Journal of Convex Analysis
Volume 17 (2010), No. 3&4, 885–913

Second-Order Analysis of Optimal Control Problems with Control and Initial-Final State Constraints

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Dedicated to Hedy Attouch on the occasion of his 60th birthday.

Received: October 25, 2008
Revised manuscript received: July 1, 2009

This paper provides an analysis of Pontryagin minima satisfying a quadratic growth condition, for optimal control problems of ordinary differential equations with constraints on initial-final state, as well as control constraints satisfying the uniform positive linear independence condition.

1. Introduction

In this paper we discuss necessary or sufficient conditions for weak or bounded strong minima of optimal control problems with control constraints and constraint on the initial-final state. There is already an important literature on this subject.

Osmolovskiĭ [10, 11, 12] analyzed second-order optimality conditions for such problems assuming linear independence of gradients of active constraints (LIG); see also Levitin, Milyutin and Osmolovskiĭ [6, p. 155–156] where these conditions were first formulated.

Malanowski [7] obtained stability and sensitivity results in the case of convex cost and constraints (including state constraints) assuming the LIG hypothesis. More recently, Hermant and the first author [1] studied similar problems, without convexity assumption except for the (local) dependence of the Hamiltonian w.r.t. the control variable, and again with the LIG hypothesis.

The main novelty is that we do not assume any more the LIG hypothesis but instead a qualification condition that implies the uniform positive linear independence of gradients of active inequality constraints. Also, we do not assume the (local) convex

*This author was supported by grants RFBR 08-01-00685 and NSh-3233.2008.1.

ISSN 0944-6532 / $ 2.50

dependence of the Hamiltonian w.r.t. the control variable, which makes the discussion of sufficient conditions more complex (since the Hessian of the Lagrangian of the problem is not a Legendre form, and so it is no more possible to pass to the limit in weakly convergent directions).

The paper is organized as follows. Section 2 sets the problem, recalls some basic concepts, and gives a decomposition principle, in the setting of abstract control constraints. Section 3 analyzes the multipliers associated with control constraints parameterized by finitely many inequalities. In Section 4, we give conditions under which weak minima that satisfy a strong version of Pontryagin's principle are bounded strong minima. Finally in Section 5 we characterize the quadratic growth condition for weak minima.

2. Pontryagin minima

2.1. Pontryagin's principle

For a given horizon $T > 0$, let $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$ and $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$ denote the control and state space. Set $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$. When needed we denote $w = (u, y)$, $\bar{w} = (\bar{u}, \bar{y})$, etc. the elements of $\mathcal{W}$. Similarly we denote when needed $\eta = (y(0), y(T))$, $\bar{\eta} = (\bar{y}(0), \bar{y}(T))$, etc. the pair of initial-final states. The cost function is defined by

$$J(w) := \int_0^T \ell(u(t), y(t))\mathrm{d}t + \phi(\eta), \tag{1}$$

where $\ell : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ (*running cost*) and $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (*initial-final cost*) are twice continuously differentiable (C^2) mappings. Consider the state equation

$$\dot{y}(t) = f(u(t), y(t)) \quad \text{for a.a. } t \in [0, T]; \tag{2}$$

where $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz and C^2 mapping. We know that the state equation (2) has for any $u \in \mathcal{U}$ and given initial condition $y(0) = y_0$ a unique solution denoted $y_{u,y_0} \in \mathcal{Y}$. We consider problems having both control constraints

$$u(t) \in U, \quad \text{for a.a. } t \in (0, T), \tag{3}$$

where U is a closed subset of $\mathbb{R}^m$, and initial-final state constraints of the form

$$\Phi(\eta) \in K, \quad \text{with } K := \{0\}_{\mathbb{R}^{r_1}} \times \mathbb{R}^{r_2}_-, \tag{4}$$

and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^r$, $r = r_1 + r_2$, r_1 and r_2 are nonnegative integers. In other words, there is a finite number of equality and inequality constraints on the initial-final state:

$$\Phi_i(\eta) = 0, \quad i = 1, \ldots, r_1, \qquad \Phi_i(\eta) \le 0, \quad i = r_1 + 1, \ldots, r. \tag{5}$$

Consider the following optimal control problem:

$$\operatorname*{Min}_{w \in \mathcal{W}} J(w) \quad \text{subject to (2)–(4).} \tag{P}$$

We call *trajectory* an element w of $\mathcal{W}$ that satisfies the state equation (2). If in addition the constraints (3)–(4) hold, we say that w is a *feasible point* of problem (P); the set of feasible points is denoted by $F(P)$.

We briefly recall several concepts of solution. Denote by $\|\cdot\|_s$ the norm of the space $L^s(0,T,\mathbb{R}^q)$ for any q, and $s \in [1,+\infty]$. A *weak (resp. strong) solution* of (P) (or *weak, strong minimum*) is an element $\bar{w} \in F(P)$ such that $J(\bar{w}) \leq J(w)$ for all $w \in F(P)$ such that $\|w-\bar{w}\|_{\mathcal{W}}$ (resp. $\|y-\bar{y}\|_\infty$) is small enough. Equivalently, $\bar{w} \in F(P)$ is a weak (resp. strong) solution of (P) if, for any sequence $w_k \in F(P)$, such that $w_k \to \bar{w}$ in $\mathcal{W}$ (resp. $y_k \to \bar{y}$ uniformly), we have that $J(\bar{w}) \leq J(w_k)$ for large enough k.

Following [6, p. 156] and [9, p. 291], we say that $\bar{w} \in F(P)$ is a *bounded strong solution (minimum)* if for any *bounded* sequence $w_k \in F(P)$, such that $y_k \to \bar{y}$ uniformly, we have that $J(\bar{w}) \leq J(w_k)$ when k is large enough. An element $\bar{w}$ of $F(P)$ is called *Pontryagin solution (minimum)* see [6, p. 156] and [9, p. 2–3], if for any sequence $w_k \in F(P)$, bounded in $\mathcal{W}$, such that $y_k \to \bar{y}$ uniformly and $\|u_k - \bar{u}\|_1 \to 0$, we have that $J(\bar{w}) \leq J(w_k)$ when k is large enough.

Equivalently, $\bar{w}$ is a bounded strong (Pontryagin) solution if for any $M > 0$, there exists $\varepsilon_M > 0$ such that if $w \in F(P)$ is such that $\|u\|_\infty \leq M$, $\|y-\bar{y}\|_\infty \leq \varepsilon_M$ (and in addition $\|u-\bar{u}\|_1 \leq \varepsilon_M$ in the case of a Pontryagin solution) we have that $J(\bar{w}) \leq J(w)$.

Obviously any of the following concepts is implied by the previous one: strong, bounded strong, Pontryagin, weak solution. We define a strong, bounded strong, Pontryagin, weak *perturbation* of $\bar{w} \in F(P)$ as a sequence w_k of trajectories in $\mathcal{W}$ associated with the corresponding optimality concept, i.e., such that $y_k \to \bar{y}$ uniformly for strong perturbation, and in addition u_k is bounded in L^∞ for the other types of perturbations, $u_k \to u$ in L^1 (uniformly) for a Pontryagin (weak) perturbation. We say that the perturbation is *feasible* if the elements of the sequence belong to $F(P)$. We call $\delta w_k := w_k - \bar{w}$ a strong, bounded strong, Pontryagin, weak *variation*.

We say that a (strong, bounded strong, Pontryagin, weak) solution $\bar{w}$ satisfies the *quadratic growth condition* if there exists $\alpha > 0$ (depending on M in the case of a bounded strong or Pontryagin solution) such that $\bar{w}$ is a solution of the same kind for the cost function

$$J_\alpha(u,y) := \int_0^T \ell_\alpha(t,u(t),y(t))\mathrm{d}t + \phi(\eta), \tag{6}$$

where

$$\ell_\alpha(t,u,y) := \ell(u,y) - \tfrac{1}{2}\alpha[|u-\bar{u}(t)|^2 + |y-\bar{y}(t)|^2]. \tag{7}$$

Then we say that the (strong, bounded strong, Pontryagin, weak) quadratic growth condition is satisfied. So for instance the quadratic growth condition for a weak solution $\bar{w}$ (we speak then of *weak quadratic growth*) means that

$$\begin{cases} \text{There exist } \alpha > 0,\ \varepsilon > 0: \ J(w) \geq J(\bar{w}) + \tfrac{1}{2}\alpha\|w-\bar{w}\|_2^2, \\ \text{for all } w \in F(P),\ \|w-\bar{w}\|_\infty < \varepsilon, \end{cases} \tag{8}$$

and the *bounded strong quadratic growth* condition means that

$$\begin{cases} \text{For any } M > 0, \text{ there exist } \alpha_M > 0, \varepsilon_M > 0: \ J(w) \geq J(\bar{w}) + \tfrac{1}{2}\alpha_M\|w-\bar{w}\|_2^2 \\ \text{for all } w \in F(P),\ \|y-\bar{y}\|_\infty < \varepsilon_M, \|u\|_\infty \leq M. \end{cases} \tag{9}$$

We now recall the formulation of Pontryagin's principle at the point $\bar{w} \in F(P)$. Let us denote by $\mathbb{R}^{q*}$ the dual of $\mathbb{R}^q$ (identified with the set of q dimensional horizontal vectors). We remind that K was defined in (4). The negative dual cone to K (set of vectors of $\mathbb{R}^{r*}$ having a nonpositive duality product with each elements of K) is $K^- = \mathbb{R}^{r_1*} \times \mathbb{R}^{r_2*}_+$. We say that $(\theta, \mu) \in K \times K^-$ is a *complementary pair* if $\mu_i \theta_i = 0$, for $i = 1, \dots, r$. The normal cone to K at the point $\theta \in K$ is the set of elements of the negative dual cone that are complementary to θ. In particular, the expression of the normal cone to K at $\Phi(\bar{\eta})$ is

$$N_K(\Phi(\bar{\eta})) := \{\mu \in \mathbb{R}^{r*};\ \mu_i \geq 0,\ \mu_i \Phi_i(\bar{\eta}) = 0,\ i > r_1\}. \tag{10}$$

Let the *end-point Lagrangian* be defined by

$$\Phi^\mu(y_0, y_T) := \phi(y_0, y_T) + \mu\Phi(y_0, y_T) = \phi(y_0, y_T) + \sum_{i=1}^{r} \mu_i \Phi_i(y_0, y_T). \tag{11}$$

Consider the *Hamiltonian function* $H : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$ defined by

$$H(u, y, p) := \ell(u, y) + pf(u, y). \tag{12}$$

Set $\mathcal{P} := W^{1,\infty}(0, T; \mathbb{R}^{n*})$. For any $\mu \in \mathbb{R}^{r*}$ and $p \in \mathcal{P}$, consider the following set of relations (reminding that $\bar{\eta} = (\bar{y}(0), \bar{y}(T))$):

$$\begin{array}{lrl} \text{(i)} & -\dot{p}(t) = & H_y(\bar{w}(t), p(t)), \quad \text{for a.a. } t \in (0, T); \\ \text{(ii)} & p(T) = & \Phi^\mu_{y_T}(\bar{\eta}); \\ \text{(iii)} & p(0) = & -\Phi^\mu_{y_0}(\bar{\eta}). \end{array} \tag{13}$$

We call *costate* associated with μ at the point $\bar{w} \in F(P)$, and denote by p^μ, the unique solution in $\mathcal{P}$ of the backward equation (13)(i–ii). Existence and uniqueness of the costate follow from the fact that this is a Cauchy problem for a linear o.d.e. with measurable and bounded coefficients. Obviously the mapping $\mu \mapsto p^\mu$ is affine. We will obtain (13)(iii) as a necessary optimality condition.

Definition 2.1. Let $\bar{w} \in F(P)$. We say that $\mu \in N_K(\Phi(\bar{\eta}))$ is a *(regular) Pontryagin multiplier* associated with $\bar{w}$ if the associated costate p^μ satisfies (13)(iii), and is such that the following Hamiltonian inequality holds:

$$H(\bar{u}(t), \bar{y}(t), p^\mu(t)) \leq H(v, \bar{y}(t), p^\mu(t)), \quad \text{for all } v \in U,\ \text{a.a. } t \in (0, T). \tag{14}$$

We denote by $M^P(\bar{w})$ the set of Pontryagin multipliers associated with $\bar{w}$; if this closed convex set is non empty, we say that $\bar{w}$ satisfies Pontryagin's principle (in qualified form), or that $\bar{w}$ is a *(regular) Pontryagin extremal.*

Remark 2.2. Let $\bar{w}$ be a Pontryagin extremal, and let $\mu \in M^P(\bar{w})$. We know (see e.g. [9, p. 24–25]) that there exists a constant $c_\mu \in \mathbb{R}$ such that

$$c_\mu := \inf_{v \in U} H(v, \bar{p}(t), p^\mu(t)), \quad \text{for all } t \in [0, T]. \tag{15}$$

By (13)(i–ii), the function $\mu \mapsto c_\mu$ is affine. Set

$$h(v, \mu, t) := H(v, \bar{y}(t), p^\mu(t)), \quad t \in (0, T). \tag{16}$$

By (14), we have that $h(\bar{u}(t), \mu, t) = c_\mu$ for a.a. $t \in (0,T)$. Let $B(0,M)$ denote the closed ball of center 0 and radius M in the appropriate space. Define

$$U_M := U \cap B(0,M), \quad \text{where } M > \|\bar{u}\|_\infty. \tag{17}$$

Let us show that there exists a representative $\tilde{u}$ of $\bar{u}$ such that

$$\tilde{u}(t) \in U_M \quad \text{and} \quad h(\tilde{u}(t), \mu, t) = c_\mu, \quad \text{for all } t \in [0,T]. \tag{18}$$

Let $\hat{t} \in [0,T]$. If $h(\bar{u}(\hat{t}), \mu, \hat{t}) = c_\mu$ and $\bar{u}(\hat{t}) \in U_M$, let $\tilde{u}(\hat{t}) := \bar{u}(\hat{t})$. Otherwise, since $h(\bar{u}(t), \mu, t) = c_\mu$ a.e., there exists a sequence $t_k \in [0,T]$, $t_k \to \hat{t}$ such that $\bar{u}(t_k) \in U_M$ and $h(\bar{u}(t_k), \mu, t_k) = c_\mu$. Extracting if necessary a subsequence, we may assume that $\bar{u}(t_k)$ converges to some limit whose value will be $\tilde{u}(\hat{t})$. Passing to the limit in the relation $h(\bar{u}(t_k), \mu, t_k) = c_\mu$, we deduce that $h(\tilde{u}(\hat{t}), \mu, \hat{t}) = c_\mu$ so that $h(\tilde{u}(t), \mu, t) = c_\mu$ for all time. Clearly $\tilde{u}(\hat{t}) \in U_M$, and hence, (18) holds. Moreover, $\tilde{u}(t) = \bar{u}(t)$ for a.a. $t \in (0,T)$, i.e., $\tilde{u}$ is a representative of $\bar{u}$.

We have stated Pontryagin's principle in the qualified form. The original Pontryagin's principle is a more general statement (that we do not need to give here) and is a necessary condition for Pontryagin solutions of (P), see [9, p. 24]. The present qualified form is is a necessary condition for Pontryagin solutions under the qualifications condition (61) used in this paper. So by Pontryagin's principle in this paper we always mean its qualified form.

Remark 2.3. Assume that, in addition to (3), problem (P) has a "control constraint" $(t, u(t)) \in \mathcal{Q}$, for a.a. $t \in (0,T)$, where $\mathcal{Q}$ is an open set in $\mathbb{R}^{m+1}$. Then, in definition of $M^P(\bar{w})$, the Hamiltonian inequality (14) changes as follows, see [9, Part 1, p. 24–25 and 148–151]: $H(\bar{u}(t), \bar{y}(t), p^\mu(t)) \le H(v, \bar{y}(t), p^\mu(t))$ for all $v \in U$ such that $(t,v) \in \mathcal{Q}$, a.a. $t \in (0,T)$. Note that $\bar{w}$ is a weak minimum in problem (P) iff there exists $\varepsilon > 0$ such that $\bar{w}$ is a strong minimum in the same problem with constraint $\mathcal{Q} = \{(t,v) : t \in [0,T],\ |v - \bar{u}(t)| < \varepsilon\}$. If the control $\bar{u}$ is a continuous function of time, then $\mathcal{Q}$ is an open set, and in this case we get a version of (regular) Pontryagin principle for a weak minima. We will say that $\bar{w}$ is a *weak extremal* if there exists $\mu \in N_K(\Phi(\bar{\eta}))$ such that for $\varepsilon > 0$ small enough the following Hamiltonian inequality holds:

$$\begin{aligned} &H(\bar{u}(t), \bar{y}(t), p^\mu(t)) \le H(v, \bar{y}(t), p^\mu(t)), \\ &\qquad \text{for all } v \in U,\ |v - \bar{u}(t)| \le \varepsilon \text{ a.a. } t \in (0,T). \end{aligned} \tag{19}$$

Again, under the qualification condition (61), any weak minimum is a weak extremal.

2.2. Hamiltonian functions with a unique minimum

In this section we study Pontryagin extremals for which, at each time, the Hamiltonian function attains its minimum at a unique point. The main result, Corollary 2.9, establishes that then a Pontryagin minimum is a bounded strong minimum.

If A is a convex subset of a finite-dimensional space, we denote by $\mathrm{ri}(A)$ its relative interior, in the sense of convex analysis (the interior of A, in the topology induced by its affine hull). We check below that a relatively interior Pontryagin multiplier (i.e., some $\mu \in \mathrm{ri}(M^P(\bar{w}))$) obtains an increase of Hamiltonian of the same growth rate as the maximum over bounded sets of Pontryagin multipliers:

Lemma 2.4. *Let $\bar{w} \in F(P)$ satisfy Pontryagin's principle, and let $M^C(\bar{w})$ be a non-empty, convex and compact subset of $M^P(\bar{w})$. Then for any $\bar{\mu} \in \operatorname{ri}(M^C(\bar{w}))$, there exists $\beta > 0$ such that, for a.a. t, and any $v \in U$:*

$$\begin{aligned} &H(v, \bar{y}(t), p^{\bar{\mu}}(t)) - H(\bar{u}(t), \bar{y}(t), p^{\bar{\mu}}(t)) \\ &\geq \beta \max_{\mu \in M^C(\bar{w})} \left(H(v, \bar{y}(t), p^{\mu}(t)) - H(\bar{u}(t), \bar{y}(t), p^{\mu}(t))\right); \end{aligned} \tag{20}$$

$$\beta \mu_i \leq \bar{\mu}_i, \quad \textit{for all } i > r_1. \tag{21}$$

Proof. Since $\bar{\mu} \in \operatorname{ri}(M^C(\bar{w}))$, and $M^C(\bar{w})$ is compact, there exists $\varepsilon_0 > 0$ such that $\bar{\mu} + \varepsilon_0(\bar{\mu} - \mu) \in M^C(\bar{w})$, for any $\mu \in M^C(\bar{w})$. The function $\mu \mapsto a(\mu) := H(v, \bar{y}(t), p^{\mu}(t)) - H(\bar{u}(t), \bar{y}(t), p^{\mu}(t))$ is affine. For given $\mu \in M^C(\bar{w})$, we have that $\mu' := \bar{\mu} + \varepsilon_0(\bar{\mu} - \mu)$ belongs to $M^C(\bar{w})$. Since $\bar{\mu} = \frac{1}{1+\varepsilon_0}\mu' + \frac{\varepsilon_0}{1+\varepsilon_0}\mu$, $\mu'_i \geq 0$ for $i > r_1$, and $a(\cdot)$ is affine and nonnegative when $v \in U$, it follows that $\bar{\mu}_i \geq \frac{\varepsilon_0}{1+\varepsilon_0}\mu_i$ for $i > r_1$, and $a(\bar{\mu}) \geq \frac{\varepsilon_0}{1+\varepsilon_0} a(\mu)$. The conclusion follows with $\beta = \varepsilon_0/(1+\varepsilon_0)$. □

We next relate the convergence of cost, for bounded strong perturbations, to some integral of difference of Hamiltonian functions. For $\mu \in \mathbb{R}^{r*}$, we define

$$J^{\mu}(w) := J(w) + \mu\Phi(\eta) = \int_0^T \ell(w(t))\mathrm{d}t + \Phi^{\mu}(\eta). \tag{22}$$

If w is a trajectory, then for any $p \in \mathcal{P}$, we have that

$$J^{\mu}(w) = \int_0^T \left[H(w(t), p(t)) - p(t)\dot{y}(t)\right] \mathrm{d}t + \Phi^{\mu}(\eta). \tag{23}$$

Lemma 2.5. *Let $\bar{w} \in \mathcal{W}$ be a trajectory, let $\mu \in \mathbb{R}^{r*}$, with associated costate p^{μ} solution of* (13)(i–ii)*, and let w be any trajectory. Denote $\eta := (y(0), y(T))$ and $\delta\eta := \eta - \bar{\eta}$. Then*

(i) *The following expansion holds:*

$$\begin{aligned} J^{\mu}(w) - J^{\mu}(\bar{w}) = &\int_0^T [H(w, p^{\mu}) - H(\bar{w}, p^{\mu}) - H_y(\bar{w}, p^{\mu})\delta y]\mathrm{d}t \\ &+ (p^{\mu}(0) + \Phi^{\mu}_{y_0}(\bar{\eta}))\delta y(0) + \tfrac{1}{2}(\Phi^{\mu})''(\bar{\eta})(\delta\eta)^2 + o(|\delta\eta|^2). \end{aligned} \tag{24}$$

(ii) *Let w_k be a bounded strong perturbation of $\bar{w}$, and $\mu \in \mathbb{R}^{r*}$. Then*

$$J(w_k) - J(\bar{w}) = \int_0^T [H(u_k, \bar{y}, p^{\mu}) - H(\bar{u}, \bar{y}, p^{\mu})]\mathrm{d}t + o(1). \tag{25}$$

(iii) *If $\bar{w}$ is a Pontryagin extremal, $\mu \in M^P(\bar{w})$, and w_k is a feasible bounded strong perturbation of $\bar{w}$, then $\liminf_k J(w_k) \geq J(\bar{w})$, with equality iff the following holds:*

$$\liminf_k \int_0^T [H(u_k, \bar{y}, p^{\mu}) - H(\bar{u}, \bar{y}, p^{\mu})]\mathrm{d}t = 0. \tag{26}$$

In particular, $J(w_k) \to J(\bar{w})$ iff the integral in (26) *converges to 0.*

Proof. Denoting $\delta y := y - \bar{y}$, observe that for any $\mu \in \mathbb{R}^{r*}$, using (23) with $p = p^\mu$, we have that

$$J^\mu(w) - J^\mu(\bar{w}) = \int_0^T [H(w, p^\mu) - H(\bar{w}, p^\mu) - p^\mu \delta\dot{y}]\mathrm{d}t + \Phi^\mu(\eta) - \Phi^\mu(\bar{\eta}). \tag{27}$$

Since by the integration by parts formula, we get

$$-\int_0^T p^\mu \delta\dot{y}\mathrm{d}t = p^\mu(0)\delta y(0) - p^\mu(T)\delta y(T) + \int_0^T \dot{p}^\mu \delta y \mathrm{d}t, \tag{28}$$

using the costate equation (13) and a second-order expansion of $\Phi^\mu(\eta)$, obtain (i). Since for a bounded strong perturbation we have that $\|y_k - \bar{y}\|_\infty \to 0$, we deduce from (24) that $J^\mu(w_k) - J^\mu(\bar{w})$ is equal to the r.h.s. of (25). Since $\eta_k \to \bar{\eta}$, $J^\mu(w_k) - J^\mu(\bar{w}) = J(w_k) - J(\bar{w}) + o(1)$. Point (ii) follows. Combining with (14), we deduce (iii). $\square$

We next show that the uniqueness of the minimum of the Hamiltonian function for all times t implies that the control is continuous.

Given a Pontryagin extremal $\bar{w}$, and $M > \|\bar{u}\|_\infty$, we say that $\mu \in M^P(\bar{w})$ satisfies the hypothesis of unique minimum of the Hamiltonian over U_M if the associated costate p^μ is such that, for all $t \in [0, T]$, the function $h(\cdot, \mu, t) = H(\cdot, \bar{y}(t), p^\mu(t))$ has a unique minimum over U_M.

Remark 2.6. Let the hypothesis of unique minimum of the Hamiltonian over U_M hold for some $\mu \in M^P(\bar{w})$. Then (i) by Lemma 2.4, it holds for any element $\bar{\mu} \in \mathrm{ri}(M^P(\bar{w}))$ (taking for $M^C(\bar{w})$ the set $M^P(\bar{w}) \cap B(0, R)$, with $R > \max\{|\mu|, |\bar{\mu}|\}$) and (ii) for given $\varepsilon > 0$ and $M > 0$, there exists $\varepsilon_M > 0$ such that

$$\begin{cases} \text{For a.a. } t \in (0, T), \text{ whenever } v \in U_M \text{ and } |v - \bar{u}(t)| \geq \varepsilon : \\ H(v, \bar{y}(t), p^\mu(t)) \geq H(\bar{u}(t), \bar{y}(t), p^\mu(t)) + \varepsilon_M. \end{cases} \tag{29}$$

Lemma 2.7. *Let $\bar{w}$ be a Pontryagin extremal, and $\mu \in M^P(\bar{w})$ satisfy the hypothesis of unique minimum of the Hamiltonian over U_M, with $M > \|\bar{u}\|_\infty$. Then (one representative of) $\bar{u}(t)$ is a continuous function of time, equal to this unique minimum.*

Proof. Let $t \in [0, T]$; then for $t_k \to t$ in $[0, T]$ the function $\tilde{u}$ constructed in Remark 2.2 is such that $h(\tilde{u}(t), \mu, t) = h(\tilde{u}(t_k), \mu, t_k) = c_\mu$. Passing to the limit obtain $h(\tilde{u}(t), \mu, t) = h(v, \mu, t) = c_\mu$ for all limit points v of $\tilde{u}(t_k)$ (they exist since $\tilde{u}(t) \in U_M$ for all $t \in [0, T]$); moreover, $v \in U_M$. Since $h(\cdot, \mu, t)$ has a unique minimum over U_M we see that $\tilde{u}(t) = v$, which proves that $\tilde{u}$ is continuous. The conclusion follows. $\square$

Consider the condition similar to (26), but with a limit instead of a lower limit:

$$\lim_k \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\mathrm{d}t = 0. \tag{30}$$

Lemma 2.8. *Let $\bar{w}$ be a Pontryagin extremal, and let $\mu \in M^P(\bar{w})$ be such that the Hamiltonian satisfies the hypothesis of unique minimum over U. If w_k is a feasible bounded strong perturbation of $\bar{w}$, then the four conditions below are equivalent: (i) $\limsup_k J(w_k) \leq J(\bar{w})$, (ii) $\lim_k J(w_k) = J(\bar{w})$, (iii) (30) holds, (iv) Any subsequence of u_k has itself a subsequence converging to $\bar{u}$ a.e.*

Proof. The equivalence of (i), (ii) and (iii) follows from Lemma 2.5. If (iii) holds, then by (29) $u_k \to \bar{u}$ in measure (i.e., for all $\varepsilon > 0$, $\operatorname{meas}(\{t \in (0,T); |u_k(t)-\bar{u}(t)| > \varepsilon\}) \to 0)$, and hence (since this holds also for an arbitrary subsequence of u_k) condition (iv) holds. Finally assume that (iv) holds, but not (iii). Taking if necessary a subsequence we may assume that there exists $\varepsilon > 0$ such that $\int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\mathrm{d}t > \varepsilon$. But then, taking again a subsequence for which $u_k \to \bar{u}$ a.e., we obtain a contradiction to Lebesgue's dominated convergence theorem. □

Corollary 2.9. *Let $\bar{w}$ be a Pontryagin extremal. Assume that, for some $\mu \in M^P(\bar{w})$, the Hamiltonian satisfies the hypothesis of unique minimum over U. Then*

(i) *any feasible bounded strong perturbation of $\bar{w}$ such that $\limsup_k J(w_k) \leq J(\bar{w})$ is a Pontryagin perturbation, and*

(ii) *$\bar{w}$ is a bounded strong minimum iff it is a Pontryagin minimum.*

Proof. (i) Let $\bar{w}$ satisfy the hypotheses of the corollary. If w_k is a feasible bounded strong perturbation of $\bar{w}$ such that $\limsup_k J(w_k) \leq J(\bar{w})$, then, by the above lemma, we deduce that $u_k \to \bar{u}$ a.e., and hence, by the dominated convergence theorem, $\|u_k - \bar{u}\|_1 \to 0$, so that w_k is a Pontryagin perturbation.

(ii) As already observed, a bounded strong minimum is a Pontryagin solution. Conversely, by point (i) a Pontryagin solution is a bounded strong solution. □

Definition 2.10. Let $\bar{w}$ be a Pontryagin extremal. We say that the Hamiltonian function satisfies a *local quadratic growth condition* for $\mu \in M^P(\bar{w})$ if there exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\begin{array}{l} \text{For a.a. } t \in (0,T), \text{ whenever } v \in U,\ |v - \bar{u}(t)| \leq \varepsilon: \\ H(v, \bar{y}(t), p^\mu(t)) \geq H(\bar{u}(t), \bar{y}(t), p^\mu(t)) + \alpha |v - \bar{u}(t)|^2. \end{array} \tag{31}$$

Remark 2.11. In view of Lemma 2.4, we have that, if $M^C(\bar{w})$ is a nonempty convex and compact subset of $M^P(\bar{w})$, and there exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\begin{array}{l} \text{For a.a. } t \in (0,T), \text{ whenever } v \in U,\ |v - \bar{u}(t)| \leq \varepsilon: \\ \max\limits_{\mu \in M^C(\bar{w})} [H(v, \bar{y}(t), p^\mu(t)) - H(\bar{u}(t), \bar{y}(t), p^\mu(t))] \geq \alpha |v - \bar{u}(t)|^2. \end{array} \tag{32}$$

then the Hamiltonian function satisfies the local quadratic growth condition, for any $\mu \in \operatorname{ri}(M^C(\bar{w}))$.

If $\mu \in M^P(\bar{w})$ is such that (29) and (31) hold, for $M > \|\bar{u}\|_\infty$, then $\bar{u}$ has a (continuous) representative $\tilde{u}$ such that, for all $t \in [0,T]$, $v \mapsto h(v, \mu, t)$ has a unique minimum at $\tilde{u}(t)$ over U_M, and we have that

$$\begin{cases} H(v, \bar{y}(t), p^\mu(t)) \geq H(\tilde{u}(t), \bar{y}(t), p^\mu(t)) + \min(\alpha |v - \tilde{u}(t)|^2, \varepsilon_M), \\ \text{whenever } v \in U_M, \text{ for all } t \in [0,T]. \end{cases} \tag{33}$$

2.3. Second-order expansion of the (weighted) cost function

Having in mind Pontryagin perturbations, we now introduce the natural notion of linearization of the state equation in the framework of the study of Pontryagin minima. Given a Pontryagin perturbation w_k of $\bar{w} \in \mathcal{W}$, satisfying the state equation (2), reminding that $\delta y_k := y_k - \bar{y}$ is the variation of states, denote by $\delta_L y_k$ the solution of the *Pontryagin linearization* of the state equation at the point $\bar{w} = (\bar{u}, \bar{y})$:

$$\delta_L \dot{y}_k = f_y(\bar{u}, \bar{y})\delta_L y_k + f(u_k, \bar{y}) - f(\bar{u}, \bar{y}); \qquad \delta_L y_k(0) = y_k(0) - \bar{y}(0). \tag{34}$$

As the following lemma shows, $\delta_L y_k$ gives a good approximation of δy_k. We denote the remaining terms in a second-order Taylor expansion in L^1 as

$$R_{1k} := O(\|u_k - \bar{u}\|_1^2 + |y_k(0) - \bar{y}(0)|^2); \qquad r_{1k} = o(\|u_k - \bar{u}\|_1^2 + |y_k(0) - \bar{y}(0)|^2).$$

A lemma similar to the one below can be found in Milyutin and Osmolovskiĭ [9, p. 40–42, Prop. 8.1 to 8.3]. We give the (short) proof in order to make the paper self-contained.

Lemma 2.12. *Let $\bar{w} \in \mathcal{W}$ satisfy the state equation* (2)*, and w_k be a Pontryagin perturbation of $\bar{w}$. Then*

$$\|\delta_L y_k - y_k - \bar{y}\|_\infty = R_{1k}. \tag{35}$$

Proof. Set $\Delta_k = f(u_k, y_k) - f(u_k, \bar{y}) - f_y(\bar{u}, \bar{y})\delta y_k$. We may write

$$\delta \dot{y}_k = f_y(\bar{u}, \bar{y})\delta y_k + f(u_k, \bar{y}) - f(\bar{u}, \bar{y}) + \Delta_k \tag{36}$$

and by the mean value theorem, for some $\theta : [0, T] \to (0, 1)$:

$$|\Delta_k| = |[f_y(u_k, \bar{y} + \theta\delta y_k) - f_y(\bar{u}, \bar{y})]\delta y_k| = O((|u_k - \bar{u}| + |\delta y_k|)|\delta y_k|), \tag{37}$$

so that $\|\Delta_k\|_1 = R_{1k}$. The conclusion follows then with Gronwall's Lemma. $\square$

Lemma 2.13. *Let $\bar{w} \in \mathcal{W}$ be a trajectory, and w_k be a Pontryagin perturbation of $\bar{w}$. Then for any $\mu \in \mathbb{R}^{r*}$, denoting by p^μ the costate associated with μ at the point $\bar{w}$ (solution of* (13)(i–ii)*), we have that:*

$$\begin{aligned} J^\mu(w_k) - J^\mu(\bar{w}) = &\int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) + \tfrac{1}{2} H_{yy}(\bar{u}, \bar{y}, p^\mu)(\delta_L y_k)^2]\mathrm{d}t \\ &+ \int_0^T [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta_L y_k \,\mathrm{d}t \\ &+ (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) \\ &+ \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_L y_k(0), \delta_L y_k(T))^2 + r_{1k}. \end{aligned} \tag{38}$$

Proof. Expanding w.r.t. $\delta y_k := y_k - \bar{y}$ the r.h.s. of the expression below:

$$\begin{aligned} &H(u_k, y_k, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) \\ &= [H(u_k, y_k, p^\mu) - H(u_k, \bar{y}, p^\mu)] + [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)], \end{aligned} \tag{39}$$

and since (all derivatives of f and ℓ being uniformly bounded and continuous over bounded sets)

$$\begin{aligned} & H(u_k, y_k, p^\mu) - H(u_k, \bar{y}, p^\mu) \\ &= H_y(u_k, \bar{y}, p^\mu)\delta y_k + \tfrac{1}{2} H_{yy}(u_k, \bar{y}, p^\mu)\delta y_k^2 + \tilde{r}_{1k}(t), \end{aligned} \tag{40}$$

where $\|\tilde{r}_{1k}\|_1 = r_{1k}$, and using the relation

$$\int_0^T H_{yy}(u_k, \bar{y}, p^\mu)\delta y_k^2 \mathrm{d}t = \int_0^T H_{yy}(\bar{u}, \bar{y}, p^\mu)\delta y_k^2 \mathrm{d}t + r_{1k}(t),$$

obtain with (24):

$$\begin{aligned} J^\mu(w_k) - J^\mu(\bar{w}) = & \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) + \tfrac{1}{2} H_{yy}(\bar{u}, \bar{y}, p^\mu)\delta y_k^2]\mathrm{d}t \\ & + \int_0^T [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta y_k \mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta y_k(0), \delta y_k(T))^2 + r_{1k}. \end{aligned} \tag{41}$$

The conclusion follows by Lemma 2.12. □

2.4. A decomposition principle

Now given $\bar{w} \in \mathcal{W}$ and a Pontryagin perturbation w_k, both satisfying the state equation (2), we start to use second-order expansions of the cost function with the remainder

$$r_{2k} := o(\|\delta u_k\|_2^2 + |y_k(0) - \bar{y}(0)|^2).$$

Consider a sequence of "measurable partitions" of $[0, T]$, i.e., measurable subsets A_k and B_k of $(0, T)$ such that

$$[0, T] = A_k \cup B_k; \qquad \operatorname{meas}(A_k \cap B_k) = 0, \tag{42}$$

such that $\operatorname{meas}(B_k) \to 0$, and a Pontryagin perturbation w_k of $\bar{w} \in \mathcal{W}$. Denote the restriction of variations of control variables to the set A_k by

$$\delta_A u_k = \mathbf{1}_{A_k}(u_k - \bar{u}); \qquad u_{A,k} := \bar{u} + \delta_A u_k, \tag{43}$$

and similarly for $\delta_B u_k$ and $u_{B,k}$. The associated states are defined as solution of the Cauchy problems

$$\dot{y}_{A,k} = f(u_{A,k}, y_{A,k}); \qquad y_{A,k}(0) = y_k(0), \tag{44}$$

$$\dot{y}_{B,k} = f(u_{B,k}, y_{B,k}); \qquad y_{B,k}(0) = \bar{y}(0), \tag{45}$$

that is, the variation in the initial condition is "absorbed" by the "A" part. We set

$$\delta_A y_k = y_{A,k} - \bar{y}; \qquad \delta_B y_k = y_{B,k} - \bar{y}. \tag{46}$$

Theorem 2.14 (Decomposition principle). *Let $\bar{w}$ be a trajectory in $\mathcal{W}$, w_k be a Pontryagin perturbation, and (A_k, B_k) be a measurable partition of $(0,T)$, such that* $\operatorname{meas}(B_k) \to 0$. *Then, for all $\mu \in \mathbb{R}^{r*}$, we have a decomposition principle*

$$J^\mu(w_k) = J^\mu(w_{A,k}) + (J^\mu(w_{B,k}) - J^\mu(\bar{w})) + r_{2k}, \tag{47}$$

and, p^μ being the costate associated with μ at the point $\bar{w}$, $J^\mu(w_{B,k})$ has the following expansion:

$$J^\mu(w_{B,k}) = J^\mu(\bar{w}) + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\mathrm{d}t + O(\|\delta_B u_k\|_1^2). \tag{48}$$

If in addition $\|\delta_A u_k\|_\infty \to 0$, then setting $r_{A2k} := o(\|\delta_A u_k\|_2^2 + |y_k(0) - \bar{y}(0)|^2)$, we have that

$$J^\mu(w_{A,k}) = J^\mu(\bar{w}) + \delta_A J_k^\mu + r_{A2k}, \tag{49}$$

with

$$\begin{aligned} \delta_A J_k^\mu := & \int_0^T [\tfrac{1}{2} H_{ww}(\bar{w}, p^\mu)(\delta_A w_k)^2 + H_u(\bar{w}, p^\mu)\delta_A u_k]\mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2. \end{aligned} \tag{50}$$

Proof. The Pontryagin linearization (34) being the sum of the ones for perturbations $u_{A,k}$ and $u_{B,k}$, it follows from Lemma 2.12 that $\|\delta y_k - \delta_A y_k - \delta_B y_k\|_\infty = R_{1,k}$, and we have

$$\text{(i)}\ \ \|\delta_A y_k\|_\infty = O(\|\delta_A u_k\|_1 + |\delta_A y_k(0)|); \qquad \text{(ii)}\ \ \|\delta_B y_k\|_\infty = O(\|\delta_B u_k\|_1). \tag{51}$$

Relation (48) follows then from (38) and (51)(ii). Now by the Cauchy-Schwarz inequality:

$$\|\delta_B u_k\|_1 \le \operatorname{meas}(B_k)^{1/2}\|\delta_B u_k\|_2 = o(\|\delta_B u_k\|_2), \tag{52}$$

so that $\|\delta_B y_k\|_\infty = o(\|\delta_B u_k\|_2)$, and using $r_{1k} = O(r_{2k})$, obtain with (38):

$$\begin{aligned} J^\mu(w_k) - J^\mu(\bar{w}) = & \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu) + \tfrac{1}{2} H_{yy}(\bar{w}, p^\mu)\delta_A y_k^2 \\ & + (H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{w}, p^\mu))\delta_A y_k]\mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) \\ & + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2 + r_{2k}. \end{aligned} \tag{53}$$

Using (51) and (52), get

$$\int_{B_k} [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta_A y_k \mathrm{d}t = O(\|\delta_B u_k\|_1 \|\delta_A y_k\|_\infty) = r_{2k}. \tag{54}$$

We deduce with (52)–(54) that

$$J^\mu(w_k) - J^\mu(\bar{w}) = \hat{\delta}_A J_k^\mu + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\mathrm{d}t + r_{2k}, \tag{55}$$

where

$$\begin{aligned}\hat{\delta}_A J_k^\mu := &\int_{A_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu)]\mathrm{d}t + \tfrac{1}{2}\int_0^T H_{yy}(\bar{w}, p^\mu)\delta_A y_k^2 \mathrm{d}t \\ &+ \int_{A_k} (H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{w}, p^\mu))\delta_A y_k \mathrm{d}t \\ &+ (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2.\end{aligned} \tag{56}$$

In particular, taking $\delta_B u_k = 0$, we obtain

$$J^\mu(w_{A,k}) = J^\mu(\bar{w}) + \hat{\delta}_A J_k^\mu + r_{A2k}, \tag{57}$$

which combined with (48), (52) and (55), proves (47). Finally when $\|\delta_A u_k\|_\infty \to 0$, it follows from a Taylor expansion that $\hat{\delta}_A J_k^\mu = \delta_A J_k^\mu + r_{A2k}$, proving (49). The conclusion follows. □

3. Inequality control constraints

We assume in this section that the control constraints are parameterized by finitely many inequalities:

$$U := \{u \in \mathbb{R}^m;\ g(u) \le 0\}, \tag{58}$$

where $g : \mathbb{R}^m \to \mathbb{R}^q$ is a C^2 mapping. In other words, the control constraints are defined by

$$g_i(u(t)) \le 0, \quad \text{for a.a. } t \in (0, T),\ i = 1, \dots, q. \tag{59}$$

We consider the "abstract" formulation where the state is a function of initial state and control. So we may write the state as $y_{u,y_0}(t)$, and define $G : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^r$ by

$$G_i(u, y_0) := \Phi_i(y_0, y_{u,y_0}(T)), \quad i = 1, \dots, r. \tag{60}$$

By $G_{1:r_1}(u, y_0)$ we denote the (vertical) vector of components 1 to r_1 of $G(u, y_0)$. We say that the following *qualification condition* [13] (a natural infinite dimension generalization of the Mangasarian-Fromovitz condition [8]; see also [2, Section 2.3.4])) holds at $\bar{w} \in F(P)$ if

$$\left\{\begin{array}{l} G'_{1:r_1}(\bar{u}, \bar{y}_0) \text{ is onto,} \\ \text{There exists } \beta > 0 \text{ and } (\bar{v}, \bar{z}_0) \in \operatorname{Ker} G'_{1:r_1}(\bar{u}, \bar{y}_0); \\ g_i(\bar{u}(t)) + g_i'(\bar{u}(t))\bar{v}(t) \le -\beta, \quad \text{for a.a. } t \in [0, T],\ i = 1 \dots, q, \\ G'_i(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) \le -\beta, \quad \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{array}\right. \tag{61}$$

We also define for future reference

$$\hat{J}(u, y_0) := J(u, y_{u,y_0}); \qquad \hat{J}^\mu(u, y_0) := \hat{J}(u, y_0) + \mu G(u, y_0), \tag{62}$$

as well as the *augmented Hamiltonian function* by

$$H^a(u, y, p, \lambda) := H(u, y, p) + \lambda g(u) = \ell(u, y) + p f(u, y) + \lambda g(u), \tag{63}$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $p \in \mathbb{R}^{n*}$, and $\lambda \in \mathbb{R}^{q*}$. Given $\bar{w} = (\bar{u}, \bar{y}) \in F(P)$, we recall that the set of normal directions to K at the point $\Phi(\bar{\eta})$ was defined in (10). The costate

$p^\mu \in \mathcal{P}$ associated with $\mu \in N_K(\Phi(\bar\eta))$ was defined as the solution of (13)(i–ii). For $\bar w \in F(P)$, $\mu \in N_K(\Phi(\bar\eta))$ such that (13)(iii) holds and $t \in [0,T]$, define

$$\Lambda_t(\bar w,\mu) := \{\lambda \in \mathbb{R}^{q*}_+ \cap g(\bar u(t))^\perp;\ H^a_u(\bar w(t), p^\mu(t), \lambda) = 0\},$$
$$LM^L(\bar w) := \{(\lambda,\mu);\ \mu \in N_K(\Phi(\bar\eta));\ (13) \text{ holds};\ \lambda \in L^\infty(0,T,\Lambda_t(\bar w,\mu))\}.$$

We call $LM^L(\bar w)$ (the superscript L is reminiscent of Lagrange) the set of first-order multipliers. If the latter is non empty, then we say that $\bar w$ is a *stationary point*. By $M^L(\bar w)$ we denote the projection on the second component, The Lagrangian for the abstract formulation is

$$\mathcal{L}(u,y_0,\lambda,\mu) := \hat J^\mu(u,y_0) + \langle \lambda, g(u)\rangle, \tag{64}$$

the last duality product being in the space $L^\infty(0,T,\mathbb{R}^q)$.

Theorem 3.1. *Let $\bar w$ be a weak solution of (P), satisfying the qualification hypothesis* (61)*. Then*

(*i*) *$LM^L(\bar w)$ is nonempty and bounded, and*
(*ii*) *$M^P(\bar w)$ is a (possibly empty) subset of $M^L(\bar w)$.*

Proof. (*i*) An abstract formulation of problem (P) is

$$\operatorname*{Min}_{u,y_0} \hat J(u,y_0); \qquad g(u) \le 0; \qquad G(u,y_0) \in K. \tag{65}$$

Let $K_g := L^\infty(0,T,\mathbb{R}^q_-)$, with associated normal cone at the point $g(\bar u)$ denoted $N_{K_g}(g(\bar u))$. The corresponding set of Lagrange multipliers at point $(\bar u, \bar y_0)$ is defined as

$$(LM^L)^\sharp(\bar w) := \{(\lambda,\mu) \in N_{K_g}(g(\bar u)) \times N_K(G(\bar u,\bar y_0));\ D_{(u,y_0)}\mathcal{L}(\bar u,\bar y_0,\lambda,\mu) = 0\}. \tag{66}$$

The qualification hypothesis (61) being a particular case of Robinson's qualification condition [13], we know that $(LM^L)^\sharp(\bar w)$ is nonempty and bounded in $L^\infty(0,T,\mathbb{R}^q)^* \times \mathbb{R}^{r*}$. It remains to prove that any $(\lambda,\mu) \in (LM^L)^\sharp(\bar w)$ is such that λ can be identified with some $\tilde\lambda$ in $L^\infty(0,T,\mathbb{R}^{q*})$, the norm in $L^\infty(0,T,\mathbb{R}^{q*})$ of $\tilde\lambda$ being uniformly bounded over all $(\lambda,\mu) \in (LM^L)^\sharp(\bar w)$. More precisely, we have to check the existence of $c > 0$ such that, $|\langle\lambda,a\rangle| \le c\|a\|_{L^1(0,T,\mathbb{R}^q)}$, for all $a \in L^\infty(0,T,\mathbb{R}^q)$. If this holds then, since $L^\infty(0,T,\mathbb{R}^q)$ is a dense subset of $L^1(0,T,\mathbb{R}^q)$, λ has a unique extension $\tilde\lambda$ in the dual space of $L^1(0,T,\mathbb{R}^q)$, i.e., $L^\infty(0,T,\mathbb{R}^{q*})$.

Since the norm of $a \in L^1(0,T,\mathbb{R}^q)$ is the sum of the norms of its positive and negative parts, it suffices to check this inequality when $a \ge 0$, i.e., since $\lambda \ge 0$, to prove that $\langle\lambda,a\rangle \le c\|a\|_{L^1(0,T,\mathbb{R}^q)}$. We can write $a(t) = \alpha(t)\bar a(t)$, with $\alpha(t) = |a(t)|$ and $|\bar a(t)| = 1$. Set $h := -(g(\bar u) + g'(\bar u)\bar v)$. By (61), $\beta \le h_i(t)$, $i = 1,\dots,q$, for a.a. t. Since $a_i(t) \le \alpha(t)$, $i = 1,\dots,q$, for a.a. t, we have that $\beta a(t) \le \alpha(t)h(t)$, and so

$$\beta\langle\lambda,a\rangle = \langle\lambda,\beta a\rangle \le \langle\lambda,\alpha h\rangle. \tag{67}$$

Since $\lambda \ge 0$, $a \ge 0$ and $g(\bar u) \le 0$, the maximal ratio between the uniform and Euclidean norm being $\sqrt{q}$, we have that:

$$0 \ge \langle\lambda,\alpha g(\bar u)\rangle \ge \sqrt{q}\langle\lambda,\|a\|_\infty g(\bar u)\rangle = \sqrt{q}\|a\|_\infty\langle\lambda,g(\bar u)\rangle = 0, \tag{68}$$

the last equality being the complementarity condition between elements of a convex cone and elements of the corresponding normal cone. It follows that $\langle \lambda, \alpha g(\bar{u}) \rangle = 0$. Combining with (67), and using (66), obtain

$$\begin{aligned} \beta\langle \lambda, a\rangle \leq & -\langle \lambda, \alpha g'(\bar{u})\bar{v}\rangle = -\langle \lambda, g'(\bar{u})\alpha\bar{v}\rangle = D_u\hat{J}^\mu(\bar{u}, \bar{y}_0)(\alpha\bar{v}) \\ & \leq \|D_u\hat{J}^\mu(\bar{u}, \bar{y}_0)\|_\infty \|\alpha\|_1 \|\bar{v}\|_\infty = \|D_u\hat{J}^\mu(\bar{u}, \bar{y}_0)\|_\infty \|\bar{v}\|_\infty \|a\|_1, \end{aligned} \tag{69}$$

which, since μ remains in a bounded set, gives the desired estimate. Point (i) follows.

(ii) Let $\mu \in M^P(\bar{w})$. Then $\bar{u}$ is solution of the problem

$$\operatorname*{Min}_{u \in L^\infty(0,T,\mathbb{R}^m)} \int_0^T H(u(t), \bar{y}(t), p^\mu(t))\mathrm{d}t; \qquad g(u(t)) \leq 0, \quad \text{for a.a. } t \in (0, T). \tag{70}$$

In view of the qualification condition, there exists some λ such that $(\lambda, \mu) \in (LM^L)(\bar{w})$. The conclusion follows. □

4. Quadratic growth with initial-final state constraints

The main result of this section is Theorem 4.8. Assuming that the restoration property (78) holds, it shows that a Pontryagin extremal with a unique minimum of the Hamiltonian satisfies the bounded strong quadratic growth condition iff the weak minimum quadratic growth condition (8) and the local quadratic growth condition for Hamiltonian (31) hold.

When dealing with initial-final state constraints we need to combine the previous decomposition principle with a certain restoration hypothesis, for which we will give sufficient conditions. In most statements of the section we will assume that

$$\bar{w} \text{ is a stationary point, and } \bar{\mu} \text{ denotes an element of } \mathrm{ri}(M^L(\bar{w})), \tag{71}$$

where $\mathrm{ri}(M^L(\bar{w}))$ is the relative interior of $M^L(\bar{w})$. Denote the set of strictly (non strictly) complementary active constraints by

$$\begin{cases} I_+ := \{1, \dots, r_1\} \cup \{r_1 < i \leq r;\ \bar{\mu}_i > 0\}, \\ I_0 := \{r_1 < i \leq r;\ \Phi_i(\bar{\eta}) = 0\} \setminus I_+. \end{cases} \tag{72}$$

Similarly to (21), all $\mu \in \mathrm{ri}(M^L(\bar{w}))$ have the same set of positive components, as can be easily checked, so that the definition does not depend on the choice of the particular $\bar{\mu}$. Define

$$K_+ := \{\theta \in \mathbb{R}^r;\ \theta_i = 0,\ i \in I_+,\ \theta_i \leq 0,\ i \in I_0\}. \tag{73}$$

Note that $K_+ = K \cap \bar{\mu}^\perp$. The function

$$d(\eta) := \sum_{i \in I_0} \Phi_i(\eta)_+ + \sum_{i \in I_+} |\Phi_i(\eta)| \tag{74}$$

is the distance of the initial-final state constraint to the set K_+, in the $L^1(\mathbb{R}^r)$ norm (the unique projection of $\theta \in \mathbb{R}^r$ in this norm being θ' defined by $\theta'_i = 0$ if $i \in I_+$, and $\theta'_i = \min(\theta_i, 0)$ otherwise). We have that

$$d(\eta) = O(-\bar{\mu}\Phi(\eta)) \text{ whenever } w \in F(P), \tag{75}$$

since then $\Phi_{1:r_1}(\eta) = 0$ and $\Phi_{r_1+1:r}(\eta) \le 0$. Call *Pontryagin norm* the following one:

$$\|w\|_P := \|u\|_1 + \|y\|_\infty. \tag{76}$$

For given $\varepsilon_D > 0$, and $u \in \mathcal{U}$, define the set of times of ε_D-deviation of $\bar{u}$ as

$$B_{\varepsilon_D}(u) := \left\{ t \in (0,T);\ \ |u(t) - \bar{u}(t)| \ge \frac{\|u - \bar{u}\|_1}{\varepsilon_D} \right\}. \tag{77}$$

We have the following relation.

Lemma 4.1. *For any $\bar{u}$ and u in $\mathcal{U}$, $u \ne \bar{u}$, we have that* $\operatorname{meas}(B_{\varepsilon_D}(u)) \le \varepsilon_D$.

Proof. This follows from

$$\operatorname{meas}(B_{\varepsilon_D}(u)) = \int_0^T \mathbf{1}_{\{|u(t)-\bar{u}(t)| \ge \frac{\|u-\bar{u}\|_1}{\varepsilon_D}\}} \mathrm{d}t \le \frac{\varepsilon_D}{\|u - \bar{u}\|_1} \int_0^T |u(t) - \bar{u}(t)| \mathrm{d}t = \varepsilon_D.$$

$\square$

Definition 4.2. We say that the *restoration property* (for the initial-final state constraints) is satisfied at $\bar{w} \in F(P)$ for $\mu \in M^L(\bar{w})$ in the Pontryagin (resp. weak) sense if there exists $\varepsilon_P > 0$ and $\varepsilon_B > 0$ such that, for any trajectory w such that $\|w - \bar{w}\|_P \le \varepsilon_P$ (resp. $\|w - \bar{w}\| \le \varepsilon_P$) and $u(t) \in U$ a.e., and any measurable set $B \subset (0,T)$ such that $\operatorname{meas}(B) \le \varepsilon_B$ over which u and $\bar{u}$ coincide, there exists $w' \in F(P)$ such that $u' = \bar{u}$ on B and

$$\|w' - w\|_\infty = O(d(\eta)); \qquad J(w') = J^\mu(w) + O(\|w - \bar{w}\|_P d(\eta)). \tag{78}$$

If $M^L(\bar{w})$ is a singleton, then the concept of restoration property will be used without indication of unique element μ of $M^L(\bar{w})$.

Denote the kernel of derivatives of almost active control constraints (relative to $\bar{w} \in \mathcal{W}$), parameterized by $\varepsilon_R > 0$ (this notation is a reminder of "restoration"), by

$$\mathcal{U}_{\varepsilon_R} := \left\{ \begin{array}{l} v \in \mathcal{U};\ g_i'(\bar{u}(t))v(t) = 0, \text{ whenever } g_i(\bar{u}(t)) \ge -\varepsilon_R, \\ i = 1, \dots, q, \text{ for a.a. } t \in (0,T) \end{array} \right\}. \tag{79}$$

We will call *special qualification condition* the Mangasarian-Fromovitz qualification condition [8] for constraints on initial-final state in K_+, over the Banach space $E_{\varepsilon_R} := \mathcal{U}_{\varepsilon_R} \times \mathbb{R}^n$. Setting $\bar{e} := (\bar{u}, \bar{y}(0))$, it can be formulated as:

$$\left\{ \begin{array}{ll} \text{(i)} & \text{There exist } e^j \subset E_{\varepsilon_R},\ j = 1, \dots, |I_+|, \text{ such that} \\ & \{G'_{I_+}(\bar{e})e^j\}_{j=1,\dots,|I_+|} \text{ is an independent family,} \\ \text{(ii)} & \text{There exists } \varepsilon_R > 0 \text{ and } e^0 \in E_{\varepsilon_R} \cap \operatorname{Ker} G'_{I_+}(\bar{e});\ G'_{I_0}(\bar{e})e^0 < 0. \end{array} \right. \tag{80}$$

This condition implies the uniqueness of the "μ component" of the Lagrange multiplier.

Lemma 4.3. *Let $\bar{w}$ satisfy* (71) *and the special qualification condition* (80). *Then $M^L(\bar{w})$ is a singleton.*

Proof. Let $(\lambda^i, \mu^i) \in LM^L(\bar{w})$, for $i = 1, 2$. Set $\lambda := \lambda^2 - \lambda^1$ and $\mu := \mu^2 - \mu^1$. Let $e = (v, z_0) \in E_{\varepsilon_R}$. Then $\lambda(t) g'(\bar{u}(t)) v(t) = 0$ for a.a. t, and therefore, taking the difference of equations of stationarity of Lagrangians (last relation in (66)), $\mu G'(\bar{u}, \bar{y}_0)(v, z_0) = 0$. Since by (80) $G'_{I_+}(\bar{e})$ is onto from E_{ε_R} onto $\mathbb{R}^{|I_+|}$, it follows that $\mu = 0$, as was to be proved. $\square$

We need the following notation:

$$\mathcal{U}_{\varepsilon_R}(B) := \{v \in \mathcal{U}_{\varepsilon_R};\ v(t) = 0 \text{ a.e. on } t \in B\}; \qquad E_{\varepsilon_R}(B) := \mathcal{U}_{\varepsilon_R}(B) \times \mathbb{R}^n. \tag{81}$$

Denote by $e^j = (u^j, z_0^j)$ the components of vectors e^j in (80), $j = 0, \dots, |I_+|$, and define $e_B^j = (u_B^j, z_{0B}^j)$ in $E_{\varepsilon_R}(B)$, for $j = 1, \dots, |I_+|$, by

$$z_{0B}^j = z_0^j; \qquad u_B^j(t) = u^j(t) \ \text{ if } t \notin B, \ \ u_B^j(t) = 0 \ \text{ otherwise.} \tag{82}$$

Since the functions u^j, $j = 1, \dots, |I_+|$, are essentially bounded, we have that

$$|G'(\bar{e})(e_B^j - e^j)| = O(\text{meas}(B)), \quad j = 1, \dots, |I_+|. \tag{83}$$

For $j = 0$, we proceed in two steps. First define $\hat{e}_B^0 = (\hat{u}_B^0, z_{0B}^0)$ in the same way, i.e.,

$$z_{0B}^0 = z_0^0; \qquad \hat{u}_B^0(t) = u^0(t) \ \text{ if } t \notin B, \ \ \hat{u}_B^0(t) = 0 \ \text{ otherwise.} \tag{84}$$

Then

$$|G'(\bar{e})(\hat{e}_B^0 - e^0)| = O(\text{meas}(B)). \tag{85}$$

Now define

$$e_B^0 := \hat{e}_B^0 + \sum_{j=1}^{|I_+|} \bar{\alpha}_j e_B^j, \tag{86}$$

where the coefficient $\bar{\alpha}$ is the solution of

$$G'_{I_+}(\bar{e}) \left(\hat{e}_B^0 + \sum_{j=1}^{|I_+|} \alpha_j e_B^j \right) = 0. \tag{87}$$

The L^1 norm over $E_{\varepsilon_R}(B)$ is defined as the sum of the L^1 norms over $\mathcal{U}_{\varepsilon_R}$ and $\mathbb{R}^n$.

Lemma 4.4. *Let $\bar{w}$ satisfy* (71) *and the special qualification condition* (80). *Then there exists $\varepsilon_B > 0$ such that, for any measurable subset B of $[0, T]$, if* $\text{meas}(B) \leq \varepsilon_B$, *then* (87) *has a unique solution, and $\{e_B^0, \dots, e_B^{|I_+|}\}$ are such that*

$$\begin{cases} \text{(i)} & |\bar{\alpha}| = O(\text{meas}(B)); \\ \text{(ii)} & \|e_B^0 - e^0\|_1 = O(\text{meas}(B)), \\ \text{(iii)} & G'_{I_0}(\bar{e}) e_B^0 < \frac{1}{2} G'_{I_0}(\bar{e}) e^0 < 0. \end{cases} \tag{88}$$

Proof. It suffices to prove (88)(i). Over the set of square invertible matrices of size $|I_+|$, the operator that to a matrix associates its inverse is known to be locally Lipschitz. When $\text{meas}(B) = 0$, the solution is $\bar{\alpha} = 0$; otherwise the perturbation of the matrix and of the r.h.s. is, by (83) and (85), of order $O(\text{meas}(B))$. The result follows. $\square$

Lemma 4.5. *Let $\bar{w}$ satisfy* (71) *and the special qualification condition* (80), *for some $\varepsilon_R > 0$. Let $M > \|\bar{u}\|_\infty$. Then, if $\hat{\varepsilon}_B > 0$ and $\varepsilon_P > 0$ are small enough, for any measurable $\hat{B} \subset (0,T)$ such that* $\text{meas}(\hat{B}) \le \hat{\varepsilon}_B$, *any trajectory w such that $\|u\|_\infty \le M$ and $\|w - \bar{w}\|_P \le \varepsilon_P$, there exists a trajectory $w'' \in \mathcal{W}$ (that does not in general satisfy the control constraints) such that*

$$\|w'' - w\|_\infty = O(d(\eta)); \qquad G(u'', y_0'') \in K_+; \qquad u'' - u \in \mathcal{U}_{\varepsilon_R}(\hat{B}). \tag{89}$$

Proof. Denote by $F_{\hat{B}}$ the finite dimensional space spanned by $e^0_{\hat{B}}, \dots, e^{|I_+|}_{\hat{B}}$, endowed with the norm of $\mathcal{U} \times \mathbb{R}^n$. Consider the mapping $\mathcal{T}$ from $\mathcal{U} \times \mathbb{R}^n$ into itself, that with a given $e := (u, y_0) \in \mathcal{U} \times \mathbb{R}^n$ associates $e + \delta e$, where $\delta e = (\delta u, \delta y_0)$ is a vector of $F_{\hat{B}}$ satisfying the following conditions:

$$\text{(i)} \;\; G_{I_+}(e) + G'_{I_+}(\bar{e})\delta e = 0; \qquad \text{(ii)} \;\; G_{I_0}(e) + G'_{I_0}(\bar{e})\delta e \le 0, \tag{90}$$

so we can write $\delta e = \sum_{j=0}^{|I_+|} \theta_j e^j_{\hat{B}}$. Coefficients θ_j, for $j = 1$ to $|I_+|$, are uniquely determined by (90)(i), and for θ_0 we choose the smallest possible nonnegative value. In view of Lemma 4.4, we have that, for some $c_1 > 0$, if $\varepsilon_{\hat{B}}$ is small enough:

$$\|\delta e\|_\infty \le c_1 \left(|G_{I_+}(e)| + |G_{I_0}(e)_+| \right). \tag{91}$$

Over the set $\mathcal{V}_{1,M} := \{w \in \mathcal{W};\; \|w - \bar{w}\|_P \le 1;\; \|u\|_\infty \le M\}$, the mapping $e \mapsto G'(e)$ is, when E_{ε_R} is endowed with the L^1 norm, Lipschitz from E_{ε_R} into $L(E_{\varepsilon_R}, \mathbb{R}^r)$. Using $G(e+\delta e) = G(e) + G'(e)\delta e + \int_0^1 [G'(e + \sigma\delta e) - G(e)]\mathrm{d}\sigma$, it follows that, for some $c_2 > 0$:

$$|G(e + \delta e) - G(e) - G'(e)\delta e| \le c_2 \|\delta e\|_P^2. \tag{92}$$

In view of (90) and (91), and since (making the abuse of notation of denoting also by P the induced norm for $G'(e)$) $\|G'(\bar{e}) - G'(e)\|_P \to 0$ when $\|u\|_\infty \le M$ and $\|w - \bar{w}\|_P \to 0$, it follows that when $\|w - \bar{w}\|_P$ is small enough, say $\|w - \bar{w}\|_P \le \varepsilon_{P_1}$, we have that

$$|G_{I_+}(e + \delta e)| + |G_{I_0}(e + \delta e)_+| \le \tfrac{1}{2}(|G_{I_+}(e)| + |G_{I_0}(e)_+|). \tag{93}$$

Consider the sequence defined by $e^{k+1} := \mathcal{T}(e^k)$, for $k \in I\!N$. It follows from (91) and (93) that, if the trajectory w corresponding to e^0 satisfies the hypotheses of the lemma for small enough ε_P, then the sequence w^k is well-defined, remains in $\mathcal{V}_{1,M}$, and converges to some e'' such that $\|e'' - e^0\| \le 2c_1 \left(|G_{I_+}(e)| + |G_{I_0}(e)_+| \right)$. The corresponding associated state y'' is such that $w'' := (u'', y'')$ satisfies (89). $\square$

Given $w \in \mathcal{W}$ satisfying the state equation (2), we consider the following measure of constraint defect:

$$D(w) := \|g(u)_+\|_\infty + \sum_{i=1}^{r_1} |\Phi_i(\eta)| + \sum_{i=r_1+1}^{r} \Phi_i(\eta)_+. \tag{94}$$

It is well-known that Robinson's qualification condition (61) implies the following local metric regularity result, due to [14] (see also [2, Prop. 2.89]): there exists $c > 0$ such that, if $w \in \mathcal{W}$ satisfies the state equation (2), and is close enough to $\bar{w}$, then there exists $\hat{w} \in F(P)$ such that

$$\|\hat{w} - w\|_{\mathcal{W}} \le cD(w). \tag{95}$$

We now check that this inequality still holds in some cases if $w \in \mathcal{W}$ is close enough to $\bar{w}$ in the Pontryagin norm.

Lemma 4.6. *Let $\bar{w}$ satisfy* (71) *and Robinson's qualification condition* (61). *Then the following metric regularity result holds. For any $M > \|\bar{u}\|_\infty$, there exists $c > 0$ and $\varepsilon_M > 0$ such that, for any measurable subset B of $[0,T]$ and any trajectory w such that $g(u(t)) \leq 0$ a.e. on B, setting $A := [0,T] \setminus B$, if*

$$\|u\|_\infty \leq M; \qquad \|w - \bar{w}\|_P \leq \varepsilon_M; \qquad \text{meas}(B) \leq \varepsilon_M; \qquad \|(u - \bar{u})\mathbf{1}_A\|_\infty \leq \varepsilon_M, \tag{96}$$

there exists $\hat{w} \in F(P)$ such that (95) *holds, and $\hat{u}(t) = u(t)$ a.e. on B.*

Proof. The proof is somewhat in the spirit of the one of Lemma 4.5, but including the control constraints. The first step is similar to Lemma 4.4. Denote $\bar{e} := (\bar{u}, \bar{y}(0))$. By Robinson's condition (61) there exist $\{e^1, \dots, e^{r_1}\}$ in $\mathcal{U} \times \mathbb{R}^n$ such that $\{G'_{1:r_1}(\bar{e})e^i\}_{1 \leq i \leq r_1}$ is of rank r_1. Denote by e^i_B the vector obtained by setting to zero the components of the control over the measurable subset B of $[0,T]$. Then

$$|G'(\bar{e})(e^j_B - e^j)| = O(\text{meas}(B)), \quad j = 1, \dots, r_1. \tag{97}$$

Let $e^0 := (\bar{v}, \bar{z}_0)$ and $\beta > 0$ be the direction and constant stated in (61). By arguments similar to those in the proof of Lemma 4.4 we obtain the existence of a direction $e^0_B = (v^0_B, z^0_B) \in \mathcal{U} \times \mathbb{R}$ such that

$$\begin{cases} \text{(i)} & \|e^0_B - e^0\|_P = O(\text{meas}(B)); \\ \text{(ii)} & g(\bar{u}(t)) + g'(\bar{u}(t))v^0_B(t) \leq -\frac{1}{2}\beta, \quad \text{for a.a. } t \in [0,T] \setminus B, \\ \text{(iii)} & v^0_B(t) = 0, \quad \text{for a.a. } t \in B, \\ \text{(iv)} & G'_i(\bar{e})e^0_B = 0, \quad i = 1, \dots, r_1, \\ \text{(v)} & G'_i(\bar{e})e^0_B \leq -\frac{1}{2}\beta, \quad \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{cases} \tag{98}$$

Let $\mathcal{T}$ be the mapping from $\mathcal{U} \times \mathbb{R}^n$ into itself, that with a given $(u, y_0) \in \mathcal{U} \times \mathbb{R}^n$ associates $(u, y_0) + \delta e$, where $\delta e = (\delta u, \delta y_0)$ is a vector of $\text{Span}\{e^0_B, \dots, e^{r_1}_B\}$ of minimum norm satisfying the following conditions:

$$\begin{cases} \text{(i)} & g(u(t)) + g'(\bar{u}(t))\delta u(t) \leq 0, \quad \text{for a.a. } t \in [0,T], \\ \text{(ii)} & G_i(e) + G'_i(\bar{e})\delta e = 0, \quad i = 1, \dots, r_1, \\ \text{(iii)} & G_i(e) + G'_i(\bar{e})\delta e \leq 0, \quad \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{cases} \tag{99}$$

We obtain by arguments similar to those at the end of the proof of Lemma 4.5 that the sequence computed by the mapping $\mathcal{T}$, with initial point (u, y_0) such that $\|u\|_\infty \leq M$ and $\|w - \bar{w}\|_P$ is small enough converges to a point $(\hat{u}, \hat{y}_0)$ such that the corresponding associated state $\hat{y}$ satisfies the conclusion of the lemma. □

We remind that (80) implies that $M^L(\bar{w})$ is a singleton (see Lemma 4.3).

Lemma 4.7. *Let $\bar{w}$ satisfy* (61), (71) *and* (80). *Then the restoration property (Definition* 4.2*) is satisfied in the Pontryagin sense.*

Proof. Let the trajectory w satisfy $g(u(t)) \leq 0$ a.e. on $[0,T]$, Given a constant $M > \max\{\|u\|_\infty, \|\bar{u}\|_\infty\}$, choose $\hat{\varepsilon}_B$ such that Lemma 4.5 applies. Let B be a measurable subset of $(0,T)$, over which $u = \bar{u}$ a.e., of measure at most $\varepsilon_B := \frac{1}{2}\hat{\varepsilon}_B$. Let B_{ε_D} be the deviation set introduced in (77), with $\varepsilon_D := \frac{1}{2}\hat{\varepsilon}_B$. By Lemma 4.1, the set $\hat{B} := B \cup B_{\varepsilon_D}$ is of measure at most $\hat{\varepsilon}_B$. Set $\hat{A} := (0,T) \setminus \hat{B}$. Since $B_{\varepsilon_D} \cap \hat{A} = \emptyset$, we have that

$$|u(t) - \bar{u}(t)| \leq {\varepsilon_D}^{-1} \|w - \bar{w}\|_P, \quad \text{a.e. on } \hat{A}. \tag{100}$$

By Lemma 4.5, there exists a trajectory $w'' \in W$ satisfying (89). Using (89) and (100) we get that, for some $c_1 > 0$ and for a.a. $t \in \hat{A}$:

$$\begin{aligned} g_i(u''(t)) &= g_i(u(t)) + g_i'(u(t))(u''(t) - u(t)) + O(d(\eta)^2), \\ &\leq g_i(u(t)) + g_i'(\bar{u}(t))(u''(t) - u(t)) + c_1 \left(\|w - \bar{w}\|_P d(\eta) + d(\eta)^2 \right). \end{aligned} \tag{101}$$

If $t \in \hat{B}$ then $g_i(u''(t)) = g_i(u(t)) \leq 0$. Otherwise, if $g_i(\bar{u}(t)) \geq -\varepsilon_R$ (we remind that ε_R was introduced in (79)), since $g_i(u(t)) \leq 0$ and $g_i(\bar{u}(t))(u''(t) - u(t)) = 0$, we get

$$g_i(u''(t)) \leq c_1 \left(\|w - \bar{w}\|_P d(\eta) + d(\eta)^2 \right). \tag{102}$$

If $g_i(\bar{u}(t)) < -\varepsilon_R$ and $t \notin \hat{B}$, then by (100), whenever ε_P is small enough, we have $g_i(u(t)) < -\frac{1}{2}\varepsilon_R$. Using (101) and the estimate $d(\eta) = O(\|w - \bar{w}\|_P)$, again whenever ε_P is small enough,we obtain:

$$g_i(u''(t)) \leq -\tfrac{1}{2}\varepsilon_R + O(d(\eta)) \leq -\tfrac{1}{2}\varepsilon_R + O(\varepsilon_P) \leq 0, \tag{103}$$

so that finally with (102), and using again $d(\eta) = O(\|w - \bar{w}\|_P)$:

$$g(u''(t)) \leq c_2 \|w - \bar{w}\|_P d(\eta), \quad \text{a.e. on } (0,T). \tag{104}$$

Let μ be the element of the singleton $M^L(\bar{w})$. We next apply Lemma 2.13 at point w, denoting therefore $\hat{p}^\mu$ the costate evaluated at the point w (and not $\bar{w}$). Note that $o(\cdot)$ and $O(\cdot)$ in the statement of this theorem come from Taylor expansions on bounded sets, and hence, are uniform over the reference point. Since $\|w'' - w\|_\infty = O(d(\eta))$, we deduce that, denoting by p^μ the costate evaluated at the point $\bar{w}$ and associated with μ:

$$\begin{aligned} J^\mu(w'') - J^\mu(w) = &\int_0^T H_u(w, \hat{p}^\mu)(u'' - u)\mathrm{d}t \\ &+ (\hat{p}^\mu(0) + (\Phi^\mu_{y_0})(\eta))(y''(0) - y(0)) + O(d(\eta)^2). \end{aligned} \tag{105}$$

Since $\|w - \bar{w}\|_P \leq \varepsilon_P$, using

$$\begin{aligned} \left| \left(\hat{p}^\mu(0) + \Phi^\mu_{y_0}(\eta)\right) - \left(p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta})\right) \right| &= O(\|w - \bar{w}\|_P), \\ \|H_u(w, \hat{p}^\mu) - H_u(\bar{w}, p^\mu)\|_1 &= O(\|w - \bar{w}\|_P) \end{aligned} \tag{106}$$

as well as $p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}) = 0$, $\|w'' - w\|_\infty = O(d(\eta))$, and $d(\eta) = O(\|w'' - w\|_P)$, we can write

$$J^\mu(w'') - J^\mu(w) = \int_0^T H_u(\bar{w}, p^\mu)(u'' - u)\mathrm{d}t + O(\|w - \bar{w}\|_P d(\eta)). \tag{107}$$

Since $u'' - u \in \mathcal{U}_{\varepsilon_R}$, we deduce from the first-order optimality conditions that

$$H_u(\bar{w}(t), p^\mu(t))(u''(t) - u(t)) = -\lambda_t g'(\bar{u}(t))(u''(t) - u(t)) = 0. \tag{108}$$

With (107)–(108), we get $J^\mu(w'') - J^\mu(w) = O(\|w - \bar{w}\|_P d(\eta))$. Now by the definition (89) of w'', we have that $J(w'') = J^\mu(w'')$. By Lemma 4.6, (89) and (104), there exists $w' \in F(P)$, such that $\|w' - w''\|_{\mathcal{W}} = O(\|w - \bar{w}\|_P d(\eta))$ and $u'(t) = u''(t) = u(t)$ a.e. on $\hat{B}$. Consequently, $J(w') - J(w'') = O(\|w - \bar{w}\|_P d(\eta))$; therefore (78) holds and $u' = \bar{u}$ on B (since $B \subset \hat{B}$). The conclusion follows. □

As shows the following theorem, the quadratic growth condition for the Hamiltonian makes a bridge between the notions of weak and bounded strong quadratic growth of the cost functional J.

Theorem 4.8. *a) Let $\bar{w} \in F(P)$ satisfy the qualification condition* (61)*. Then the bounded strong quadratic growth condition* (9) *at $\bar{w}$ implies the three following conditions:*

(*i*) *the weak minimum quadratic growth condition* (8)*,*
(*ii*) *Pontryagin's principle with a unique minimum of the Hamiltonian over U for some $\mu \in M^P(\bar{w})$, and*
(*iii*) *the local quadratic growth condition for Hamiltonian* (31)*, for some $\mu \in M^P(\bar{w})$.*

b) Conversely, if (i)–(iii) holds as well as the restoration property (78) *in the weak sense, for some $\mu \in \mathrm{ri}(M^P(\bar{w}))$, then the bounded strong quadratic growth condition holds at $\bar{w}$.*

Proof. *a*) Let $\bar{w}$ satisfy the bounded strong quadratic growth condition. Then of course the condition of weak quadratic growth (8) holds. In addition, by the definition (9) of bounded strong quadratic growth, for any $M > \|\bar{u}\|_\infty$, there exists $\varepsilon_M > 0$, such that $\bar{w}$ attains the minimum of $J(w) - \frac{1}{2}\alpha_M \|w - \bar{w}\|_2^2$ over the set

$$\mathcal{W}_M := \{w \in F(P);\ \|y - \bar{y}\|_\infty \le \varepsilon_M;\ |u(t)| \le M \text{ a.e.}\}, \tag{109}$$

By Pontryagin's principle applied to this problem, the local quadratic growth condition for Hamiltonian function (31) follows, as well as the uniqueness minimum of the Hamiltonian over U for some $\mu \in M^P(\bar{w})$.

b) Let $\bar{w}$ satisfy conditions (*i*)–(*iii*) of the theorem, as well as the restoration property (78). If the bounded strong quadratic growth condition is not satisfied, there exists a sequence of bounded strong perturbation w_k such that

$$J(w_k) \le J(\bar{w}) + o(\|w_k - \bar{w}\|_2^2). \tag{110}$$

By Corollary 2.9, w_k is a Pontryagin perturbation of $\bar{w}$. Let $M > \sup_k \|u_k\|_\infty$. We know that (33) holds for some constants $\alpha > 0$ and $\varepsilon_M > 0$. Let $\varepsilon > 0$ be such that $\alpha\varepsilon^2 < \varepsilon_M$, and set

$$B_k := \{t \in (0, T);\ |u_k(t) - \bar{u}(t)| > \varepsilon\}; \qquad A_k := (0, T) \setminus B_k. \tag{111}$$

In view of Lemma 2.4 we may assume that conditions (*ii*) and (*iii*) hold for the same $\mu \in \mathrm{ri}(M^P(\bar{w}))$, as well as the (weak) restoration property (78). By (31) and (33), as well as by Theorem 2.14 and Lemma 4.1, since $\mathrm{meas}(B_k) \to 0$, we have that

$$\begin{aligned} J^\mu(w_k) &= J^\mu(w_{A,k}) + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu)]\mathrm{d}t + r_{2k} \\ &\geq J^\mu(w_{A,k}) + \alpha\varepsilon^2 \,\mathrm{meas}(B_k) + r_{2k}. \end{aligned} \tag{112}$$

Applying the (weak) restoration property (78) to $w_{A,k}$, get the existence of $w'_{A,k} \in F(P)$ such that

$$\begin{cases} \text{(i)} & \|w'_{A,k} - w_{A,k}\| = O(d(\eta_{A,k})); \\ \text{(ii)} & J(w'_{A,k}) = J^\mu(w_{A,k}) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})). \\ \text{(iii)} & u'_{A,k}(t) = u_{A,k}(t) = \bar{u}(t) \quad \text{a.e. on } B_k. \end{cases} \tag{113}$$

Combining (113)(ii) with (112), we obtain

$$J^\mu(w_k) \geq J(w'_{A,k}) + \alpha\varepsilon^2 \,\mathrm{meas}(B_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})) + r_{2k}. \tag{114}$$

On the other hand, since $\|y_{B,k} - \bar{y}\|_\infty = O(\mathrm{meas}(B_k))$, we have that

$$\Phi(\eta_k) = \Phi(\eta_{A,k}) + O(\mathrm{meas}(B_k)). \tag{115}$$

We deduce that

$$d(\eta_{A,k}) = d(\eta_k) + O(\mathrm{meas}(B_k)). \tag{116}$$

Also, since $w_k \in F(P)$, we have $\Phi_{1:r_1}(\eta_k) = 0$ and $\Phi_{r_1+1:r}(\eta_k) \leq 0$. Combining (114) and (116) and using (75), we get

$$\begin{aligned} J(w_k) &\geq J(w'_{A,k}) + \alpha\varepsilon^2 \,\mathrm{meas}(B_k) - \mu\Phi(\eta_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})) + r_{2k} \\ &\geq J(w'_{A,k}) + \tfrac{1}{2}\alpha\varepsilon^2 \,\mathrm{meas}(B_k) - \mu\Phi(\eta_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_k)) + r_{2k} \\ &\geq J(w'_{A,k}) + \tfrac{1}{2}\alpha\varepsilon^2 \,\mathrm{meas}(B_k) - \tfrac{1}{2}\mu\Phi(\eta_k) + r_{2k}. \end{aligned} \tag{117}$$

Since $w'_{A,k}$ is feasible, by (113)(i), we have that

$$\begin{aligned} \|w'_{A,k} - \bar{w}\| &= O\left(\|u'_{A,k} - \bar{u}\|_\infty + |y'_{A,k,0} - \bar{y}_0|\right) \\ &= O\left(\|u_{A,k} - \bar{u}\|_\infty + |y_{A,k,0} - \bar{y}_0| + d(\eta_{A,k})\right) = O(\varepsilon) + o(1). \end{aligned} \tag{118}$$

Since $\bar{w}$ is a weak minimum, for small enough $\varepsilon > 0$ we get $J(w'_{A,k}) \geq J(\bar{w})$. Thus, taking into account that $\frac{1}{2}\mu\Phi(\eta_k) \leq 0$, we find that (117) implies

$$J(w_k) \geq J(\bar{w}) + \tfrac{1}{2}\alpha\varepsilon^2 \,\mathrm{meas}(B_k) + r_{2k}. \tag{119}$$

In view of (110), $\mathrm{meas}(B_k) = r_{2k}$. By Lemma 4.6 and (116), there exists $w'_k \in F(P)$ such that $\|w'_k - w_{A,k}\|_\infty = r_{2k}$. Therefore, $\|w'_k - \bar{w}\|_2^2 = \|w_{A,k} - \bar{w}\|_2^2 + r_{2k}$ so that, using Theorem 2.14 (with here $\mu = 0$) for the first equality, when $\varepsilon > 0$ is small enough,

$$\begin{aligned} J(w_k) = J(w_{A,k}) + r_{2k} = J(w'_k) + r_{2k} &\geq J(\bar{w}) + \tfrac{1}{2}\alpha\|w_{A,k} - \bar{w}\|_2^2 + r_{2k}, \\ &= J(\bar{w}) + \tfrac{1}{2}\alpha\|w_k - \bar{w}\|_2^2 + r_{2k}, \end{aligned} \tag{120}$$

contradicting (110). ◻

5. Second-order necessary or sufficient conditions

5.1. Critical directions

Since the qualification hypothesis (61) is a particular case of Robinson's qualification condition, the second-order optimality condition due to Cominetti [3] (see also [2, Thm. 3.45]) holds. We denote $\bar{y}(0)$ as $\bar{y}_0$, and recall that $\hat{J}(u, y_0) := J(u, y_{u,y_0})$, where y_{u,y_0} is the solution of the state equation (2), with initial condition $y(0) = y_0$, and that G was defined in (60). Reminding the notations in (62) and (64), define the following set of active inequalities at time t and for the initial-final state constraints:

$$I_t := \{1 \leq i \leq q;\ g_i(\bar{u}(t)) = 0\}; \qquad I_F := \{r_1 + 1 \leq j \leq r;\ G_j(\bar{u}, \bar{y}_0) = 0\}. \tag{121}$$

Set $\mathcal{U}_2 := L^2(0, T, \mathbb{R}^m)$. The linear mappings $\hat{J}'(\bar{u}, \bar{y}_0)$ and $G'(\bar{u}, \bar{y}_0)$, defined over $\mathcal{U} \times \mathbb{R}^n$, have a unique extension into $\mathcal{U}_2 \times \mathbb{R}^n$, denoted in the same way. We define the set of *extended* tangent directions to the control and initial-final state constraints (they are extended in the sense that we take L^2 spaces instead of L^∞):

$$T_g(\bar{u}) := \{v \in \mathcal{U}_2;\ g'_{I_t}(\bar{u}(t))v(t) \leq 0,\ \text{a.a. } t \in (0, T)\}, \tag{122}$$

$$T_\Phi(\bar{u}, \bar{y}_0) := \{(v, z_0) \in \mathcal{U}_2 \times \mathbb{R}^n;\ G'_{1:r_1}(\bar{u}, \bar{y}_0)(v, z_0) = 0;\ G'_{I_F}(\bar{u}, \bar{y}_0)(v, z_0) \leq 0\}, \tag{123}$$

the set of *extended* critical directions:

$$C_2(\bar{u}, \bar{y}_0) := \{(v, z_0) \in T_\Phi(\bar{u}, \bar{y}_0);\ v \in T_g(\bar{u});\ \hat{J}'(\bar{u}, \bar{y}_0)(v, z_0) \leq 0\}. \tag{124}$$

The set of critical directions (in the original space) is

$$C_\infty(\bar{u}, \bar{y}_0) := \{(v, z_0) \in C_2(\bar{u}, \bar{y}_0);\ v \in L^\infty(0, T, \mathbb{R}^m)\}. \tag{125}$$

Finally $T^2_-(g(\bar{u}), g'(\bar{u})v)$ stands for the second-order tangent set to $L^\infty(0, T, \mathbb{R}^q_-)$ at the point $g(\bar{u})$, in the direction $g'(\bar{u})v$, i.e., for $s > 0$:

$$T^2_-(g(\bar{u}), g'(\bar{u})v) = \{\hat{v} \in L^\infty(0, T, \mathbb{R}^q);\ \operatorname*{supess}_t(g(\bar{u}) + sg'(\bar{u})v + \tfrac{1}{2}s^2\hat{v}) \leq o(s^2)\}. \tag{126}$$

If K is a subset of a Banach space $\mathcal{X}$, its support function $\sigma(\cdot, K) : \mathcal{X}^* \to \bar{\mathbb{R}}$ is defined by $\sigma(x^*, K) := \sup\{\langle x^*, x\rangle;\ x \in K\}$. By Cominetti [3], we know that

Theorem 5.1. *Let $\bar{w} = (\bar{u}, \bar{y})$ be a weak solution of (P), satisfying the qualification hypothesis* (61)*. Then*

$$\begin{aligned} &\max_{(\lambda,\mu)\in LM^L(\bar{w})} \big(\mathcal{L}_{(u,y_0)^2}(\bar{u}, \bar{y}_0, \lambda, \mu)(v, z_0)^2 - \sigma(\lambda, T^2_-(g(\bar{u}), g'(\bar{u})v)\big) \geq 0, \\ &\qquad \textit{for all } (v, z_0) \in C_\infty(\bar{u}, \bar{y}_0). \end{aligned} \tag{127}$$

The amount $\sigma(\lambda, T^2_-(g(\bar{u}), g'(\bar{u})v)$ is called the "sigma-term". A practical characterization of the second-order tangent set to L^∞ is not known (see however [4]). We do not detail the proof of Theorem 5.1 since it is a standard application of [3]; note that, since the constraint on the initial-final state consists in a finite number of inequalities they have no contribution to the sigma-term. We next prove a stronger result. Let us denote

$$\Omega(v, z_0) := \max_{(\lambda,\mu)\in LM^L(\bar{w})} \mathcal{L}_{(u,y_0)^2}(\bar{u}, \bar{y}_0, \lambda, \mu)(v, z_0)^2. \tag{128}$$

Note that the maximum is indeed attained since the set of multipliers is, by the Banach-Alaoglu theorem, compact for the weak∗ topology of L^∞, and the function to be maximized is a continuous linear form for that topology. Since the sigma term is nonpositive (e.g. [2, Equation (3.110)]), and $C_\infty(\bar{u}, \bar{y}_0) \subset C_2(\bar{u}, \bar{y}_0)$, a sufficient condition for (127) is

$$\Omega(v, z_0) \geq 0, \quad \text{for all } (v, z_0) \in C_2(\bar{u}, \bar{y}_0). \tag{129}$$

Note that the above condition makes sense since, for any $(\lambda, \mu) \in LM^L(\bar{w})$, the quadratic form $(v, z_0) \mapsto \mathcal{L}_{(u,y_0)^2}(\bar{u}, \bar{y}_0, \lambda, \mu)(v, z_0)^2$, defined over $\mathcal{U} \times \mathbb{R}^n$, has a unique extension into $\mathcal{U}_2 \times \mathbb{R}^n$.

Theorem 5.2. *Let $\bar{w} = (\bar{u}, \bar{y})$ be a weak solution of (P), satisfying the qualification hypotheses* (61) *and* (80). *Then condition* (129) *holds.*

Proof. Define the set of times with small negative constraints as

$$I_\varepsilon = \{t \in (0, T);\ -\varepsilon \leq g_i(\bar{u}(t)) < 0, \text{ for some } 1 \leq i \leq q\}. \tag{130}$$

Let $(v, z_0) \in C_2(\bar{u}, \bar{y}_0)$ be an extended critical direction. For given $\varepsilon > 0$, consider the perturbed direction $(v'_\varepsilon, z_0) \in \mathcal{U} \times \mathbb{R}^n$, defined by

$$v'_\varepsilon(t) = 0 \text{ if either } t \in I_\varepsilon, \text{ or } |v(t)| > \varepsilon^{-1}; \quad v'_\varepsilon(t) = v(t) \text{ otherwise.} \tag{131}$$

By (80), where vectors e^k were defined, there exists $(v_\varepsilon, z_{0\varepsilon}) \in C_\infty(\bar{u}, \bar{y}_0)$ of the form

$$(v_\varepsilon, z_{0\varepsilon}) = (v'_\varepsilon, z_0) + \sum_{k=0}^{|I_+|} \alpha_{\varepsilon,k} e^k \tag{132}$$

such that $v_\varepsilon - v'_\varepsilon \in \mathcal{U}_{\varepsilon_R}$, $|\alpha_\varepsilon| \to 0$ when $\varepsilon \downarrow 0$, and so $\|(v_\varepsilon, z_{0\varepsilon}) - (v'_\varepsilon, z_0)\|_\infty \to 0$. When $\varepsilon < \varepsilon_R$, if $-\varepsilon \leq g_i(\bar{u}(t))$, we have that $v'_\varepsilon(t) = 0$ and $g'_i(\bar{u}(t))e^k(t) = 0$. It follows that $g(\bar{u}) + \rho g'(\bar{u})v_\varepsilon \leq 0$ when $\rho > 0$ is small enough. In that case we know (e.g. [2, Remark 3.47]) that $\sigma(\lambda, T^2_-(g(\bar{u}), g'(\bar{u})v_\varepsilon)) = 0$. Therefore, by Theorem 5.1, $\Omega(v_\varepsilon, z_{0\varepsilon}) \geq 0$, so that there exists $\lambda_\varepsilon \in L^\infty(0, T, \mathbb{R}^{q*})$ such that $(\lambda_\varepsilon, \mu) \in LM^L(\bar{w})$ and

$$\mathcal{L}_{(u,y_0)^2}(\bar{u}, \bar{y}_0, \lambda_\varepsilon, \mu)(v_\varepsilon, z_{0\varepsilon})^2 \geq 0. \tag{133}$$

Since the set of Lagrange multipliers is bounded in L^∞, λ_ε has weak∗ limit points when $\varepsilon \downarrow 0$. As $(v_\varepsilon, z_{0\varepsilon}) \to (v, z_0)$ in the L^2 norm (so that the term in product of λ_ε strongly converges in the L^1 norm), we may pass to the limit in this inequality when $\varepsilon \to 0$. The conclusion follows. □

Remark 5.3. The method of proof is a variant of the one used for "extended polyedricity", see [2, Section 3.2.3]. The basic concept there is the one of *radial* critical directions, i.e., critical directions v for which there exists $\kappa > 0$ such that (in our notations) $g(u) + \kappa g'(u)v \leq 0$. Note that the L^∞ regularity of the multiplier compensates the fact that extended critical directions belong to L^2 spaces.

We next discuss relations with weak quadratic growth, defined in (8).

Corollary 5.4. *Let $\bar{w}$ be a weak solution of (P), satisfying the qualification hypotheses* (61) *and* (80) *and the weak quadratic growth condition* (8) *with parameter α. Then*

$$\Omega(v, z_0) \geq \alpha(\|v\|_2^2 + |z_0|^2), \quad \textit{for all } (v, z_0) \in C_2(\bar{u}, \bar{y}_0). \tag{134}$$

This is a consequence of the second-order necessary condition for the problem of minimizing over $F(P)$ the perturbed cost function (the proof follows the one of Theorem 5.2)

$$\hat{J}_\alpha(u, y_0) := \hat{J}(u, y_0) - \tfrac{1}{2}\alpha\left(\|u - \bar{u}\|_2^2 + |y_0 - \bar{y}_0|^2\right). \tag{135}$$

5.2. Characterization of the weak quadratic growth condition

We next characterize the weak quadratic growth condition. The difficult part will be in the proof of the sufficient condition, in which, due to the nonconvexity of the Hamiltonian, so that Ω is not necessarily weakly l.s.c., we have to avoid to pass to weak limit.

Let $\bar{w} \in F(P)$ satisfy the special qualification hypothesis (80). By Lemma 4.3, the projection $M^L(\bar{w})$ of the set $LM^L(\bar{w})$ on the second component is a singleton $\{\bar{\mu}\}$. We consider the following two conditions: *uniform local quadratic growth of Hamiltonian functions along the trajectory* (analogous to (31)) i.e.,

$$\begin{cases} \text{There exists } c_H > 0, \ \varepsilon_\infty > 0; \quad \text{for a.a. } t \in (0, T), \\ H(u, \bar{y}(t), \bar{p}(t)) \geq H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) + c_H |u - \bar{u}(t)|^2, \\ \text{for all } u \in \mathbb{R}^m; \ g(u) \leq 0; \ |u - \bar{u}(t)| \leq \varepsilon_\infty, \\ \text{where } \bar{p} \text{ is the costate associated with } \bar{\mu}, \end{cases} \tag{136}$$

and *uniform quadratic growth along critical directions* (134).

Theorem 5.5. *Let $\bar{w} \in F(P)$ satisfy the qualification hypotheses* (61) *and* (80). *Then $\bar{w}$ satisfies the weak quadratic growth condition iff* (134) *and* (136) *hold.*

Proof. Let $\bar{w} \in F(P)$ satisfy (61) and (80). If $\bar{w}$ satisfies the weak quadratic growth condition, by Corollary 5.4, (134) holds for some $\alpha > 0$, and (reducing α if necessary) we may assume that $\bar{w}$ is a weak solution of the perturbed problem obtained by replacing the cost function J with $J(w) - \frac{1}{2}\alpha\|w - \bar{w}\|_2^2$. Applying Remark 2.3 to this perturbed problem, we obtain (136).

Conversely, if $\bar{w}$ satisfies (134) and (136), let us show by contradiction that the weak quadratic growth condition holds. So assume that there exist a sequence $w_k \in F(P)$ such that $w_k \to \bar{w}$ in $\mathcal{W}$, $w_k \neq \bar{w}$ for all k, and

$$J(w_k) \leq J(\bar{w}) + o(\|w_k - \bar{w}\|_2^2). \tag{137}$$

Define the *local critical cone* C_t at time $t \in [0, T]$ as (I_t was defined in (121)):

$$C_t \ := \{v \in \mathbb{R}^m; \ H_u(\bar{u}(t), \bar{y}(t), \bar{p}(t))v \leq 0; \ g_i'(\bar{u}(t))v \leq 0, i \in I_t\}. \tag{138}$$

Note that, by (124):

$$C_2(\bar{u}, \bar{y}_0) = \{(v, z_0) \in T_\Phi(\bar{u}, \bar{y}_0);\ v(t) \in C_t \text{ for a.a. } t;\ \bar{\mu} G'(\bar{u}, \bar{y}_0)(v, z_0) = 0\}. \tag{139}$$

Since C_t is a polyhedron, by Hoffman's Lemma [5], there exists a constant κ_t such that

$$\operatorname{dist}(v, C_t) \leq \kappa_t \left[[H_u(\bar{w}(t), \bar{p}(t))v]_+ + \sum_{i \in I_t} [g_i'(\bar{u}(t))v]_+ \right]. \tag{140}$$

This constant κ_t can be estimated as a finite maximum of those for projections over vector subspaces corresponding to a subset of active inequalities (for the problem of projection over the local critical cone), and therefore is a measurable function of time. Take a sequence of positive numbers $\varepsilon_k \downarrow 0$ (we will be more precise later) and consider the measurable partition

$$B_k := \{t \in (0, T);\ \kappa_t \geq 1/\varepsilon_k\}; \qquad A_k := (0, T) \setminus B_k. \tag{141}$$

Denote

$$\begin{cases} u_k' := \bar{u} + (u_k - \bar{u})\mathbf{1}_{A_k}; & u_k'' := \bar{u} + (u_k - \bar{u})\mathbf{1}_{B_k}, \\ \gamma_k' := \|u_k' - \bar{u}\|_2^2 + |y_{k0} - \bar{y}_0|^2; & \gamma_k'' := \|u_k'' - \bar{u}\|_2^2; \qquad \gamma_k := \gamma_k' + \gamma_k''. \end{cases} \tag{142}$$

Note that $\gamma_k = \|u_k - \bar{u}\|_2^2 + |y_{k0} - \bar{y}_0|^2$. Let y_k' be the state associated with control u_k' and initial state y_{k0}. Since $\varepsilon_k \downarrow 0$, we have that $\operatorname{meas}(B_k) \to 0$. The decomposition principle (Theorem 2.14) combined with (136) implies

$$\begin{aligned} J^{\bar{\mu}}(w_k) &= J^{\bar{\mu}}(w_k') + \int_{B_k} [H(u_k'', \bar{y}, \bar{p}) - H(\bar{u}, \bar{y}, \bar{p})]\mathrm{d}t + o(\gamma_k). \\ &\geq J^{\bar{\mu}}(w_k') + c_H \gamma_k'' + o(\gamma_k). \end{aligned} \tag{143}$$

Let $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$. Adding $0 \geq \int_0^T \lambda(t) g(u_k'(t))\mathrm{d}t$ to (38), obtain by a second-order expansion that

$$J^{\bar{\mu}}(w_k') \geq J^{\bar{\mu}}(w_k') + \langle \lambda, g(u_k') \rangle = J(\bar{w}) + O(\gamma_k'). \tag{144}$$

If (for a subsequence) $\gamma_k' = o(\gamma_k)$, deduce then with (143) that

$$J(w_k) \geq J^{\bar{\mu}}(w_k) \geq J(\bar{w}) + c_H \gamma_k'' + o(\gamma_k) = J(\bar{w}) + c_H \gamma_k + o(\gamma_k), \tag{145}$$

which gives the desired contradiction to (137). If remains to consider the (converse) case when

$$\gamma_k = O(\gamma_k'). \tag{146}$$

By Lemma 4.3, the set $M^L(\bar{w})$ is a singleton, and by Lemma 4.7, the (weak) restoration property (Definition 4.2) is satisfied. We apply it to the sequence u_k'. There exists $\hat{w}_k \in F(P)$ such that, η_k' denoting the initial-final state associated with w_k' (and reminding the definition (76) of the Pontryagin norm:

$$\begin{cases} \text{(i)} & \|\hat{w}_k - w_k'\|_\infty = O(d(\eta_k')); \\ \text{(ii)} & J(\hat{w}_k) = J^{\bar{\mu}}(w_k') + O\left(\|w_k' - \bar{w}\|_P d(\eta_k')\right). \\ \text{(iii)} & \hat{u}_k(t) = \bar{u}(t), \quad \text{for a.a. } t \in B_k. \end{cases} \tag{147}$$

Combining (143) and (147)(ii), obtain

$$J^{\bar\mu}(w_k) \geq J(\hat w_k) + O\left(\|w_k' - \bar w\|_P d(\eta_k')\right) + o(\gamma_k). \tag{148}$$

Since $\text{meas}(B_k) \to 0$, and hence, $\|u_k'' - \bar u\|_1 = o(\|u_k'' - \bar u\|_2)$, we have that

$$|\Phi(\eta_k') - \Phi(\eta_k)| = O(\|u_k'' - \bar u\|_1) = o((\gamma_k'')^{1/2}). \tag{149}$$

Consequently

$$d(\eta_k') = d(\eta_k) + o((\gamma_k'')^{1/2}). \tag{150}$$

Since $\|w_k' - \bar w\|_P = O((\gamma_k')^{1/2})$, we deduce from (148) that

$$J^{\bar\mu}(w_k) \geq J(\hat w_k) + O\left((\gamma_k)^{1/2} d(\eta_k)\right) + o(\gamma_k). \tag{151}$$

Using (75), we deduce that, for large enough k,

$$J(w_k) \geq J(\hat w_k) - \tfrac{1}{2}\bar\mu\Phi(\eta_k) + o(\gamma_k). \tag{152}$$

In view of (137), we obtain

$$J(\hat w_k) - \tfrac{1}{2}\bar\mu\Phi(\eta_k) \leq J(\bar w) + o(\gamma_k). \tag{153}$$

We will end the proof by obtaining a contradiction to (153). Set $\hat\gamma_k := \|\hat u_k - \bar u\|_2^2 + |\hat y_{k0} - \bar y_0|^2$. By (147)(i), we have that

$$\hat\gamma_k^{1/2} = (\gamma_k')^{1/2} + O(d(\eta_k')) = O((\gamma_k')^{1/2}) = O((\gamma_k)^{1/2}). \tag{154}$$

Since $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ (for any $a, b \in \mathbb{R}$), by the first equality above, we have that for some $c > 0$ independent of k:

$$\hat\gamma_k \geq \tfrac{1}{2}\gamma_k' - c d(\eta_k')^2. \tag{155}$$

This relation will be used later (for proving (181)). Combining (38) and (154), obtain

$$J(\hat w_k) \geq J^{\bar\mu}(\hat w_k) = J(\bar w) + \int_0^T H_u(\bar w(t), \bar p(t))(\hat u_k(t) - \bar u(t))\mathrm{d}t + O(\gamma_k). \tag{156}$$

In view of (153), we deduce that

$$\int_0^T H_u(\bar w(t), \bar p(t))(\hat u_k(t) - \bar u(t))\mathrm{d}t \leq O(\gamma_k). \tag{157}$$

Consider, for each time t, the projection $v_k(t)$ of the displacement $\hat u_k(t) - \bar u(t)$ over the local critical cone, and the difference between this projection and the projected direction:

$$v_k(t) := P_{C_t}(\hat u_k(t) - \bar u(t)); \qquad \hat v_k := v_k - (\hat u_k - \bar u). \tag{158}$$

Note that v_k is measurable and that, in view of the restoration property, $\hat u_k = \bar u$ a.e. on B_k, and hence, $v_k = \hat v_k(t) = 0$ a.e. on B_k. If $i \in I_t$, since $g_i(\bar u(t)) = 0$ and $g_i(\hat u_k(t)) \leq 0$, we have a.e. by a second-order Taylor expansion:

$$g_i'(\bar u(t))(\hat u_k(t) - \bar u(t)) \leq g_i(\hat u_k(t)) + O(|\hat u_k(t) - \bar u(t)|^2) \leq O(|\hat u_k(t) - \bar u(t)|^2). \tag{159}$$

By the definition of sets A_k and B_k, a.e. on $[0,T]$, we have that

$$|\hat{v}_k(t)| \leq \frac{1}{\varepsilon_k}\left([H_u(\bar{w}(t),\bar{p}(t))(\hat{u}_k(t)-\bar{u}(t))]_+ + \sum_{i\in I_t}[g_i'(\bar{u}(t))(\hat{u}_k(t)-\bar{u}(t))]_+\right), \quad (160)$$

so that by (154) and (157)–(160):

$$\|\hat{v}_k\|_1 = \|v_k - (\hat{u}_k - \bar{u})\|_1 = O\left(\frac{\gamma_k}{\varepsilon_k}\right). \quad (161)$$

Since a projection is nonexpansive, we also have

$$\text{(i)}\ \ |v_k(t)| \leq |\hat{u}_k(t)-\bar{u}(t)|, \ \text{ for a.a. } t; \qquad \text{(ii)}\ \ \|v_k\|_\infty = O(\|\hat{u}_k - \bar{u}\|_\infty). \quad (162)$$

We now fix $\varepsilon_k := (\|u_k - \bar{u}\|_\infty + |y_{k0} - \bar{y}_0|)^{1/2}$. Using (147)(i), (150), and $\gamma_k^{1/2} = O(\varepsilon_k^2)$, and denoting by $\mathbf{1}_{A_k}(t)$ the characteristic function of the set A_k, obtain

$$\|\hat{u}_k - u_k\|_\infty = \|(\hat{u}_k - u_k')\mathbf{1}_{A_k}\|_\infty = O(d(\eta_k')) = O(\gamma_k^{1/2}) = O(\varepsilon_k^2), \quad (163)$$

so that, using again $\gamma_k^{1/2} = O(\varepsilon_k^2)$:

$$\|\hat{u}_k - \bar{u}\|_\infty \leq \|\hat{u}_k - u_k\|_\infty + \|u_k - \bar{u}\|_\infty = O(\varepsilon_k^2). \quad (164)$$

Combining with (161), get

$$\int_0^T |\hat{u}_k(t) - \bar{u}(t)||\hat{v}_k(t)|\mathrm{d}t \leq \frac{\|\hat{u}_k - \bar{u}\|_\infty}{\varepsilon_k}\varepsilon_k\|\hat{v}_k\|_1 = O(\varepsilon_k\gamma_k) = o(\gamma_k). \quad (165)$$

By (158), (162), and (164), we have that $\|\hat{v}_k\|_\infty \leq 2\|\hat{u}_k - \bar{u}\|_\infty = O(\varepsilon_k^2)$. Therefore we obtain in a similar way

$$\|\hat{v}_k\|_2^2 \leq \|\hat{v}_k\|_\infty\|\hat{v}_k\|_1 = O\left(\frac{\|\hat{u}_k - \bar{u}\|_\infty}{\varepsilon_k}\right)\varepsilon_k\|\hat{v}_k\|_1 = o(\gamma_k), \quad (166)$$

$$\|v_k\|_2^2 = \|\hat{u}_k - \bar{u} + \hat{v}_k\|_2^2 = \|\hat{u}_k - \bar{u}\|_2^2 + o(\gamma_k). \quad (167)$$

Since $\hat{w}_k \in F(P)$, setting $\hat{e}_k := (\hat{u}_k, \hat{y}_{k0})$, and using (154), we have that (I_F was defined in (121))

$$G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) = G_i(\hat{e}_k) - G_i(\bar{e}) + O(\gamma_k)\begin{cases} = O(\gamma_k), & 1 \leq i \leq r_1, \\ \leq O(\gamma_k), & i \in I_F. \end{cases} \quad (168)$$

In view of (161), using $\gamma_k/\varepsilon_k = o(\gamma_k^{1/2})$, we have that

$$\begin{aligned} & G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) \\ &= G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) + o((\gamma_k)^{1/2}) = o(\gamma_k^{1/2}), \quad 1 \leq i \leq r_1, \\ & G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) \\ &= G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) + o((\gamma_k)^{1/2}) \leq o(\gamma_k^{1/2}), \quad i \in I_F. \end{aligned} \quad (169)$$

Also, when $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$, using (153), (154), and (159), for some $c > 0$:

$$\begin{aligned} \bar{\mu} G'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) = & -\hat{J}'(\bar{u}, \bar{y}_0)(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) - \langle \lambda, g'(\bar{u})(\hat{u}_k - \bar{u}) \rangle \\ \geq & -c\gamma_k. \end{aligned} \tag{170}$$

Therefore, using again (161) and $\gamma_k/\varepsilon_k = o(\gamma_k^{1/2})$ we get

$$-\bar{\mu} G'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) \leq o(\gamma_k^{1/2}). \tag{171}$$

It follows with (169) that

$$G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) = o((\gamma_k)^{1/2}), \quad \text{for all } i \in I_+. \tag{172}$$

Therefore, by the special qualification condition (80), there exists a critical direction (v_k', z_k') such that, setting $z_{k0} := \hat{y}_{k0} - \bar{y}_0$

$$\|v_k' - v_k\|_\infty + |z_{k0}' - z_{k0}| = o((\gamma_k)^{1/2}). \tag{173}$$

We recall that $\Omega(v, z_0)$ was defined in (128). Note that

$$\Omega(v_k, z_{k0}) := \max_{(\lambda,\bar{\mu}) \in LM^L(\bar{w})} \int_0^T H_{ww}^a(\bar{w}, \bar{p}, \lambda)(v_k, z_k)^2 \mathrm{d}t + (\Phi^{\bar{\mu}})''(\bar{\eta})(z_{k0}, z_k(T))^2. \tag{174}$$

By (134), we obtain that

$$\Omega(v_k', z_{k0}') \geq \alpha \left(\|v_k'\|_2^2 + |z_{k0}'|^2 \right), \tag{175}$$

and hence in view of (173)

$$\Omega(v_k, z_{k0}) \geq \alpha \left(\|v_k\|_2^2 + |z_{k0}|^2 \right) + o(\gamma_k). \tag{176}$$

By (165)–(167), and since $LM^L(\bar{w})$ is bounded, we have

$$\Omega(v_k, z_{k0}) = \Omega(\hat{u}_k - \bar{u} + \hat{v}_k, z_{k0}) = \Omega(\hat{u}_k - \bar{u}, z_{k0}) + o(\gamma_k), \tag{177}$$

and hence, in view of (176) and (165)–(167) again:

$$\begin{aligned} \Omega(\hat{u}_k - \bar{u}, z_{k0}) &= \Omega(v_k, z_{k0}) + o(\gamma_k) \\ &\geq \alpha(\|v_k\|_2^2 + |z_{k0}|^2) + o(\gamma_k) = \alpha \hat{\gamma}_k + o(\gamma_k). \end{aligned} \tag{178}$$

For a given λ such that $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$, adding to (38) (written for the sequence $\hat{w}_k$) the inequality $0 \geq \int_0^T \lambda(t)(g(\hat{u}_k(t)) - g(\bar{u}(t)))\,\mathrm{d}t$ and using $H_u^a(\bar{w}, \bar{p}, \lambda) = 0$, obtain

$$\begin{aligned} J^{\bar{\mu}}(\hat{w}_k) - J(\bar{w}) \geq \tfrac{1}{2} \int_0^T & H_{ww}^a(\bar{w}, \bar{p}, \lambda)((\hat{u}_k - \bar{u}), (\hat{y}_k - \bar{y}))^2 \mathrm{d}t \\ & + \tfrac{1}{2}(\Phi^{\bar{\mu}})''(\bar{\eta}))(\hat{\eta}_k - \bar{\eta})^2 + (\|\lambda\|_\infty + 1) o(\gamma_k), \end{aligned} \tag{179}$$

so that, maximizing w.r.t. the bounded set $LM^L(\bar{w})$, we obtain in view of (178)

$$J(\hat{w}_k) - J(\bar{w}) \geq J^{\bar{\mu}}(\hat{w}_k) - J(\bar{w}) \geq \Omega(\hat{u}_k - \bar{u}, z_{k0}) + o(\gamma_k) \geq \alpha \hat{\gamma}_k + o(\gamma_k). \tag{180}$$

Combining with (146) and (155), we get for some $\alpha' > 0$:

$$J(\hat{w}_k) - J(\bar{w}) \geq \tfrac{1}{2}\alpha\gamma'_k - \alpha' d(\eta'_k)^2 + o(\gamma'_k). \tag{181}$$

By (150), $d(\eta'_k)^2 \leq 2d(\eta_k)^2 + o(\gamma_k)$ and so using also (146)

$$J(\hat{w}_k) - J(\bar{w}) \geq \tfrac{1}{2}\alpha\gamma'_k - 2\alpha' d(\eta_k)^2 + o(\gamma'_k). \tag{182}$$

Using (75) and (146), we see that this gives a contradiction to (153), as it was to be shown. □

Acknowledgements. The authors thank the two referees for their useful remarks.

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