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On Existence and Asymptotic Stability of Solutions of a Functional-Integral Equation of Fractional Order

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We study the solvability of a quadratic functional-integral equation of fractional order. This equation is considered in the Banach space of real functions defined, bounded and continuous on an unbounded interval. Moreover, we will obtain some asymptotic characterization of solutions.

Keywords: Functional integral equation, fractional integral, existence, asymptotic behaviour, measure of noncompactness, Schauder fixed point principle

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1. Introduction

In this paper, we will study the quadratic functional-integral equation of fractional order

$$x(t) = f(t) + g\left(t, \frac{h(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}}\, ds\right), \quad t \in \mathbb{R}_+ \text{ and } \alpha \in (0, 1). \tag{1}$$

Throughout $f : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy special assumptions (see Section 3). Let us recall that the functions $g = g(t, v)$ and $h = h(t, x)$ involved in Eq. (1) generate the superposition operators G and H defined by

$$(Gv)(t) = g(t, v(t)) \tag{2}$$

and

$$(Hx)(t) = h(t, x(t)), \tag{3}$$

respectively, where $v = v(t)$ and $x = x(t)$ are arbitrary functions defined on $\mathbb{R}_+$ (see [1]).

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If $g(t, v) = v$ we obtain a quadratic Urysohn-Volterra integral equation of fractional order studied by Banaś and O'Regan in [11] while in the case $g(t, v) = v$ and $u(t, s, x) = w(t, x)$ we get a fractional quadratic integral equation of Hammerstein-Volterra type studied by Darwish in [22].

In the case $\alpha = 1$, $g(t, v) = v$, $h(t, x) = -x$ and $u(t, s, x) = k(t, s)\, x$, Eq. (1) becomes the Volterra counterpart of the following equation

$$x(t) + x(t) \int_0^t k(t, s)\, x(s)\, ds = f(t), \quad t \in \mathbb{R}_+. \tag{4}$$

Eq. (4) is the nonlinear particle transport equation when removal effects are dominant, where t is the particle speed, the known term $f(t)$ is the intensity of the external source and the unknown function $x(t)$ is related to the particle distribution function $y(t)$ by

$$x(t) = Q(t)\, y(t),$$

where, Q is the positive macroscopic removal collision frequency of the host medium. Finally, the kernel $k(t, s)$ is given by

$$k(t, s) = \frac{1}{2tQ(t)Q(s)} \int_{|t-s|}^{t+s} v\, q(v)\, dv,$$

where q is the macroscopic removal collision frequency by the particles between themselves, see [14, 15, 20, 38]. On the other hand, Eq. (4) is a generalization of a famous equation in the transport theory, the so-called Chandrasekhar $H-$equation in which t ranges from 0 to 1, $f(t) = 1$, x must be identified with the $H-$function, and

$$k(t, s) = -\frac{t\phi(s)}{t + s}$$

for a nonnegative characteristic function ϕ, see [21, 31, 33, 38].

Moreover, quadratic integral equations have many other useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in kinetic theory of gases, in the theory of neutron transport, and in the traffic theory, see [14, 15, 20, 28, 30, 31].

The theory of quadratic integral equations with nonsingular kernels has received a lot of attention. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels. For example, Argyros [2], Banaś et al. [5, 8, 10], Banaś and Martinon [9], Benchohra and Darwish [13], Caballero et al. [16, 17, 18, 19], Darwish [24], Leggett [33], Stuart [37] and Spiga et al. [38]. On the other hand, after the appearance of Darwish's paper [22] there is a great interest of the study of singular quadratic integral equations, see [7, 11, 12, 23, 25, 26, 27].

It is worthwhile mentioning that up to now only the two papers by Banaś and D. O'Regan [11] and by Darwish [27] concerning with the study of quadratic integral equation of singular kernel in the space of real functions defined, continuous and bounded on an *unbounded interval*.

The aim of this paper is to prove the existence of solutions of Eq. (1) in the space of real functions, defined, continuous and bounded on an unbounded interval. Moreover, we will obtain some asymptotic characterization of solutions of Eq. (1). Our proof depends on suitable combination of the technique of measures of noncompactness and the Schauder fixed point principle.

2. Notation and Auxiliary Facts

This section is devoted to collecting some definitions and results which will be needed further on. First we recall the definition of the Riemann-Liouville fractional integral, see [29, 32, 34, 35, 36] for more information.

Definition 2.1. Let $f \in L_1(a,b)$, $0 \le a < b < \infty$, and let $\alpha > 0$ be a real number. The Riemann-Liouville fractional integral of order α of the function $f(t)$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad a < t < b,$$

where $\Gamma(\alpha)$ denotes the gamma function.

Now, let $(E, \|.\|)$ be an infinite dimensional Banach space with zero element 0. Let $B(x,r)$ denote the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(0,r)$.

If X is a subset of E, then $\overline{X}$ and $\operatorname{Conv} X$ denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the definition of the concept of a measure of noncompactness [3]:

Definition 2.2. A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.
2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
3) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.
4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $0 \le \lambda \le 1$.
5) If $X_n \in \mathcal{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \ne \phi$.

The family $\ker \mu$ described above is called the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from 5) belongs to $\ker \mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for every n then we have that $\mu(X_\infty) = 0$.

In what follows we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. This space is equipped with the standard norm

$$\| x \| = \sup\{|x(t)| : t \ge 0\}.$$

Now, we recollect the construction of the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be used in the next section (see [4, 6]).

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$ let us denoted by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in [0, T],\ |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon)$$

and

$$\omega_0^\infty(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Moreover, for a fixed number $t \in \mathbb{R}_+$ let us define

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x,\ y \in X\}.$$

Finally, let us define the function μ on the family $\mathcal{M}_{BC(\mathbb{R}_+)}$ by

$$\mu(X) = \omega_0^\infty(X) + c(X), \tag{5}$$

where $c(X) = \limsup_{t \to \infty} \operatorname{diam} X(t)$. The function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$, see [4].

Let us mention that the kernel $\ker \mu$ of the measure μ consists of all sets $X \in \mathcal{M}_{BC(\mathbb{R}_+)}$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions belonging to the set X tends to zero at infinity.

For further purposes we recall the definition of the concept of the asymptotic stability which will be used in our considerations.
To this end assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$. Let $Q : \Omega \to BC(\mathbb{R}_+)$ be a given operator. Consider the following operator equation

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+.$$

Definition 2.3. We say that solutions of the above equation are asymptotically stable if there exists a ball $B(x_0, r)$ such that $\Omega \cap B(x_0, r) \neq \emptyset$ and such that for each $\varepsilon > 0$ there exists $T > 0$ such that for arbitrary solutions $x = x(t),\ y = y(t)$ of this equation belonging to $\Omega \cap B(x_0, r)$ the inequality $|x(t) - y(t)| \leq \varepsilon$ is satisfied for any $t \geq T$.

3. Main Theorem

In this section we will study Eq. (1) assuming that the following hypotheses are satisfied:

(h_1) $f : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function on $\mathbb{R}_+$.

(h_2) $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to g(t, 0)$ is bounded on $\mathbb{R}_+$ with $g^* = \sup\{|g(t, 0)| : t \in \mathbb{R}_+\}$. Moreover, there exists a continuous function $m(t) = m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, x) - g(t, y)| \leq m(t)|x - y|,$$

for all $x,\ y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_3) $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $n(t) = n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t, x) - h(t, y)| \leq n(t)|x - y|,$$

for all $x,\ y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_4) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, there exist a function $l(t) = l : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ and such that

$$|u(t, s, x) - u(t, s, y)| \leq l(t)\Phi(|x - y|)$$

for all $t,\ s \in \mathbb{R}_+$ such that $t \geq s$ and for all $x \in \mathbb{R}$.
For further purpose let us define the function $u^* : \mathbb{R}_+ \to \mathbb{R}_+$ by $u^*(t) = \max\{|u(t, s, 0)| : 0 \leq s \leq t\}$.

(h_5) The functions $\phi,\ \psi,\ \xi,\ \eta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\phi(t) = m(t)n(t)l(t)t^\alpha$, $\psi(t) = m(t)n(t)u^*(t)t^\alpha$, $\xi(t) = m(t)l(t)|h(t, 0)|t^\alpha$ and $\eta(t) = m(t)u^*(t)|h(t, 0)|t^\alpha$ are bounded on $\mathbb{R}_+$ and the functions ϕ and ξ vanish at infinity, i.e. $\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \xi(t) = 0$.

(h_6) There exists a positive solution r_0 of the inequality

$$(\|f\| + g^*)\Gamma(\alpha + 1) + [\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*] \leq r\,\Gamma(\alpha + 1), \tag{6}$$

and $\phi^*\Phi(r_0) + \psi^* < \Gamma(\alpha + 1)$, where $\phi^* = \sup\{\phi(t) : t \in \mathbb{R}_+\}$, $\psi^* = \sup\{\psi(t) : t \in \mathbb{R}_+\}$, $\xi^* = \sup\{\xi(t) : t \in \mathbb{R}_+\}$ and $\eta^* = \sup\{\eta(t) : t \in \mathbb{R}_+\}$.

Now, we are in a position to state and prove our main result.

Theorem 3.1. Let the hypotheses $(h_1) - (h_6)$ be satisfied. Then Eq. (1) has at least one solution $x \in BC(\mathbb{R}_+)$ and all solutions of this equation belonging to the ball B_{r_0} are asymptotically stable.

Proof. Denote by $\mathcal{F}$ the operator associated with the right-hand side of equation (1), i.e., Equation (1) takes the form

$$x = \mathcal{F}x,$$

where

$$\mathcal{F}x = f + G\mathcal{H}x, \tag{7}$$

$$(\mathcal{H}x)(t) = (Hx)(t) \cdot (\mathcal{U}x)(t) \tag{8}$$

and

$$(\mathcal{U}x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t - s)^{1-\alpha}}\, ds, \quad t \in \mathbb{R}_+. \tag{9}$$

Solving Eq. (1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $BC(\mathbb{R}_+)$.

We claim that for any function $x \in BC(\mathbb{R}_+)$ the operator $\mathcal{F}x$ is continuous on $\mathbb{R}_+$. To establish this claim it suffices to show that if $x \in BC(\mathbb{R}_+)$ then $\mathcal{U}x$ is continuous function on $\mathbb{R}_+$, thanks (h_1), (h_2) and (h_3). In fact, take an arbitrary $x \in BC(\mathbb{R}_+)$ and fix $\varepsilon > 0$ and $T > 0$. Assume that $t_1,\ t_2 \in \mathbb{R}_+$ are such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we get

$$\begin{aligned}
&|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
= & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\,ds\right| \\
\leq & \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\,ds\right| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|u(t_2,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(t_2-s)^{1-\alpha}}\,ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |u(t_1,s,x(s))|[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds.
\end{aligned}$$

Therefore, if we denote

$$\begin{aligned}
\omega_d^T(u,\varepsilon) = \ & \sup\{|u(t_2,s,y) - u(t_1,s,y)| :\ s,\ t_1,\ t_2 \in [0,T], \\
& t_1 \geq s,\ t_2 \geq s,\ |t_2 - t_1| \leq \varepsilon,\ \text{and } y \in [-d,d]\},
\end{aligned}$$

then we obtain

$$\begin{aligned}
&|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{|u(t_2,s,x(s)) - u(t_2,s,0)| + |u(t_2,s,0)|}{(t_2-s)^{1-\alpha}}\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{\omega_{\|x\|}^T(u,\varepsilon)}{(t_2-s)^{1-\alpha}}\,ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} [|u(t_1,s,x(s)) - u(t_1,s,0)| + |u(t_1,s,0)|] \\
& \times[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{l(t_2)\Phi(|x(s)|) + u^*(t_2)}{(t_2-s)^{1-\alpha}}\,ds + \frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)}[t_2^\alpha - (t_2-t_1)^\alpha] \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} [l(t_1)\Phi(|x(s)|) + u^*(t_1)] \times [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
\leq & \frac{l(t_2)\Phi(\|x\|) + u^*(t_2)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha + \frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)}t_2^\alpha \\
& + \frac{l(t_1)\Phi(\|x\|) + u^*(t_1)}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha].
\end{aligned}$$

Thus

$$\omega^T(\mathcal{U}x,\varepsilon) \leq \frac{1}{\Gamma(\alpha+1)}\left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha\omega_{\|x\|}^T(u,\varepsilon)\right\}, \tag{10}$$

where

$$\hat{l}(T) = \max\{l(t) : t \in [0,T]\}$$

and

$$\hat{u}(T) = \max\{u^*(t) : t \in [0,T]\}.$$

In view of the uniform continuity of the function u on $[0,T] \times [0,T] \times [-\|x\|, \|x\|]$ we have that $\omega^T_{\|x\|}(u,\varepsilon) \to 0$ as $\varepsilon \to 0$. From the above inequality we infer that the function $\mathcal{U}x$ is continuous on the interval $[0,T]$ for any $T > 0$. This yields the continuity of $\mathcal{U}x$ on $\mathbb{R}_+$ and, consequently, the function $\mathcal{F}x$ is continuous on $\mathbb{R}_+$.

Now, we show that $\mathcal{F}x$ is bounded on $\mathbb{R}_+$. Indeed, in view of our hypotheses, for arbitrary $x \in BC(\mathbb{R}_+)$ and for a fixed $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|(\mathcal{F}x)(t)| &\leq \left| f(t) + g\left(t, \frac{h(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right)\right| \\
&\leq \|f\| + \frac{m(t)}{\Gamma(\alpha)}[|h(t,x(t)) - h(t,0)| + |h(t,0)|] \\
&\quad \times \int_0^t \frac{|u(t,s,x(s)) - u(t,s,0)| + |u(t,s,0)|}{(t-s)^{1-\alpha}}\, ds + |g(t,0)| \\
&\leq \|f\| + g^* + \frac{m(t)[n(t)\|x\| + |h(t,0)|}{\Gamma(\alpha)} \int_0^t \frac{l(t)\Phi(|x(s)|) + u^*(t)}{(t-s)^{1-\alpha}}\, ds \\
&\leq \|f\| + g^* + \frac{m(t)[n(t)\|x\| + |h(t,0)|]}{\Gamma(\alpha+1)}\, [l(t)\ \Phi(\|x\|) + u^*(t)]\ t^\alpha \\
&= \|f\| + g^* + \frac{1}{\Gamma(\alpha+1)}[\phi(t)\|x\|\Phi(\|x\|) + \psi(t)\|x\| + \xi(t)\Phi(\|x\|) + \eta(t)].
\end{aligned}$$

Hence, $\mathcal{F}x$ is bounded on $\mathbb{R}_+$, thanks hypothesis (h_5). This assertion in conjunction with the continuity of $\mathcal{F}x$ on $\mathbb{R}_+$ allows us to conclude that the operator $\mathcal{F}$ maps $BC(\mathbb{R}_+)$ into itself. Moreover, from the last estimate we have

$$\|\mathcal{F}x\| \leq \|f\| + g^* + \frac{1}{\Gamma(\alpha+1)}\, [\phi^*\|x\|\Phi(\|x\|) + \psi^*\|x\| + \xi^*\Phi(\|x\|) + \eta^*].$$

Linking this estimate with hypothesis (h_6) we deduce that there exists $r_0 > 0$ such that the operator $\mathcal{F}$ transforms the ball B_{r_0} into itself.

In what follows let us take a nonempty set $X \subset B_{r_0}$. Then, for arbitrary $x,\ y \in X$ and for a fixed $t \in \mathbb{R}_+$, we obtain

$$
\begin{aligned}
& |(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
\leq\ & \frac{m(t)}{\Gamma(\alpha)} \left| h(t, x(t)) \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}}\, ds - h(t, y(t)) \int_0^t \frac{u(t, s, y(s))}{(t-s)^{1-\alpha}}\, ds \right| \\
\leq\ & \frac{m(t)|h(t, x(t)) - h(t, y(t))|}{\Gamma(\alpha)} \int_0^t \frac{|u(t, s, x(s))|}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)|h(t, y(t))|}{\Gamma(\alpha)} \int_0^t \frac{|u(t, s, x(s)) - u(t, s, y(s))|}{(t-s)^{1-\alpha}} ds \\
\leq\ & \frac{m(t)n(t)|x(t) - y(t)|}{\Gamma(\alpha)} \int_0^t \frac{|u(t, s, x(s)) - u(t, s, 0)| + |u(t, s, 0)|}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)[n(t)|y(t)| + |h(t, 0)|]}{\Gamma(\alpha)} \int_0^t \frac{l(t)\ \Phi(|x(s) - y(s)|)}{(t-s)^{1-\alpha}} ds \\
\leq\ & \frac{m(t)n(t)|x(t) - y(t)|}{\Gamma(\alpha)} \int_0^t \frac{l(t)\ \Phi(|x(s)|) + u^*(t)}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)[n(t)|y(t)| + |h(t, 0)|]}{\Gamma(\alpha)} \int_0^t \frac{l(t)\ \Phi(|x(s)| + |y(s)|)}{(t-s)^{1-\alpha}} ds \\
\leq\ & \frac{m(t)n(t)l(t)(|x(t)| + |y(t)|)}{\Gamma(\alpha)} \int_0^t \frac{\Phi(|x(s)|)}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)n(t)u^*(t)|x(t) - y(t)|}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
& + \frac{m(t)n(t)l(t)|y(t)|}{\Gamma(\alpha)} \int_0^t \frac{\Phi(|x(s)| + |y(s)|)}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)l(t)|h(t, 0)|}{\Gamma(\alpha)} \int_0^t \frac{\Phi(|x(s)| + |y(s)|)}{(t-s)^{1-\alpha}}\, ds \\
\leq\ & \frac{2m(t)n(t)l(t)r_0\Phi(r_0)}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{m(t)n(t)u^*(t)\operatorname{diam} X(t)}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
& + \frac{m(t)n(t)l(t)r_0\Phi(2r_0)}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} + \frac{m(t)l(t)|h(t, 0)|\Phi(2r_0)}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
\leq\ & \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\alpha+1)} + \frac{\psi(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t) + \frac{\phi(t)r_0\Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{\xi(t)\Phi(2r_0)}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Hence, we can easily deduce the following inequality

$$\operatorname{diam}(\mathcal{F}X)(t) \leq \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\alpha+1)} + \frac{\psi(t)}{\Gamma(\alpha+1)} \operatorname{diam} X(t) + \frac{\phi(t)r_0\Phi(2r_0)}{\Gamma(\alpha+1)} + \frac{\xi(t)\Phi(2r_0)}{\Gamma(\alpha+1)}.$$

Now, taking into account hypothesis (h_5), we obtain

$$c(\mathcal{F}X) \leq k\, c(X), \tag{11}$$

where $k = \frac{\phi^*\Phi(r_0)+\psi^*}{\Gamma(\alpha+1)} \geq \frac{\psi^*}{\Gamma(\alpha+1)}$. Obviously, in view of hypothesis (h_6) we have that $k < 1$.

In what follows, let us take arbitrary numbers $\varepsilon > 0$ and $T > 0$. Choose a function $x \in X$ and take $t_1,\ t_2 \in [0,T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then, taking into account our hypotheses and (10), we have

$$
\begin{aligned}
& |(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \\
\leq\ & |f(t_2) - f(t_1)| + m(t_2)\,|(Hx)(t_2)(\mathcal{U}x)(t_2) - (Hx)(t_1)(\mathcal{U}x)(t_2)| \\
& \quad + m(t_2)\,|(Hx)(t_1)(\mathcal{U}x)(t_2) - (Hx)(t_1)(\mathcal{U}x)(t_1)| \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))| \\
\leq\ & \omega^T(f,\varepsilon) + \frac{m(t_2)|h(t_2,x(t_2)) - h(t_1,x(t_1))|}{\Gamma(\alpha)} \\
& \quad \times \int_0^{t_2} \frac{|u(t_2,s,x(s)) - u(t_2,s,0)| + |u(t_2,s,0)|}{(t_2-s)^{1-\alpha}}\,ds \\
& \quad + \frac{m(t_2)[|h(t_1,x(t_1)) - h(t_1,0)| + |h(t_1,0)|]}{\Gamma(\alpha+1)} \\
& \qquad \left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha \omega^T_{\|x\|}(u,\varepsilon)\right\} \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))| \\
\leq\ & \omega^T(f,\varepsilon) + \frac{m(t_2)[n(t_2)|x(t_2) - x(t_1)| + \omega^T_h(\varepsilon)]}{\Gamma(\alpha)} \int_0^{t_2} \frac{l(t_2)\Phi(|x(s)|) + u^*(t_2)}{(t_2-s)^{1-\alpha}}\,ds \\
& \quad + \frac{m(t_2)[n(t_1)|x(t_1)| + |h(t_1,0)|]}{\Gamma(\alpha+1)} \left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha \omega^T_{\|x\|}(u,\varepsilon)\right\} \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))| \\
\leq\ & \omega^T(f,\varepsilon) + \frac{t_2^\alpha}{\Gamma(\alpha+1)} m(t_2)[n(t_2)\omega^T(x,\varepsilon) + \omega^T_h(\varepsilon)][l(t_2)\Phi(r_0) + u^*(t_2)] \\
& \quad + \frac{\hat{m}(T)[n(t_1)r_0 + \hat{h}(T)]}{\Gamma(\alpha+1)} \left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha \omega^T_{\|x\|}(u,\varepsilon)\right\} \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))| \\
\leq\ & \omega^T(f,\varepsilon) + \frac{[\phi(t_2)\Phi(r_0) + \psi(t_2)]}{\Gamma(\alpha+1)} \omega^T(x,\varepsilon) + \frac{T^\alpha\ \omega^T_h(\varepsilon)}{\Gamma(\alpha+1)} \hat{m}(T)[\hat{l}(T)\Phi(r_0) + \hat{u}(T)] \\
& \quad + \frac{\hat{m}(T)[\hat{n}(T)r_0 + \hat{h}(T)]}{\Gamma(\alpha+1)} \left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha \omega^T_{\|x\|}(u,\varepsilon)\right\} \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))| \\
\leq\ & \omega^T(f,\varepsilon) + \frac{[\phi^*\Phi(r_0) + \psi^*]}{\Gamma(\alpha+1)} \omega^T(x,\varepsilon) + \frac{T^\alpha\ \omega^T_h(\varepsilon)}{\Gamma(\alpha+1)} \hat{m}(T)[\hat{l}(T)\Phi(r_0) + \hat{u}(T)] \\
& \quad + \frac{\hat{m}(T)[\hat{n}(T)r_0 + \hat{h}(T)]}{\Gamma(\alpha+1)} \left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)] + T^\alpha \omega^T_{\|x\|}(u,\varepsilon)\right\} \\
& \quad + |g\,(t_2, (Hx)(t_1)(\mathcal{U}x)(t_1)) - g\,(t_1, (Hx)(t_1)(\mathcal{U}x)(t_1))|,
\end{aligned}
$$

where

$$\omega^T_h(\varepsilon) = \sup\{|h(t_2,x) - h(t_1,x)| : t_1,\ t_2 \in [0,T],\ |t_2 - t_1| \leq \varepsilon,\ x \in [-r_0, r_0]\},$$

$$\hat{n}(T) = \max\{n(t) : t \in [0,T]\},$$
$$\hat{m}(T) = \max\{m(t) : t \in [0,T]\}$$

and

$$\hat{h}(T) = \max\{|h(t,0)| : t \in [0,T]\}.$$

Hence,

$$\begin{aligned}
&\omega^T(\mathcal{F}x, \varepsilon) \\
\leq\ & \omega^T(f,\varepsilon) + \frac{[\phi^*\Phi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon) + \frac{T^\alpha\, \omega_h^T(\varepsilon)}{\Gamma(\alpha+1)}\hat{m}(T)[\hat{l}(T)\Phi(r_0)+\hat{u}(T)] \\
&+\frac{\hat{m}(T)[\hat{n}(T)r_0+\hat{h}(T)]}{\Gamma(\alpha+1)}\left\{2\varepsilon^\alpha[\hat{l}(T)\Phi(\|x\|)+\hat{u}(T)]+T^\alpha\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&+\sup\left\{|g\left(t_2,(Hx)(t_1)(\mathcal{U}x)(t_1)\right)\right. \\
&\qquad \left.-g\left(t_1,(Hx)(t_1)(\mathcal{U}x)(t_1)\right)| : t_1,\ t_2 \in [0,T],\ \|x\| \leq r\right\}.
\end{aligned}$$

Since the function $g(t,y)$ is uniformly continuous on the set $[0,T]\times[-N,N]$, the function $h(t,x)$ is uniformly continuous on the set $[0,T]\times[-r_0,r_0]$ and the function $u(t,s,x)$ is uniformly continuous on the set $[0,T]\times[0,T]\times[-r_0,r_0]$, where

$$N = \sup\left\{\int_0^{t_1}\frac{|u(t_1,s,x(s))|}{(t_1-s)^{1-\alpha}}\,ds : t_1\in[0,T],\ \|x\|\leq r_0\right\},$$

(Obviously, $N<\infty$ because $u(t,s,x)$ is bounded on $[0,T]\times[0,T]\times[-r_0,r_0]$ and $\int_0^{t_1}\frac{1}{(t_1-s)^{1-\alpha}}ds \leq \frac{T^\alpha}{\alpha}$), we have

$$\sup\{|g(t_2,y)-g(t_1,y)| : t_1,\ t_2\in[0,T],\ |t_2-t_1|\leq\varepsilon,\ |y|\leq N\}\to 0 \ \text{ as } \varepsilon\to 0.$$

Therefore, from the last estimate we derive the following one

$$\omega_0^T(\mathcal{F}X) \leq k\,\omega_0^T(X).$$

Hence we have

$$\omega_0^\infty(\mathcal{F}X) \leq k\,\omega_0^\infty(X). \tag{12}$$

From (11) and (12) and the definition of the measure of noncompactness μ given by formula (5), we obtain

$$\mu(\mathcal{F}X) \leq k\,\mu(X). \tag{13}$$

In the sequel let us put $B^1_{r_0} = \operatorname{Conv}\mathcal{F}(B_{r_0})$, $B^2_{r_0} = \operatorname{Conv}\mathcal{F}(B^1_{r_0})$ and so on. In this way we have constructed a decreasing sequence of nonempty, bounded, closed and convex subsets $(B^n_{r_0})$ of B_{r_0} such that $\mathcal{F}(B^n_{r_0})\subset B^n_{r_0}$ for $n=1,\ 2,\ldots$. Since the above reasons leading to (13) holds for any subset X of B_{r_0} we have

$$\mu(B^n_{r_0}) \leq k^n\mu(B_{r_0}), \ \text{ for any } n=1,\ 2,\ 3,\ldots.$$

This implies that $\lim_{n\to\infty}\mu(B^n_{r_0}) = 0$. Hence, taking into account Definition 2.2 we infer that the set $Y=\bigcap_{n=1}^\infty B^n_{r_0}$ is nonempty, bounded, closed and convex subset of B_{r_0}. Moreover, $Y\in\ker\mu$. Also, the operator $\mathcal{F}$ maps Y into itself.

We will prove that the operator $\mathcal{F}$ is continuous on the set Y. In order to do this let us fix a number $\varepsilon > 0$ and take arbitrary functions $x,\ y \in Y$ such that $\|x-y\| \leq \varepsilon$. Keeping in mind the facts that $Y \in \ker\mu$ and the structure of sets belong to $\ker\mu$ we can find a number $T > 0$ such that for each $z \in Y$ and $t \geq T$ we have that $|z(t)| \leq \varepsilon$. Since $\mathcal{F}$ maps Y into itself we have that $\mathcal{F}x,\ \mathcal{F}y \in Y$. Thus, for $t \geq T$ we get

$$|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \leq |(\mathcal{F}x)(t)| + |(\mathcal{F}y)(t)| \leq 2\varepsilon. \tag{14}$$

On the other hand, let us assume $t \in [0, T]$. Then we obtain

$$\begin{aligned}
& |(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
\leq\ & \frac{m(t)n(t)|x(t)-y(t)|}{\Gamma(\alpha)} \int_0^t \frac{l(t)\ \Phi(|x(s)|) + u^*(t)}{(t-s)^{1-\alpha}}\, ds \\
& + \frac{m(t)[n(t)|y(t)| + |h(t,0)|]}{\Gamma(\alpha)} \int_0^t \frac{l(t)\ \Phi(|x(s)-y(s)|)}{(t-s)^{1-\alpha}} ds \\
\leq\ & \frac{[m(t)n(t)l(t)\Phi(r_0) + m(t)n(t)u^*(t)]\varepsilon}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
& + \frac{[m(t)n(t)l(t)r_0 + m(t)l(t)|h(t,0)|]\Phi(\varepsilon)}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
\leq\ & \frac{\phi(t)\Phi(r_0) + \psi(t)}{\Gamma(\alpha+1)}\, \varepsilon + \frac{\phi(t)r_0 + \xi(t)}{\Gamma(\alpha+1)} \Phi(\varepsilon) \\
\leq\ & \frac{\phi^*\Phi(r_0) + \psi^*}{\Gamma(\alpha+1)}\, \varepsilon + \frac{\phi^* r_0 + \xi^*}{\Gamma(\alpha+1)} \Phi(\varepsilon).
\end{aligned} \tag{15}$$

Now, taking into account (14), (15) and hypothesis (h_5) we conclude that the operator $\mathcal{F}$ is continuous on the set Y.
Finally, linking all above obtained facts about the set Y and the operator $\mathcal{F} : Y \to Y$ and using the classical Schauder fixed point principle we deduce that the operator $\mathcal{F}$ has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of the quadratic integral equation (1). Moreover, since $B_{r_0} \subset \ker\mu$ we have that all solutions of Eq. (1) belonging to B_{r_0} are asymptotically stable.
This completes the proof.

In what follows we present an example where our Theorem 3.1 can be applied.

Consider the following integral equation of fractional order

$$x(t) = e^{-t} + \frac{1}{1+t^2} + \sin\left[\frac{1}{t^3+1} \cdot \frac{\arctan(x(t))}{\Gamma(\frac{1}{2})} \int_0^t \frac{\sqrt{1+\lambda|x(s)|}}{(t-s)^{\frac{1}{2}}} ds\right],$$

where $t \in \mathbb{R}_+$, and $\lambda > 0$.

Observe that the above equation is a special case of Eq. (1). Indeed, if we put $\alpha = \frac{1}{2}$ and

$$f(t) = e^{-t},$$
$$g(t,x) = \frac{1}{1+t^2} + \sin\left(\frac{1}{t^3+1} \cdot x\right),$$
$$h(t,x) = \arctan x,$$
$$u(t,s,x) = \sqrt{1+\lambda|x|},$$

then we can easily check that the assumptions of Theorem 3.1 are satisfied for any $0 < \lambda \le 0,02$. In fact, we have that the function $f(t) = e^{-t}$ is continuous and bounded on $\mathbb{R}_+$. Moreover, $\|f\| = 1$.

Observe that the function $g(t,x) = \frac{1}{1+t^2} + \sin\left(\frac{1}{t^3+1} \cdot x\right)$ satisfies assumption (h_2) with $m(t) = \frac{1}{t^3+1}$ and $|g(t,0)| = g(t,0) = \frac{1}{1+t^2}$, being $g^* = 1$.

Next, let us notice that the function $h(t,x) = \arctan x$ satisfies assumption (h_3) with $n(t) = 1$. The function $u(t,s,x) = \sqrt{1+\lambda|x|}$ satisfies assumption (h_4) with $l(t) = 1$, $\Phi(r) = \frac{\lambda r}{2}$ and $u(t,s,0) = 1$, being $u^* = 1$.

To check that assumption (h_5) is satisfied let us observe that the functions ϕ,ψ, ξ, η appearing in that assumption take the form

$$\phi(t) = \frac{t^{\frac{1}{2}}}{t^3+1}; \qquad \psi(t) = \frac{t^{\frac{1}{2}}}{t^3+1}; \qquad \xi(t) = 0; \qquad \eta(t) = 0,$$

and it is easily seen that

$$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \psi(t) = 0.$$

Moreover it is easily seen that $\phi^* = \psi^* \cong 0.637$ and, as $\Gamma(\frac{3}{2}) \cong 0.886$, the inequality in assumption (h_6) has the form

$$2 \cdot 0.886 + \left(\frac{0.637\lambda}{2} \cdot r^2 + 0.637 \cdot r\right) \le 0.886 \cdot r.$$

We can easily check that the above inequality is satisfied for any $r_0 \in (15.6, 17.8)$. Moreover, taking $\lambda \le 0.02$ and $r_0 \in (15.6, 17.8)$ it can be proved that $\phi^*\Phi(r_0) + \psi^* < \Gamma(\alpha+1)$. Now, by Theorem 3.1, we infer that our equation has a solution in $B_{r_0} \subset BC(\mathbb{R}_+)$ with $r_0 \in (15.6, 17.8)$ and the solutions in B_{r_0} are asymptotically stable.

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