 2.3.** Let $a_i\in\mathbb{R}$, $i=1,2,.....N$, $a_i\neq 0$ for every $i$ and

$$\Omega=\left\{x\in\mathbb{R}^N:\sum_{i=1}^N a_i^2x_i^2<1\right\} \tag{22}$$

then

$$u(x)=\frac{1}{2}\left(\sum_{i=1}^N a_i^2\right)^{-1}\left(1-\sum_{i=1}^N a_i^2x_i^2\right) \tag{23}$$

and

$$\lambda(\Omega)=\frac{1}{2}\left(\sum_{i=1}^N a_i^2\right)^{-1}. \tag{24}$$

In particular if $a_i=R^{-1}$ for every $i=1,2,...N$, then $\Omega=B_R(0)$ and $\lambda(\Omega)=R^2/2N$. When $N=1$ and $\Omega=(0,L)$ then $\lambda(\Omega)=L^2/8$.

If $\Omega=B_R(0)\setminus B_{\tau R}(0)$ $(0<\tau<1)$, then direct calculations show that when $N=2$

$$\lambda(\Omega)=\frac{R^2}{4}\left\{1+\frac{\tau^2-1}{2\log\tau}\left[\log\left(\frac{\tau^2-1}{2\log\tau}\right)-1\right]\right\} \tag{25}$$

and when $N>2$

$$\lambda(\Omega)=\frac{R^2}{2n}\left\{-\left(\frac{N-2}{2}\right)^{\frac{2}{N}}\frac{N}{N-2}K(\tau)^{\frac{2}{N}}+1+K(\tau)\right\} \tag{26}$$

where

$$K(\tau) = \frac{1-\tau^2}{\tau^{2-N}-1}.$$

We remark explicitly that (24), (25), (26) yield

$$\lambda(B_R(0) \setminus B_{\tau R}(0)) + \lambda(B_{\tau R}(0)) \leq \lambda(B_R(0)) \tag{27}$$

for every $N \geq 2$.

We first consider the simplest case $\Sigma = \emptyset$ and we show the following

**Theorem 2.4.** *Let $v \in BH_0(\Omega)$. Then*

$$\int_\Omega |v|\, dx \leq \lambda(\Omega)|\Delta v|_\Omega \tag{28}$$

*and constant in inequality* (28) *is as best as possible.*

**Proof.** Let $v \in BH(\Omega)$ and $z \in H_0^1(\Omega)$ such that $\Delta z = \operatorname{sign} v$ in $\Omega$: it is readily seen that $z \in W^{2,p}(\Omega)$ for every $p > 1$ hence it is continuous in $\overline{\Omega}$. An integration by parts yields

$$\int_\Omega z\, d(\Delta v) = -\int_\Omega \nabla z \cdot \nabla v\, dx = \int_\Omega v\Delta z\, dx = \int_\Omega |v|\, dx. \tag{29}$$

By the maximum principle

$$0 \leq \max_{\overline{\Omega}} |z| \leq \lambda(\Omega)$$

and therefore (29) yields

$$\int_\Omega |v|\, dx \leq (\max_{\overline{\Omega}} |z|)|\Delta v|_\Omega \leq \lambda(\Omega)|\Delta v|_\Omega. \tag{30}$$

In order to show optimality of the constant in inequality (28), let $u \in H_0^1(\Omega)$ be as in Definition 2.1 and let $\Omega \ni \overline{x} \in \operatorname{argmax} u$. Let $r_0 > 0$ such that $B_r = B_r(\overline{x}) \subset \Omega$ for every $r \leq r_0$ and let $v_r \in H_0^1(\Omega)$ be the unique solution of

$$\Delta v_r = \frac{1}{\omega_N r^N}\mathbf{1}_{B_r} \tag{31}$$

in $\Omega$. It is readily seen that $v_r \in BH(\Omega)$ and by (31)

$$\int_\Omega \Delta v_r\, dx = |\Delta v_r|_\Omega = 1. \tag{32}$$

Since $v_r \leq 0$ in $\Omega$, then $\Delta u = \operatorname{sign} v_r$ in $\Omega$. Therefore

$$\begin{aligned}\int_\Omega |v_r|\, dx &= \int_\Omega v_r \Delta u\, dx = \int_\Omega u\Delta v_r\, dx \\ &= \frac{1}{\omega_N r^N}\int_{B_r}(u - u(\overline{x}))\, dx + u(\overline{x})\int_\Omega \Delta v_r\, dx \\ &= \lambda(\Omega)|\Delta v_r|_\Omega - \epsilon_r = \lambda(\Omega) - \epsilon_r\end{aligned} \tag{33}$$

where

$$\epsilon_r = \frac{1}{\omega_N r^N} \int_{B_r} (u(\overline{x}) - u)\, dx.$$

Clearly $0 \le \epsilon_r \to 0$ as $r \to 0^+$ and if there exists $\lambda'(\Omega) < \lambda(\Omega)$ such that inequality (28) holds true with $\lambda'(\Omega)$ in place of $\lambda(\Omega)$ for every function $v \in BH(\Omega)$ null on the boundary, then by (32) and (33)

$$\lambda(\Omega) - \epsilon_r = \int_\Omega |v_r|\, dx \le \lambda'(\Omega)|\Delta v_r|_\Omega = \lambda'(\Omega) \tag{34}$$

hence

$$0 < \lambda(\Omega) - \lambda'(\Omega) \le \epsilon_r, \tag{35}$$

a contradiction when $r$ is small enough. □

**Definition 2.5.** *We say that a bounded lipschitz open set $\Omega \subset \mathbb{R}^N$ satisfies the oscillation condition if for every $h \in L^\infty(\Omega)$, $|h| = 1$ a.e. and for every $v \in H^1_0(\Omega)$ such that $\Delta v = h$ in $\Omega$ we have*

$$\operatorname{osc}_{\overline{\Omega}} v \le \lambda(\Omega). \tag{36}$$

We are now in a position to state and prove the main result of this subsection which deals with the case $\Sigma = \partial\Omega$.

**Theorem 2.6.** *Let $\mathcal{P}(\Omega)$ be the set of $C > 0$ such that the inequality*

$$\int_\Omega |v|\, dx \le C \left\{ |\Delta v|_\Omega + \int_{\partial\Omega} \left| \frac{\partial v}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1} \right\} \tag{37}$$

*holds true for every $v \in BH_0(\Omega)$. Then*

$$\mathcal{P}(\Omega) \neq \emptyset \tag{38}$$

*and*

$$\frac{1}{2}\lambda(\Omega) \le \inf \mathcal{P}(\Omega) \le \lambda(\Omega). \tag{39}$$

*Moreover if $\Omega$ satisfies the oscillation condition then*

$$\min \mathcal{P}(\Omega) = \frac{1}{2}\lambda(\Omega). \tag{40}$$

**Proof.** The right inequality in (39) and (38) are an obvious consequence of Theorem 2.4. In order to prove the left inequality let $u \in H^1_0(\Omega)$ be as in Definition 2.1 and let $\Omega \ni \overline{x} \in \operatorname{argmax} u$. Let $r_0 > 0$ such that $B_r = B_r(\overline{x}) \subset \Omega$ for every $r \le r_0$ and let $v_r \in H^1_0(\Omega)$ be the unique solution of

$$\Delta v_r = \frac{1}{\omega_N r^N} \mathbf{1}_{B_r} \tag{41}$$

in $\Omega$. It is readily seen that $v_r \le 0$, $v_r \in BH_0(\Omega)$, $\Delta u = \operatorname{sign} v_r$ in $\Omega$ and

$$|\Delta v_r|_\Omega + \int_{\partial\Omega} \left| \frac{\partial v_r}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1} = 1 + \int_{\partial\Omega} \left| \frac{\partial v_r}{\partial \mathbf{n}} \right|. \tag{42}$$

By using (41), by taking into account that $v_r \leq 0$ in $\Omega$ yields

$$\frac{\partial v_r}{\partial \mathbf{n}} \geq 0 \;\; \mathcal{H}^{N-1} \;\; \text{a.e. on } \partial\Omega$$

and by noticing that

$$\min_{\overline{\Omega}} u = 0$$

we get

$$c_u = \frac{1}{2}(\max_{\overline{\Omega}} u + \min_{\overline{\Omega}} u) = \frac{1}{2}u(\overline{x}) = \frac{1}{2}\lambda(\Omega)$$

and

$$\begin{aligned}
\int_\Omega |v_r|\, dx = & \int_\Omega (u - c_u)\Delta v\, dx + \int_{\partial\Omega} (u - c_u)\frac{\partial v_r}{\partial \mathbf{n}}\, d\mathcal{H}^{N-1} \\
= & \frac{1}{\omega_N r^N}\int_{B_r} (u - c_u)\, dx + c_u \int_{\partial\Omega} \frac{\partial v_r}{\partial \mathbf{n}}\, d\mathcal{H}^{N-1} \\
= & \frac{1}{\omega_N r^N}\int_{B_r} (u - u(\overline{x}))\, dx + \frac{1}{\omega_N r^N}\int_{B_r} (u(\overline{x}) - c_u)\, dx \\
& + c_u \int_{\partial\Omega} \frac{\partial v_r}{\partial \mathbf{n}}\, d\mathcal{H}^{N-1} \\
= & \frac{1}{2}\lambda(\Omega)\left(1 + \int_{\partial\Omega}\left|\frac{\partial v_r}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1}\right) - \epsilon_r
\end{aligned} \tag{43}$$

where

$$\epsilon_r = \frac{1}{\omega_N r^N}\int_{B_r} (u(\overline{x}) - u)\, dx.$$

Clearly $0 \leq \epsilon_r \to 0$ as $r \to 0^+$ and if there exists $\lambda'(\Omega) < \frac{1}{2}\lambda(\Omega)$ such that inequality (37) holds for every function $v \in BH(\Omega)$ null on the boundary, then by (43)

$$\begin{aligned}
& \frac{1}{2}\lambda(\Omega)\left(1 + \int_{\partial\Omega}\left|\frac{\partial v_r}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1}\right) - \epsilon_r \leq \int_\Omega |v_r|\, dx \leq \\
& \leq \lambda'(\Omega)\left(1 + \int_{\partial\Omega}\left|\frac{\partial v_r}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1}\right)
\end{aligned} \tag{44}$$

hence

$$0 < \frac{1}{2}\lambda(\Omega) - \lambda'(\Omega) \leq \epsilon_r, \tag{45}$$

a contradiction when $r$ is small enough. Assume now that $\Omega$ satisfies the oscillation condition: let $v \in BH(\Omega),\ v = 0$ in $\partial\Omega$ and $u \in H_0^1(\Omega),\ \Delta u = \operatorname{sign} v$ in $\Omega$. Then by setting

$$c_u = \frac{1}{2}(\max_{\overline{\Omega}} u + \min_{\overline{\Omega}} u)$$

we get

$$\begin{aligned}
& \int_\Omega (u - c_u)\, d(\Delta v) - \int_{\partial\Omega} (u - c_u)\frac{\partial v}{\partial \mathbf{n}}\, d\mathcal{H}^{N-1} \\
= & -\int_\Omega \nabla(u - c_u)\cdot\nabla v\, dx = \int_\Omega v\Delta u\, dx = \int_\Omega |v|\, dx.
\end{aligned} \tag{46}$$

Hence by taking into account that

$$\max_{\overline{\Omega}} |u - c_u| = \frac{1}{2} \operatorname{osc}_{\overline{\Omega}} u$$

an by recalling that $\Omega$ satisfies the *oscillation condition* we obtain

$$\begin{aligned} \int_\Omega |v|\, dx &\leq \frac{1}{2}(\operatorname{osc}_{\overline{\Omega}} u) \left( |\Delta v|_\Omega + \int_{\partial\Omega} \left| \frac{\partial v}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1} \right) \\ &\leq \frac{1}{2}\lambda(\Omega) \left( |\Delta v|_\Omega + \int_{\partial\Omega} \left| \frac{\partial v}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1} \right) \end{aligned} \tag{47}$$

thus concluding the proof. $\square$

**Remark 2.7.** Although a complete characterization of domains verifying Definition 2.5 seems very hard to tackle, nevertheless the following counterexamples, suggested by S. Solimini, show that oscillation property does not hold for every bounded Lipschitz open set. Indeed let $N \geq 3$,

$$\Omega = \prod_{i=0}^{N-1} \left(0, 2^{\frac{i}{N}}\right) \tag{48}$$

and

$$\begin{aligned} \Omega_1 &= \prod_{i=0}^{N-2} \left(0, 2^{\frac{i}{N}}\right) \times (0, 2^{-\frac{1}{N}}), \\ \Omega_2 &= \prod_{i=0}^{N-2} \left(0, 2^{\frac{i}{N}}\right) \times (2^{-\frac{1}{N}}, 2^{1-\frac{1}{N}}). \end{aligned} \tag{49}$$

Since

$$(1, 2^{1/N}, ..., 2^{1-\frac{2}{N}}, 2^{-\frac{1}{N}}) = 2^{-\frac{1}{N}}(2^{\frac{1}{N}}, 2^{\frac{2}{N}}, ..., 2^{1-\frac{1}{N}}, 1) \tag{50}$$

then there exists a rotation $\mathcal{R}$ such that $\Omega_1 = 2^{-\frac{1}{N}}\mathcal{R}(\Omega)$, $\Omega_2 = 2^{-\frac{1}{N}}(\mathbf{e}_1 + \mathcal{R}(\Omega))$. Let $u \in H_0^1(\Omega)$ such that $\Delta u = -1$ in $\Omega$ and define $w_i \in H_0^1(\Omega_i)$, $i = 1, 2$, $w_1(x) = 2^{-\frac{2}{N}}u(2^{-\frac{1}{N}}\mathcal{R}^T x)$, $w_2(x) = -2^{-\frac{2}{N}}u(\mathcal{R}^T(2^{-\frac{1}{N}}x + \mathbf{e}_1)$. It is readily seen that $\Delta w_1 = -1$ in $\Omega_1$ and $\Delta w_1 = 1$ in $\Omega_2$, that $\frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu}$ on $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Then by setting $w = \mathbf{1}_{\Omega_1} w_1 + \mathbf{1}_{\Omega_1} w_2$, $h = -\mathbf{1}_{\Omega_1} + \mathbf{1}_{\Omega_1}$ we have $w \in H_0^1(\Omega)$, $\Delta w = h$ in $\Omega$ and therefore

$$\operatorname{osc}_{\overline{\Omega}} w = \lambda(\Omega_1) + \lambda(\Omega_2) = 2^{\frac{N-2}{N}}\lambda(\Omega) > \lambda(\Omega) \tag{51}$$

thus violating the oscillation condition.

The case $N = 2$ requires some preliminary remark. For every $a, b > 0$ let us consider the class of rectangles $R_{a,b} = (-a, a) \times (-b, b)$ : we claim that $\lambda(R_{2,1}) \leq 1/2$. Indeed let $0 < \alpha < 1/2$, $E_\alpha = \{(x, y) : \alpha^2 x^2 + (1 - 4\alpha^2)y^2 \leq 1\}$: then $R_{2,1} \subset E_\alpha$ hence

$$\lambda(R_{2,1}) \leq \lambda(E_\alpha) = \frac{1}{2}(1 - 3\alpha^2)^{-1}$$

for every $0 < \alpha < 1/2$ and therefore

$$\begin{aligned} ll\lambda(R_{2,1}) &\leq \inf_{0<\alpha<1/2} \frac{1}{2}(1-3\alpha^2)^{-1} = \frac{1}{2} = \lambda(B_{\sqrt{2}}(0)) \\ &< \lambda(R_{\sqrt{2},\sqrt{2}}) = 2\lambda(R_{1,1}). \end{aligned} \tag{52}$$

Let us assume now that $R_{2,1}$ satisfies the oscillation condition. Then by noticing that $R_{2,1} = (R_{1,1} - \frac{1}{2}\mathbf{e_1}) \cup (\frac{1}{2}\mathbf{e_1} + R_{1,1}) = Q_- \cup Q_+$, let $h = \mathbf{1}_{Q_-} - \mathbf{1}_{Q_+}$ and $u \in H^1_0(R_{2,1})$, $\Delta u = h$ in $R_{2,1}$. Then by symmetry $u \in H^1_0(Q_\pm)$, $u(-1,0) = -u(1,0) = \min u = -\max u$, $2\lambda(R_{1,1}) = \operatorname{osc} u$ and by taking into account that $R_{2,1}$ is assumed to satisfy the oscillation condition, (52) yields

$$2\lambda(R_{1,1}) = \operatorname{osc} u \leq \lambda(R_{2,1}) < \lambda(R_{\sqrt{2},\sqrt{2}}) = 2\lambda(R_{1,1}),$$

a contradiction.

**Remark 2.8.** We notice that Remark 2.7 does not imply the existence of a domain $\Omega$ such that $\inf \mathcal{P}(\Omega) > \lambda(\Omega)/2$ which remains an open question.

The case $N = 1$ is completely described by the following theorem which shows that the class of subsets satisfying the oscillation condition is not empty.

**Theorem 2.9.** *Let $v \in BH_0(0,L)$ then*

$$\int_0^L |v|\,dx \leq \frac{L^2}{16}\left\{|v''|_{(0,L)} + |v'_-(L)| + |v'_+(0)|\right\} \tag{53}$$

*and equality in* (53) *occurs if*

$$v(x) = \left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^+. \tag{54}$$

**Proof.** Let $h \in L^\infty(0,L)$, $|h| \leq 1$ a.e. and let $z \in W^{2,\infty}(0,L) \cap H^1_0(\Omega)$ such that $z'' = h$ in $(0,L)$. We claim that

$$\operatorname{osc}_{[0,L]} z \leq \frac{L^2}{8} = \lambda(0,L). \tag{55}$$

Since $z'' - u'' = h + 1 \geq 0$ the maximum principle yields $z \leq u$ and (55) is satisfied if $z \geq 0$; by changing $z$ into $-z$ the same holds when $z \leq 0$. If $z$ changes its sign then we may assume without restriction that there exists $\tau \in (0,1)$ such that $z(\tau L) = 0$ and $z$ attains its maximum and its minimum in $[0,\tau L]$ and in $[\tau l, L]$ respectively. Hence

$$\operatorname{osc}_{[0,L]} z \leq \lambda(0,\tau L) + \lambda(\tau L, L) = \frac{L^2}{8}(\tau^2 + (1-\tau)^2) \leq \frac{L^2}{8} \tag{56}$$

thus proving that $(0,L)$ satisfies the oscillation condition. Then (53) follows by Theorem 2.5, while substitution of (54) concludes the proof. $\square$

The ideas of the previous theorem work also for radial functions in a sphere, namely we have the following

**Theorem 2.10.** *Let $v \in BH_0(B_R(0))$ a radial function. Then*

$$\int_{B_R(0)} |v|\,dx \le \frac{R^2}{4N}\left\{|\Delta v|_{B_R(0)} + \int_{\partial B_R(0)} \left|\frac{\partial v}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1}\right\}. \tag{57}$$

**Proof.** Since $v$ is radial then $\operatorname{sign} v$ is radial; hence it will be enough to prove that the oscillation condition holds for every radial function $h$. Therefore let $h \in L^\infty(B_R(0))$ a radial function such that $|h| = 1$ a.e. and let $z \in H_0^1(B_R(0))$ such that $\Delta z = h$ in $B_R(0)$. As in the previous theorem the maximum principle shows that if $z$ does not change its sign then $\operatorname{osc} z \le \lambda(B_R(0))$. Otherwise there exists $\tau \in (0,1)$ such that $z(\tau R) = 0$ and $z$ attains its maximum and its minimum in $B_\tau(0)$ and in $B_L(0) \setminus B_{\tau L}(0)$ respectively. Hence (27) implies

$$\operatorname{osc} z \le \lambda(B_R(0) \setminus B_{\tau R}(0)) + \lambda(B_{\tau R}(0)) \le \lambda(B_R(0)) = R^2/2N$$

which proves that the oscillation condition holds for every radial function $h$, $|h| = 1$ a.e. and concludes the proof. □

### 2.2. The general case

Let $\Sigma \subset \Omega$ such that (16) is satisfied and let

$$\mathcal{A}(\Sigma) = \left\{u \in H^1(\Omega) \cap C(\overline{\Omega}) : -\Delta u = 1 \text{ in } \Omega \ u = 0 \text{ on } \partial\Omega \setminus \Sigma\right\} \tag{58}$$

if $\Sigma \ne \partial\Omega$, and

$$\mathcal{A}(\partial\Omega) = \left\{u \in H^1(\Omega) \cap C(\overline{\Omega}) : -\Delta u = 1 \text{ in } \Omega\right\}. \tag{59}$$

Clearly $\mathcal{A}(\partial\Omega) \ne \emptyset$ and when $\Sigma \ne \partial\Omega$ any solution of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{60}$$

belongs to $\mathcal{A}(\Sigma)$ hence $\mathcal{A}(\Sigma) \ne \emptyset$ for every $\Sigma$ satisfying (16). We then define

$$\lambda_*(\Sigma, \Omega) = \inf\left\{\max_{\overline{\Omega}} |u(x)| : \ u \in \mathcal{A}(\Sigma)\right\} \tag{61}$$

and we show that

**Lemma 2.11.** *Assume that* (16) *holds and let $\Sigma \ne \emptyset$. Then there exists $u_h \in \mathcal{A}(\Sigma)$ such that*

$$0 \le \max_{\overline{\Omega}} u_h = -\min_{\overline{\Omega}} u_h \tag{62}$$

$$u_h(x) = \min_{\overline{\Omega}} u_h \quad \forall x \in \Sigma \tag{63}$$

$$\max_{x \in \overline{\Omega}} u_h(x) \to \lambda_*(\Sigma, \Omega). \tag{64}$$

**Proof.** Let $v_h \in \mathcal{A}(\Sigma)$ such that

$$\max_{x\in\ \overline{\Omega}} |v_h(x)| \to \lambda_*(\Sigma,\Omega)$$

and $v_h^* \in H^1(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v_h^* = 1 & \text{in } \Omega \\ v_h^* = 0 & \text{in } \partial\Omega\setminus\Sigma \\ v_h^* = \min\limits_{\overline{\Omega}} v_h & \text{on } \Sigma. \end{cases} \tag{65}$$

By the maximum principle (65) yields $v_h^* \le v_h,\ \min_{\overline{\Omega}} v_h^* = \min_{\overline{\Omega}} v_h \le 0$ hence

$$\max_{\overline{\Omega}} |v_h^*| \le \max_{\overline{\Omega}} |v_h|$$

and therefore

$$\max_{\overline{\Omega}} |v_h^*(x)| \to \lambda_*(\Sigma,\Omega). \tag{66}$$

Let now

$$d_h = \max_{\overline{\Omega}} v_h^* + \min_{\overline{\Omega}} v_h^*$$

and $\widetilde{v}_h^* \in H^1(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta \widetilde{v}_h^* = 1 & \text{in } \Omega \\ \widetilde{v}_h^* = 0 & \text{in } \partial\Omega\setminus\Sigma \\ \widetilde{v}_h^* = -\max\limits_{\overline{\Omega}} v_h^* & \text{on } \Sigma. \end{cases} \tag{67}$$

Then

$$\begin{cases} \Delta(v_h^* - \widetilde{v}_h^*) = 0 & \text{in } \Omega \\ v_h^* - \widetilde{v}_h^* = 0 & \text{in } \partial\Omega\setminus\Sigma \\ v_h^* - \widetilde{v}_h^* = d_h & \text{on } \Sigma \end{cases} \tag{68}$$

and again by the maximum principle

$$\min\{d_h, 0\} \le v_h^* - \widetilde{v}_h^* \le \max\{d_h, 0\}$$

which implies

$$-\max\{d_h, 0\} \le \widetilde{d}_h = \max_{\overline{\Omega}} \widetilde{v}_h^* + \min_{\overline{\Omega}} \widetilde{v}_h^* \le \min\{d_h, 0\}$$

that is

$$d_h \widetilde{d}_h \le 0. \tag{69}$$

For every $t \in [0,1]$ let $w_h^t = t\widetilde{v}_h^* + (1-t)v_h^*,\ u_h^t \in \mathcal{A}_f(\Sigma)$ such that

$$-\Delta u_h^t = f, \qquad u_h^t - w_h^t \in H_0^1(\Omega)$$

and $\eta_h : [0,1] \to \mathbb{R}$ defined by

$$\eta_h(t) = \max_{\overline{\Omega}} u_h^t + \min_{\overline{\Omega}} u_h^t.$$

It is readily seen that $\eta_h$ is continuous and that $\eta_h(0) \cdot \eta_h(1) = d_h \widetilde{d}_h \le 0$ by (69): hence for every $h \in \mathbb{N}$ there exists $t_h \in [0,1]$ such that by setting $u_h = u_h^{t_h}$ we get

$$\max_{\overline{\Omega}} u_h + \min_{\overline{\Omega}} u_h = 0$$

which proves (62). Since $u_h$ is superharmonic in $\Omega$, null on $\partial\Omega \setminus \Sigma$ then

$$\max_{\overline{\Omega}} u_h \ge 0$$

hence by (62)

$$\min_{\overline{\Omega}} u_h \le 0$$

and by noticing that $u_h$ is constant on $\Sigma$ we have also

$$u_h(x) = \min_{\overline{\Omega}} u_h$$

for every $x \in \Sigma$, thus proving (63).

If now $d_h \ge 0$ then $\widetilde{v}_h^* \le w_h^{t_h} \le v_h^*$ and therefore $\widetilde{v}_h^* = u_h^1 \le u_h \le u_h^0 = v_h^*$, that is by (62)

$$\max_{\overline{\Omega}} |u_h| = \max_{\overline{\Omega}} u_h \le \max_{\overline{\Omega}} v_h^* = \max_{\overline{\Omega}} |v_h^*|. \tag{70}$$

Analogously if $d_h \le 0$ we get

$$\max_{\overline{\Omega}} |u_h| = -\min_{\overline{\Omega}} u_h \le -\min_{\overline{\Omega}} v_h^* = \max_{\overline{\Omega}} |v_h^*| \tag{71}$$

and therefore by (66)

$$\max_{\overline{\Omega}} u_h \to \lambda_*(\Sigma, \Omega)$$

thus concluding the proof. $\square$

For every $a \in \mathbb{R}$ let $u_a^\Sigma \in H^1(\Omega)$ be the unique $u \in \mathcal{A}(\Sigma)$ such that $u = a$ on $\Sigma$ if $\Sigma \neq \emptyset$, $u_a^\Sigma \equiv u$ where $u$ is the unique solution of (60) otherwise. An immediate consequence of Lemma 2.11 is the following

**Lemma 2.12.** *Assume that* (16) *holds, then*

$$\lambda_*(\Sigma, \Omega) = \min \left\{ \max_{\overline{\Omega}} u_a^\Sigma : \; -\lambda(\Omega) \le a \le 0 \right\}. \tag{72}$$

*In particular*

$$\lambda_*(\partial\Omega, \Omega) = \frac{1}{2}\lambda(\Omega) \tag{73}$$

*and*

$$\lambda_*(\emptyset, \Omega) = \lambda(\Omega). \tag{74}$$

**Proof.** Assume first that $\Sigma \neq \emptyset$. Clearly

$$\lambda_*(\Sigma, \Omega) \leq \inf \left\{ \max_{\overline{\Omega}} |u_a^\Sigma| : \; a \in \mathbb{R} \right\}$$

and by Lemma 2.11 there exists $u_h \in \mathcal{A}(\Sigma)$ such that

$$u_h(x) = \min_{\overline{\Omega}} u_h \quad \forall x \in \Sigma \tag{75}$$

$$0 \leq \max_{\overline{\Omega}} u_h = - \min_{\overline{\Omega}} u_h \tag{76}$$

$$\max_{x \in \; \overline{\Omega}} u_h(x) \to \lambda_*(\Sigma, \Omega). \tag{77}$$

Hence by setting $a_h = \min_{\overline{\Omega}} u_h$ we get $u_{a_h}^\Sigma = u_h$ hence (76) and (77) yield

$$\lambda_*(\Sigma, \Omega) = \inf \left\{ \max_{\overline{\Omega}} u_a^\Sigma : \; a \in \mathbb{R} \right\}.$$

Moreover by recalling that $0 \geq a_h = - \max u_{a_h}^\Sigma \geq -\lambda(\Omega)$ we get

$$\lambda_*(\Sigma, \Omega) = \inf \left\{ \max_{\overline{\Omega}} u_a^\Sigma : \; -\lambda(\Omega) \leq a \leq 0 \right\}$$

and it is readily seen that this infimum is actually a minimum: indeed let $a_h \in [-\lambda(\Omega), 0]$ such that

$$\max_{\overline{\Omega}} u_{a_h}^\Sigma \to \lambda_*(\Sigma, \Omega)$$

then up to subsequences $a_h \to \overline{a}$ hence $u_{a_h}^\Sigma \to u_{\overline{a}}^\Sigma$ uniformly in $\Omega$ and therefore

$$\lambda_*(\Sigma, \Omega) = \max_{\overline{\Omega}} u_{\overline{a}}^\Sigma.$$

If now $\Sigma = \partial\Omega$ we get $u_{\overline{a}} - \overline{a} \in H_0^1(\Omega)$ and

$$-\Delta(u_{\overline{a}} - \overline{a}) = 1$$

hence by (18), (75) and (76)

$$\lambda(\Omega) = \max_{\overline{\Omega}} (u_{\overline{a}} - \overline{a}) = 2 \max u_{\overline{a}} = 2\lambda_*(\partial\Omega, \Omega)$$

thus proving (73). Eventually if $\Sigma = \emptyset$ then $\mathcal{A}(\Sigma)$ contains only the solution of (60) that is $\lambda_*(\emptyset, \Omega) = \lambda(\Omega)$ and (74) is proven. $\square$

**Remark 2.13.** We remark explicitly that if $\Sigma \neq \emptyset$ then by Lemma 2.11 and Lemma 2.12 there exists $\overline{a} \in [-\lambda(\Omega), 0]$ such that

$$\max_{\overline{\Omega}} u_{\overline{a}}^\Sigma = \min \left\{ \max_{\overline{\Omega}} u_a^\Sigma : \; -\lambda(\Omega) \leq a \leq 0 \right\}$$

and

$$\overline{a} = \min_{\overline{\Omega}} u_{\overline{a}}^\Sigma = - \max_{\overline{\Omega}} u_{\overline{a}}^\Sigma.$$

Moreover a standard regularity argument shows that $u_{\overline{a}}^\Sigma \in C^\infty(\overline{\Omega})$.

Another simple consequence of Lemma 2.11 and Lemma 2.12 is the following

**Proposition 2.14.** *Assume that $\Sigma$ satisfies* (16)*, then*

$$\frac{1}{2}\lambda(\Omega) \leq \lambda_*(\Sigma,\Omega) \leq \lambda(\Omega) \tag{78}$$

*and therefore*

$$\lambda_*(\Sigma,\Omega) = \min\left\{\max_{\overline{\Omega}} u_a^\Sigma : \ -\lambda(\Omega) \leq a \leq -\frac{\lambda(\Omega)}{2}\right\}. \tag{79}$$

**Proof.** Assume that $\Sigma \subset \partial\Omega$ satisfies (16) then by Lemma 2.7 and Lemma 2.8 we get that for every $a \in [-\lambda(\Omega), 0]$, $w = u_a^\Sigma - u_a^{\partial\Omega}$ is harmonic in $\Omega$ and $w = -a \geq 0$ on $\partial\Omega \setminus \Sigma$, $w = 0$ on $\Sigma$. Then $u_a^\Sigma \geq u_a^{\partial\Omega}$ and therefore by (73)

$$\lambda_*(\Sigma,\partial\Omega) \geq \lambda_*(\partial\Omega,\Omega) = \frac{1}{2}\lambda(\Omega).$$

By noticing that for every $\Sigma$ satisfying (16)

$$\max_{\overline{\Omega}} u_a^\Sigma \leq \lambda(\Omega)$$

we get $\lambda_*(\Sigma,\Omega) \leq \lambda(\Omega)$ thus proving (78). By recalling Remark 2.13 it is readily seen that (78) implies (79). $\square$

Let $h \in L^\infty(\Omega)$, $|h| \leq 1$ a.e. and for every $\Sigma \subset \partial\Omega$ satisfying (16) let

$$\mathcal{A}_h(\Sigma) = \left\{u \in H^1(\Omega) \cap C(\overline{\Omega}) : -\Delta u = h \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \setminus \Sigma\right\}$$

if $\Sigma \neq \partial\Omega$,

$$\mathcal{A}_h(\partial\Omega) = \left\{u \in H^1(\Omega) \cap C(\overline{\Omega}) : -\Delta u = h \text{ in } \Omega\right\}.$$

Clearly $\mathcal{A}_h(\Sigma) \neq \emptyset$ so we may define

$$\lambda_h(\Sigma,\Omega) = \inf\{\operatorname{osc}_{\overline{\Omega}} u : u \in \mathcal{A}_h(\Sigma)\} \tag{80}$$

and

$$\lambda^*(\Sigma,\Omega) = \frac{1}{2}\sup\{\lambda_h(\Sigma,\Omega) : \ |h| \leq 1 \text{ a.e. in } \Omega\}. \tag{81}$$

The main result of this subsection is the following

**Theorem 2.15.** *Assume that $\Sigma \subset \partial\Omega$ satisfies* (16) *and let $\mathcal{P}(\Sigma,\Omega)$ be the set of $C > 0$ such that the inequality*

$$\int_\Omega |v|\,dx \leq C\left(|\Delta v|_\Omega + \int_\Sigma \left|\frac{\partial v}{\partial \mathbf{n}}\right| d\mathcal{H}^{N-1}\right) \tag{82}$$

*holds true for every $v \in BH_0(\Omega)$. Then*

$$\mathcal{P}(\Sigma,\Omega) \neq \emptyset \tag{83}$$

*and*

$$\lambda_*(\Sigma,\Omega) \leq \inf \mathcal{P}(\Sigma,\Omega) \leq \lambda^*(\Sigma,\Omega). \tag{84}$$

**Proof.** Since $\mathcal{P}(\emptyset,\Omega)\subset\mathcal{P}(\Sigma,\Omega)$ then (83) is a direct consequence of Theorem 2.4. In order to prove (84) let $v\in BH_0(\Omega)$ and $u\in\mathcal{A}_h(\Sigma)$ with $h=\operatorname{sign} v$. Then by setting

$$c_u=\frac{1}{2}(\max_{\overline{\Omega}} u+\min_{\overline{\Omega}} u)$$

we get

$$\begin{aligned}&\int_\Omega (u-c_u)\,d(\Delta v)-\int_{\partial\Omega}(u-c_u)\frac{\partial v}{\partial\mathbf{n}}\,d\mathcal{H}^{N-1}\\ =\ &-\int_\Omega \nabla(u-c_u)\cdot\nabla v\,dx=\int_\Omega v\Delta u\,dx=\int_\Omega |v|\,dx.\end{aligned}\tag{85}$$

Hence by taking into account that

$$\max_{\overline{\Omega}}|u-c_u|=\frac{1}{2}\operatorname{osc}_{\overline{\Omega}} u$$

we obtain

$$\int_\Omega |v|\,dx\le\frac{1}{2}(\operatorname{osc}_{\overline{\Omega}} u)\left(|\Delta v|_\Omega+\int_\Sigma\left|\frac{\partial v}{\partial\mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right)\tag{86}$$

for every $u\in\mathcal{A}_h(\Sigma)$ and therefore

$$\begin{aligned}\int_\Omega |v|\,dx&\le\frac{1}{2}(\inf_{\mathcal{A}_f(\Sigma)}\operatorname{osc}_{\overline{\Omega}} u)\left(|\Delta v|_\Omega+\int_\Sigma\left|\frac{\partial v}{\partial\mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right)\\ &=\frac{1}{2}\lambda_f(\Sigma,\Omega)\left(|\Delta v|_\Omega+\int_\Sigma\left|\frac{\partial v}{\partial\mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right)\\ &\le\lambda^*(\Sigma,\Omega)\left(|\Delta v|_\Omega+\int_\Sigma\left|\frac{\partial v}{\partial\mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right),\end{aligned}\tag{87}$$

thus proving that inequality (82) holds with $\overline{\lambda}(\Sigma,\Omega)\le\lambda^*(\Sigma,\Omega)$, the right hand side of (84). It remains only to prove that $\lambda_*(\Sigma,\Omega)\le\overline{\lambda}(\Sigma,\Omega)$. If $\Sigma=\emptyset$ we may proceed as in Theorem 2.4, otherwise let $u_{\overline{a}}\in\mathcal{A}(\Sigma)$ as in Remark 2.13, choose $y\in\Omega$ such that $u_{\overline{a}}(y)=\max_{\overline{\Omega}} u$, $r_0>0$ such that $B_{r_0}(y)\subset\Omega$ and, for every $r\le r_0$, let $w_r\in H_0^1(\Omega)$ be the solution of

$$\Delta w_r=\frac{1}{\omega_N r^N}\mathbf{1}_{B_r(y)}\ \text{ in }\Omega.\tag{88}$$

Clearly $w_r\le 0$ in $\Omega$, hence

$$\frac{\partial w_r}{\partial\mathbf{n}}\ge 0\ \ \mathcal{H}^{N-1}-\text{a.e on }\partial\Omega$$

and as in (85) we get

$$\begin{aligned}&\int_\Omega u_{\overline{a}}\,d(\Delta w_r)-\int_{\partial\Omega}u_{\overline{a}}\frac{\partial w_r}{\partial\mathbf{n}}\,d\mathcal{H}^{N-1}\\ =\ &-\int_\Omega\nabla u_{\overline{a}}\cdot\nabla w_r\,dx=\int_\Omega w_r\Delta u_{\overline{a}}\,dx=\int_\Omega|w_r|\,dx.\end{aligned}\tag{89}$$

By (76), (77), (88) and (89)

$$\int_\Omega |w_r|\,dx = \frac{1}{\omega_N r^N}\int_{B_r(y)} u_{\overline{a}}\,dx + (\max_{\overline{\Omega}} u_{\overline{a}})\int_\Sigma \frac{\partial w_r}{\partial \mathbf{n}}\,d\mathcal{H}^{N-1} \tag{90}$$

and since $u_{\overline{a}} \in \mathcal{A}(\Sigma)$ we get that $u_{\overline{a}} + (2n)^{-1}\|x-y\|^2$ is harmonic in $\Omega$, hence

$$\begin{aligned} u_{\overline{a}}(y) &= \frac{1}{\omega_N r^N}\int_{B_r(y)} (u_{\overline{a}}(x) + (2n)^{-1}\|x-y\|^2)\,dx \\ &= \frac{1}{\omega_N r^N}\int_{B_r(y)} u_{\overline{a}}(x)\,dx + \frac{r^2}{2n(N+2)\omega_N}\mathcal{H}^{N-1}(\partial B_1). \end{aligned} \tag{91}$$

By setting now

$$\sigma_r = \frac{r^2}{2n(N+2)\omega_N}\mathcal{H}^{N-1}(\partial B_1)$$

we get

$$\begin{aligned} \int_\Omega |w_r|\,dx &= u_{\overline{a}}(y)\left\{1 + \int_\Sigma \frac{\partial w_r}{\partial \mathbf{n}}\,d\mathcal{H}^{N-1}\right\} - \sigma_r \\ &= (\max_{\overline{\Omega}} u_{\overline{a}})\left\{\int_\Omega |\Delta w_r|\,dx + \int_\Sigma \left|\frac{\partial w_r}{\partial \mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right\} - \sigma_r \\ &= \lambda_*(\Sigma,\Omega)\left\{\int_\Omega |\Delta w_r|\,dx + \int_\Sigma \left|\frac{\partial w_r}{\partial \mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right\} - \sigma_r. \end{aligned} \tag{92}$$

If there exists $\overline{\lambda}(\Sigma,\Omega) < \lambda_*(\Sigma,\Omega)$ such that (82) holds for every $v \in BH(\Omega)$ null on the boundary, then from (92) we get

$$\begin{aligned} &\overline{\lambda}(\Sigma,\Omega)\left\{\int_\Omega |\Delta w_r|\,dx + \int_\Sigma \left|\frac{\partial w_r}{\partial \mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right\} \geq \int_\Omega |w_r|\,dx \\ &= \lambda_*(\Sigma,\Omega)\left\{\int_\Omega |\Delta w_r|\,dx + \int_\Sigma \left|\frac{\partial w_r}{\partial \mathbf{n}}\right|\,d\mathcal{H}^{N-1}\right\} - \sigma_r \end{aligned} \tag{93}$$

and by taking into account that

$$\int_\Omega |\Delta w_r|\,dx = 1$$

we get

$$0 < \lambda_*(\Sigma,\Omega) - \overline{\lambda}(\Sigma,\Omega) \leq \sigma_r,$$

a contradiction when $r \to 0^+$. □

**Remark 2.16.** By Lemma 2.11 and Lemma 2.12 it is readily seen that Theorem 2.15 implies both Theorem 2.4 and Theorem 2.6 by setting $\Sigma = \partial\Omega$ and $\Sigma = \emptyset$ respectively.

**Remark 2.17.** Let $N = 1$, $\Omega = (0, L)$, $\Sigma = \{L\}$: we claim that $\lambda^*(\Sigma, \Omega) = \lambda_*(\Sigma, \Omega)$. Let $u \in \mathcal{A}(\Sigma)$ i.e.

$$u'' = -1; \qquad u(0) = 0 \tag{94}$$

that is

$$u(x) = sx - \frac{x^2}{2} \ \ \forall a \in \mathbb{R}$$

and

$$\lambda_*(\Sigma, \Omega) = \min_s \max \left\{ \left| sL - \frac{L^2}{2} \right|, \frac{s^2}{2} \right\} = \left( \frac{3}{2} - \sqrt{2} \right) L^2. \tag{95}$$

Let now $|h| \leq 1$ a.e. in $(0, L)$ and consider for every $c \in \mathbb{R}$ the follwing Dirichlet problems

$$u'' = -1; \qquad u(0) = 0, \qquad u(L) = c \tag{96}$$

$$v'' = h; \qquad v(0) = 0, \qquad v(L) = c. \tag{97}$$

We claim that $\operatorname{osc}_{\overline{\Omega}} v \leq \operatorname{osc}_{\overline{\Omega}} u$. If $\min v \geq \min u = c \wedge 0$ it is a simple consequence of the maximum principle, otherwise there exists $0 < \tau < 1$ such that $v(\tau L) = c \wedge 0$ and

$$\begin{aligned} \operatorname{osc}_{\overline{\Omega}} v &\leq |c \wedge 0| + \frac{1}{2} \left[ \left( \frac{c \wedge 0}{\tau L} + \frac{\tau L}{2} \right)^+ \right]^2 + \frac{L^2}{8} (1 - \tau)^2 \\ &\leq |c \wedge 0| + \frac{1}{2} \left[ \left( \frac{c \wedge 0}{L} + \frac{L}{2} \right)^+ \right]^2 = \operatorname{osc}_{\overline{\Omega}} u. \end{aligned} \tag{98}$$

By recalling Remark 2.13 there exists $\overline{a} \in [-\lambda(0, L), \frac{1}{2}\lambda(0, L)]$ such that

$$\lambda_*(\Sigma, \Omega) = \overline{a} = \min_{\overline{\Omega}} u^{\Sigma}_{\overline{a}} = - \max_{\overline{\Omega}} u^{\Sigma}_{\overline{a}}$$

and by choosing $c = \overline{a}$ in (97) we get

$$\operatorname{osc}_{\overline{\Omega}} v \leq \operatorname{osc}_{\overline{\Omega}} u^{\Sigma}_{\overline{a}} \leq \lambda_*(\Sigma, \Omega)$$

hence

$$\lambda_h(\Sigma, \Omega) \leq \lambda_*(\Sigma, \Omega) \tag{99}$$

that is

$$\lambda^*(\Sigma, \Omega) = \lambda_*(\Sigma, \Omega) \tag{100}$$

as claimed.

## 3. Some applications to the Calculus of Variations

Although all inequalities proven in the previous section do not provide an estimate between the $L^1$ norm of $v \in BH(\Omega)$ and the total variation of its full Hessian matrix, nevertheless they can be carefully used to obtain an almost sharp existence result in some variational problems. Let us introduce now

$$\mathcal{K} = \{ u \in SBH(\Omega) : \nabla^2 u \in L^2(\Omega), \ u = 0 \text{ in } \partial\Omega \}$$

and let $F : \mathcal{K} \to \mathbb{R}$ be defined by (see also [8], [9], [10], [11], [12])

$$F(u) = \int_\Omega W(\nabla^2 u)\, dx + \gamma \int_{S_{\nabla u}} \left| \left[ \frac{\partial u}{\partial \nu} \right] \right| d\mathcal{H}^{N-1} + \beta \mathcal{H}^{N-1}(S_{\nabla u}) - \int_\Omega f u\, dx \qquad (101)$$

where $\beta, \gamma > 0,\ f \in L^\infty(\Omega)$ and $W$ is a quadratic form such that for every $N \times N$ real matrix $A$,

$$W(A) \geq c\|A\|^2$$

for suitable $c > 0$. We will prove the following

**Theorem 3.1.** *If*

$$\|f\|_\infty < \frac{\gamma}{\lambda(\Omega)} \qquad (102)$$

*then there exists* $u \in \operatorname{argmin} F$.

**Proof.** It is well known that $F$ is sequentially lower semicontinuous in $w^* - SBH$ and since

$$\begin{aligned} \int_\Omega f u\, dx &\leq \|f\|_\infty \int_\Omega |u| \leq \|f\|_\infty \lambda(\Omega) |\Delta u|_\Omega \\ &\leq \|f\|_\infty \lambda(\Omega) \left\{ \int_\Omega |\nabla^2 u|\, dx + \int_{S_{\nabla u}} \left| \left[ \frac{\partial u}{\partial \nu} \right] \right| d\mathcal{H}^{N-1} \right\} \end{aligned} \qquad (103)$$

a standard argument shows that for every minimizing sequence $u_h$ we have

$$\begin{aligned} &\frac{c}{2} \int_\Omega |\nabla^2 u_h|^2\, dx + \beta \mathcal{H}^{N-1}(S_{\nabla u_h}) + (\gamma - \|f\|_\infty \lambda(\Omega)) \int_{S_{\nabla u_h}} \left| \left[ \frac{\partial u_h}{\partial \nu} \right] \right| d\mathcal{H}^{N-1} \\ &\leq \frac{1}{4c} \|f\|_\infty^2 \lambda^2(\Omega) \mathcal{L}^N(\Omega). \end{aligned} \qquad (104)$$

By taking into account that $u_h = 0$ on $\partial\Omega$ we get

$$\int_\Omega \nabla u_h\, dx = 0$$

hence there exists $C(\Omega) > 0$ such that for every $h \in \mathbb{N}$

$$\int_\Omega |\nabla u_h|\, dx \leq C(\Omega) |D^2 u_h|_\Omega,$$

and by (104) we deduce that up to subsequences $u_h \to u$ in $w^* - BH$ hence $u \in \operatorname{argmin} F$ by lower semicontinuity. □

The main goal of this section is to show that (102) is an almost sharp condition for existence of minimizers of (101) (see also [12] for the one dimensional case). To this aim we need the following technical result whose proof is reported only for the sake of completeness.

**Lemma 3.2.** *Assume that $\mu_h \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, $\mu_h \to \mu \in \mathcal{M}(\Omega)$ strongly in $H^{-1}(\Omega)$ and that there exists a constant $k > 0$ such that*

$$\int_\Omega \varphi\, d\mu \geq k \int_\Omega \varphi\, dx \tag{105}$$

*for every $0 \leq \varphi \in H_0^1(\Omega)$. Then there exists $h_0 \in \mathbb{N}$ such that for every $h \geq h_0$ and for every $0 \leq \varphi \in H_0^1(\Omega)$*

$$\int_\Omega \varphi\, d\mu_h \geq \frac{k}{2} \int_\Omega \varphi\, dx, \tag{106}$$

*hence $\mu_h - \frac{k}{2}\mathcal{L}^N \llcorner \Omega$ is a positive measure for $h \geq h_0$.*

**Proof.** Assume by contradiction that there exist a sequence $\mathbb{N} \ni h_j \to +\infty$ and $0 \leq \varphi_j \in H_0^1(\Omega)$ such that

$$\int_\Omega \varphi_j\, d\mu_{h_j} < \frac{k}{2} \int_\Omega \varphi_j\, dx. \tag{107}$$

We may assume without restriction that $\|\varphi_j\|_{H_0^1} \leq 1$, $\varphi_j \to \varphi$ in $w - H_0^1(\Omega)$: since $\mu \in H^{-1}(\Omega)$ then there exists a vector field $\mathbf{M} \in L^2(\Omega; \mathbb{R}^N)$ such that $\mu = \operatorname{div} \mathbf{M}$ (see [1], Thm. 3.10). Therefore

$$\begin{aligned}
\int_\Omega \varphi_j\, d\mu_{h_j} = & \int_\Omega (\varphi_j - \varphi)\, d(\mu_{h_j} - \mu) + \int_\Omega (\varphi_j - \varphi)\, d\mu \\
& + \int_\Omega \varphi\, d(\mu_{h_j} - \mu) + \int_\Omega \varphi\, d\mu \\
= & \int_\Omega (\varphi_j - \varphi)\, d(\mu_{h_j} - \mu) + \int_\Omega (\nabla\varphi_j - \nabla\varphi) \cdot \mathbf{M}\, dx \\
& + \int_\Omega \varphi\, d(\mu_{h_j} - \mu) + \int_\Omega \varphi\, d\mu
\end{aligned} \tag{108}$$

and by taking into account that $\mu_h \to \mu \in \mathcal{M}(\Omega)$ strongly in $H^{-1}(\Omega)$ and $\varphi_j \to \varphi$ in $w - H_0^1(\Omega)$ we get

$$\int_\Omega \varphi_j\, d\mu_{h_j} \to \int_\Omega \varphi\, d\mu \tag{109}$$

which together with (107) yields

$$\int_\Omega \varphi\, d\mu \leq \frac{k}{2} \int_\Omega \varphi\, dx \tag{110}$$

thus contradicting (105). By (106)

$$\mu_h - \frac{k}{2}\mathcal{L}^N \llcorner \Omega$$

is a positive distribution, hence a positive measure in $\Omega$. □

We are now ready to state and prove

**Theorem 3.3.** *If*

$$f \equiv -\frac{\gamma}{\lambda(\Omega)} - \delta, \quad \delta > 0 \tag{111}$$

*then $F$ is not bounded from below.*

**Proof.** Let $z \in H_0^1(\Omega)$ be such that $\Delta z = -1$ in $\Omega$, $\Omega \ni \overline{x} \in \operatorname{argmax} z$, $r_0 > 0$ such that $B_r = B_r(\overline{x}) \subset \Omega$ for every $r \leq r_0 < 1$,

$$\epsilon_r = \frac{1}{\omega_N r^N} \int_{B_r} (z(\overline{x}) - z)\, dx \to 0$$

as $r \to 0^+$. For every such $r$, let $\Omega_r$ be a polyhedral set such that $B_r \subset\subset \Omega_r \subset\subset \Omega$ and

$$\mathcal{L}^N(\Omega \setminus \Omega_r) \to 0, \qquad \max_{x \in \partial\Omega_r} d(x, \partial\Omega) \to 0$$

as $r \to 0^+$. Let now $v_r \in H_0^1(\Omega)$, $v_r \equiv 0$ in $\Omega \setminus \Omega_r$ be the unique solution of

$$\Delta v_r = \frac{1}{\omega_N r^N} \mathbf{1}_{B_r} + \sigma_r \mathbf{1}_{\Omega_r \setminus B_r} \ \text{ in } \Omega_r \tag{112}$$

where

$$\sigma_r = \frac{\epsilon_r}{\lambda(\Omega_r)\mathcal{L}^N(\Omega_r \setminus B_r)}.$$

It is readily seen that $v_r \leq 0$ in $\Omega$, that

$$\frac{\partial v_r}{\partial \mathbf{n}} = 0 \ \text{ in } \partial\Omega, \qquad \frac{\partial v_r}{\partial \mathbf{n_r}} \geq 0 \ \text{ in } \partial\Omega_r$$

(here $\mathbf{n_r}$ denotes the unit outer vector normal to $\partial\Omega_r$) and as in (43)

$$\begin{aligned}\int_\Omega |v_r|\, dx &= \lambda(\Omega_r) - \epsilon_r + \sigma_r \int_{\Omega_r \setminus B_r} (z - c_z)\, dx \\ &\geq \lambda(\Omega_r) - \frac{3}{2}\epsilon_r = \lambda(\Omega_r)|\Delta v_r|_\Omega - \frac{3}{2}\epsilon_r.\end{aligned} \tag{113}$$

Since $v_r \in H_0^1(\Omega)$ solves (112) in $\Omega_r$, then $v_r \in W^{2,p}(\Omega_r)$ for every $p > 1$ and therefore $v_r \in C^1(\overline{\Omega}_r)$. Since $\Omega_r$ is a polyhedral set then there exists a sequence of continuous piecewise affine functions $0 \geq v_r^h \in SBH(\Omega) \cap H_0^1(\Omega)$, $v_r^h \equiv 0$ in $\Omega \setminus \Omega_r$ such that $v_r^h \to v_r$ in $H_0^1(\Omega)$ and

$$\int_{\partial\Omega_r} \left| \frac{\partial v_r^h}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1} \to \int_{\partial\Omega_r} \left| \frac{\partial v_r}{\partial \mathbf{n}} \right| d\mathcal{H}^{N-1}. \tag{114}$$

It is readily seen that $\Delta v_r^h \to \Delta v_r$ strongly in $H^{-1}(\Omega)$ and since $\Delta v_r \geq \sigma_r$ a.e. in $\Omega_r$, by Lemma 3.2 we get

$$\Delta v_r^h \geq \frac{\sigma_r}{2} \mathcal{L}^N \llcorner \Omega_r$$

when $h$ is large enough, say $h \geq h_r$. Then

$$|\Delta v_r^h|_{\Omega_r} = \int_{\Omega_r} d(\Delta v_r^h) = \int_{\partial\Omega_r} \frac{\partial v_r^h}{\partial \mathbf{n_r}}\, d\mathcal{H}^{N-1} \tag{115}$$

for every $h \geq h_r$ and by recalling (114) we get

$$|\Delta v_r^h|_{\Omega_r} \to |\Delta v_r|_{\Omega_r}. \tag{116}$$

But

$$|\Delta v_r^h|_{\Omega} = |\Delta v_r^h|_{\Omega_r} + \int_{\partial\Omega_r} \left| \frac{\partial v_r^h}{\partial \mathbf{n_r}} \right| \, d\mathcal{H}^{N-1} \tag{117}$$

hence

$$|\Delta v_r^h|_{\Omega} \to |\Delta v_r|_{\Omega}. \tag{118}$$

Let now $\epsilon_r$ as in (43): by (43), (122), (115), (114), (116) for every $r \leq r_0/2$ there exists a continuous piecewise affine function $0 \geq w_r \in SBH(\Omega) \cap H_0^1(\Omega)$ null on $\Omega \setminus \Omega_r$ such that

$$\begin{gathered} \int_{\Omega} |w_r - v_r| \, dx \leq \frac{\epsilon_r}{6} \\ \big|\, |\Delta w_r|_{\Omega} - |\Delta v_r|_{\Omega} \,\big| \leq \frac{\epsilon_r}{6} \end{gathered} \tag{119}$$

hence

$$\int_{\Omega} w_r \, dx \leq 2\epsilon_r - \lambda(\Omega_r)|\Delta w_r|_{\Omega} = 2\epsilon_r - \lambda(\Omega_r) \int_{S_{\nabla w_r}} \left| \left[ \frac{\partial w_r}{\partial \nu} \right] \right| \, d\mathcal{H}^{N-1}. \tag{120}$$

Let $v_0 \in H_0^1(\Omega)$, $v_0 \equiv 0$ in $\Omega \setminus B_{r_0}$ such that

$$\Delta v_0 = \frac{1}{\omega_N r^N} \mathbf{1}_{B_r} \ \text{ in } B_{r_0} \tag{121}$$

then $v_r \leq v_0 \leq 0$ in $B_{r_0}$,

$$\int_{\Omega} v_r \, dx \leq \int_{B_{r_0} \setminus B_{r_0/2}} v_0 \, dx = I(r_0) < 0 \tag{122}$$

hence

$$\int_{\Omega} w_r \, dx \leq \frac{I(r_0)}{2} < 0 \tag{123}$$

when $r$ is small enough. Then, by choosing again $r$ small enough, Proposition 2.2 yields

$$2\gamma \left( \frac{1}{\lambda(\Omega_r)} - \frac{1}{\lambda(\Omega)} \right) < \frac{\delta}{2}$$

and by taking into account (120) and (123)

$$\begin{aligned} F(\frac{w_r}{\epsilon_r}) &\leq \frac{\delta\lambda(\Omega) + 4\gamma}{2\epsilon_r \lambda(\Omega_r)} \int_{\Omega} w_r \, dx + \frac{\gamma}{\epsilon_r} \int_{S_{\nabla w_r}} \left| \left[ \frac{\partial w_r}{\partial \nu} \right] \right| \, d\mathcal{H}^{N-1} \\ &\leq \frac{4\gamma}{\lambda(\Omega_r)} + \frac{\delta}{4\epsilon_r} I(r_0) \to -\infty \end{aligned} \tag{124}$$

when $r \to 0^+$, thus proving that $F$ is unbounded from below. □

**Remark 3.4.** It is worth noticing that condition (102) becomes less restrictive when $\lambda(\Omega)$ becomes smaller and smaller but it is readily seen that

$$\inf\{\lambda(\Omega) : |\Omega| = 1\} = 0.$$

Indeed it is enough to choose

$$\Omega_m = \left\{(x, y) : \frac{m^2}{\pi}x^2 + \frac{1}{m^2\pi}y^2 < 1\right\}$$

to obtain

$$\lambda(\Omega_m) = \left(m^2\pi + \frac{\pi}{m^2}\right)^{-1} \to 0$$

as $m \to \infty$: hence a domain $\Omega^*$ which minimizes $\lambda(\Omega)$ among all connected Lipschitz domains having the same $N$-dimensional Lebesgue measure does not exist. On the other hand, if for every $N$-uple $\xi = (a_1, a_2...., a_N)$ such that $p(\xi) \equiv \Pi_{i=1}^N a_i \neq 0$ we define

$$\Omega_\xi = \left\{x \in \mathbb{R}^N : \sum_{i=1}^N a_i^2 x_i^2 < 1\right\}$$

then $\lambda(\Omega_\xi) = \frac{1}{2}(\sum_{i=1}^N a_i^2)^{-1}$ and therefore

$$\max\{\lambda(\Omega_\xi) : \ p(\xi) = R^{-N}\} = \frac{R^2}{2N}$$

that is the maximum is attained when $a_i = R^{-1}$. In particular this means that when $N = 2$ the *circle* has the minimal resistance to collapse among all *ellipses* having the same area.

## A. Appendix

We show here how an optimal Poincaré inequality holds also for BH functions vanishing only on a subset of $\partial\Omega$ having positive Hausdorff measure. Unfortunately we are able to prove this result only when $N = 1$, since the general case seems very hard to tackle at least with the methods developed in this paper.

**Theorem A.1.** *Let $v \in BH(0, L)$ such that $v(L) = 0$ then*

$$\int_0^L |v|\, dx \leq \frac{L^2}{2}\left(|v''|_{(0,L)} + |v'_-(L)|\right) \tag{125}$$

*and equality holds in* (125) *if $v(x) = c(x - L), \ c \in \mathbb{R}$.*

**Proof.** Let $v \in BH(0, L)$, $v(L) = 0$ and let $z \in W^{2,\infty}(0, L)$ such that $z'' = \operatorname{sign} v$ in $(0, L)$, $z(0) = z'(0) = 0$. Since $|z''(x)| \leq 1$ a.e. in $(0, L)$, it is readily seen that

$$|z(x)| \leq \frac{1}{2}x^2 \leq \frac{L^2}{2}$$

and integrating by parts we easily get

$$\int_0^L v z''\, dx = \int_0^L z\, d(v'') - z(L) v'_-(L) \tag{126}$$

hence

$$\int_0^L |v|\, dx \le \frac{L^2}{2} \left( |v''|_{(0,L)} + |v'_-(L)| \right). \tag{127}$$

Optimality of the constant in (125) can be obviously proven by simply substituting $v(x) = c(x - L)$ in (125). □

**Acknowledgements.** I wish to thank my friend and collegue Prof. S. Solimini for helpful discussions on this subject.

## References

[1] R. A. Adams: Sobolev Spaces, Academic Press, New York (1975).

[2] L. Ambrosio, N. Fusco, D. Pallara: Functions of Bounded Variation and Free Discontinuity Problems, Oxford Math. Mon., Oxford University Press, Oxford (2000).

[3] M. Carriero, A. Leaci, F. Tomarelli: Plastic free discontinuities and special bounded hessian, C. R. Acad. Sci., Paris, Sér. I 314(8) (1992) 595–600.

[4] M. Carriero, A. Leaci, F. Tomarelli: Special bounded Hessian and elastic-plastic plate, Rend. Accad. Naz. Sci. XL, V. Ser., Mem. Mat. 16 (1992) 223–258.

[5] M. Carriero, A. Leaci, F. Tomarelli: Strong solution for an elastic-plastic late, Calc. Var. Partial Differ. Equ. 2(2) (1994) 219–240.

[6] M. Carriero, A. Leaci, F. Tomarelli: Second order variational problems with free discontinuity and free gradient discontinuity, in: Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi, D. Pallara (ed.), Quad. Mat. 14, Aracne, Rome (2004) 135–186.

[7] D. Percivale: Perfectly plastic plates: a variational definition, J. Reine Angew. Math. 411 (1990) 39–50.

[8] D. Percivale, F. Tomarelli: Scaled Korn-Poincaré inequality in BD and a model of elastic plastic cantilever, Asymptotic Anal. 23(3–4) (2000) 291–311.

[9] D. Percivale, F. Tomarelli: From SBD to SBH: the elastic plastic plate, Interfaces Free Bound. 4(2) (2002) 137–165.

[10] D. Percivale, F. Tomarelli: From special bonded deformation to special bounded Hessian: the elastic plastic beam, Math. Models Methods Appl. Sci. 15(7) (2005) 1009–1058.

[11] D. Percivale, F. Tomarelli: Smooth and creased equilibria for elastic-plastic plates and beams, in: Variational Problems in Material Science (Trieste, 2004), G. dal Maso et al. (ed.), Progr. Nonlinear Differential Equations Appl. 68, Birkhäuser, Basel (2006) 127–136.

[12] D. Percivale, F. Tomarelli: A variational principle for plastic hinges in a beam, Math. Models Methods Appl. Sci. 19(12) (2009) 1–34.

[13] D. Percivale, F. Tomarelli: Regular minimizers of free discontinuity problems, to appear.

[14] D. Gilbarg, N. S. Trudinger: Elliptic Partial Differential Equations of Second Order, Springer, Berlin (1983).