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Quasiconvexity and Uniqueness of Stationary Points on a Space of Measure Preserving Maps

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Let $\Omega \subset \mathbb{R}^n$ be a bounded starshaped domain and consider the energy functional

$$\mathbb{F}[u;\Omega] := \int_\Omega \mathbf{F}(\nabla u(x))\,dx,$$

over the space of measure preserving maps

$$\mathcal{A}_p(\Omega) = \left\{ u \in \bar{\xi}x + W_0^{1,p}(\Omega,\mathbb{R}^n) : \det \nabla u = 1 \text{ a.e. in } \Omega \right\},$$

with $p \in [1,\infty[$, $\bar{\xi} \in \mathbb{M}_{n\times n}$ and $\det \bar{\xi} = 1$. In this short note we address the question of *uniqueness* for solutions of the corresponding system of Euler-Lagrange equations. In particular we give a new proof of the celebrated result of Knops & Stuart [3] using a method based on *comparison* with homogeneous degree-one extensions as introduced by the second author in [6].

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded *starshaped* domain. In this short note we consider the energy functional

$$\mathbb{F}[u;\Omega] := \int_\Omega \mathbf{F}(\nabla u(x))\,dx, \tag{1}$$

over the space of *admissible* maps

$$\mathcal{A}_p(\Omega) := \left\{ u \in W_v^{1,p}(\Omega,\mathbb{R}^n) : \det \nabla u = 1 \text{ a.e. in } \Omega \right\}, \tag{2}$$

with $p \in [1,\infty[$, where

$$W_v^{1,p}(\Omega,\mathbb{R}^n) = \left\{ u \in W^{1,p}(\Omega,\mathbb{R}^n) : u|_{\partial\Omega} = v|_{\partial\Omega} \right\},$$

while $v = \bar{\xi}x$ and $\bar{\xi} \in \mathbb{M}_{n\times n}$ with $\det \bar{\xi} = 1$. Here the integrand $\mathbf{F} : \mathbb{R}^{n\times n} \to \mathbb{R}$ is of class C^1 and for future reference we associate with it the following set of hypotheses.

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(H1) (*Growth* condition) There exists $c_1 > 0$ such that for all $\xi \in \mathbb{R}^{n\times n}$ we have that

$$|\mathbf{F}(\xi)| \le c_1(1 + |\xi|^p).$$

(H2) (*Coercivity* condition) There exists $c_2 > 0$ such that for all $\xi \in \mathbb{R}^{n\times n}$ we have that

$$c_2|\xi|^p - c_1 \le \mathbf{F}(\xi).$$

(H3)$_\xi$ (*Quasiconvexity* at ξ) For *fixed* $\xi \in \mathbb{M}_{n\times n}$ and all $\varphi \in \mathrm{C}_c^\infty(\Omega, \mathbb{R}^n)$ we have that

$$\int_\Omega (\mathbf{F}(\xi + \nabla\varphi(x)) - \mathbf{F}(\xi))\, dx \ge 0.$$

If, additionally, the inequality is *strict* for $\varphi \neq 0$ then $\mathbf{F}$ is referred to as being *strictly* quasiconvex at ξ.

(H4)$_\xi$ (*Rank-one* convexity at ξ) For *fixed* $\xi \in \mathbb{M}_{n\times n}$ and all *rank-one* $\zeta \in \mathbb{M}_{n\times n}$ the function

$$\mathbb{R} \ni t \mapsto \mathbf{F}(\xi + t\zeta) \in \mathbb{R}$$

is *convex* at $t = 0$.[1]

Here we are primarily concerned with the question of *uniqueness* for solutions to the system of Euler-Lagrange equations associated with the energy functional (1) over the space (2) as well as that for its *strong* local minimizers.[2] Indeed, the *former*, under the stated C^1 regularity assumption on $\mathbf{F}$ are known to take the form

$$\begin{cases} \mathit{div}\ \mathfrak{S}[x, \nabla u(x)] = 0 & x \in \Omega, \\ \det \nabla u(x) = 1 & x \in \Omega, \\ u(x) = v(x) & x \in \partial\Omega, \end{cases}$$

where, we have set

$$\begin{aligned} \mathfrak{S}[x,\xi] = {} & \mathbf{F}_\xi(\xi) - \mathfrak{p}(x)\xi^{-t} \\ =: {} & \mathfrak{T}[x,\xi]\xi^{-t}, \end{aligned} \tag{3}$$

for $x \in \Omega$, $\xi \in \mathbb{R}^{n\times n}$ satisfying $\det\xi = 1$ and $\mathfrak{p}$ a suitable Lagrange multiplier while

$$\mathfrak{T}[x,\xi] = \mathbf{F}_\xi(\xi)\xi^t - \mathfrak{p}(x)\mathbf{I}_n. \tag{4}$$

A motivating source for this type of problem is nonlinear elasticity where (1) and (2) represent a simple *model* of a homogeneous *incompressible* hyperelastic material and solutions to the above system of equations serve as the corresponding *equilibrium* states (cf., e.g., Ball [1]).[3]

[1]For a comprehensive treatment of the *convexity* notions (H3), (H4) and their significance in the *calculus* of *variations* we refer the interested reader to [2].

[2]A map $u \in \mathcal{A}_p(\Omega)$ is a *strong* local minimizer of $\mathbb{F}$ if and only if there exists $\rho = \rho(u) > 0$ such that $\mathbb{F}[u;\Omega] \le \mathbb{F}[w;\Omega]$ for all $w \in \mathcal{A}_p(\Omega)$ satisfying $||u - w||_{W^{1,p}} \le \rho$.

[3]In the language of elasticity, the *tensor* fields (3) and (4) are referred to as the *Piola-Kirchhoff* and the *Cauchy* stress tensors respectively and the *Lagrange* multiplier $\mathfrak{p}$ is better known as the *hydrostatic* pressure.

The question of uniqueness of solutions to the above system of Euler-Lagrange equations [subject to *linear* boundary conditions] was established in a seminal paper of Knops & Stuart (see [3]). There it is shown that subject to $\mathbf{F}$ being of class C^2, *rank-one* convex everywhere and *strictly* quasiconvex at $\bar{\xi}$ any smooth solution u in a *starshaped* domain satisfying $\det \nabla u = 1$ in Ω and $u = \bar{\xi}x$ on $\partial\Omega$ satisfies $u = \bar{\xi}x$ on $\bar{\Omega}$.

Without *further* restriction on the domain Ω uniqueness in general may *fail*. Indeed one can construct domains Ω for which *any* energy functional $\mathbb{F}$ whose integrand $\mathbf{F}$ satisfies (H1)–(H3) admits multiple [*infinitely* many for $n = 2$] *strong* local minimizers as well as multiple [*infinitely* many for *n even*] smooth solutions to the corresponding system of Euler-Lagrange equations. (See [7], [8] and [4], [5].)[4]

In this short note we give a *new* proof of the *aforementioned* uniqueness result of Knops & Stuart [3]. This is based on *firstly* removing the measure preserving condition $\det \nabla u = 1$ and considering instead a suitable *unconstrained* functional [with the aid of the Lagrange multiplier $\mathfrak{p}$] and *secondly* utilising the so-called *stationarity* condition followed by comparison with *homogeneous* degree-one extensions as introduced in [6]. This approach has the advantage of *extending* the uniqueness result to all *weak* solutions u of class C^1 satisfying the *weak* form of the stationarity condition (see (8) below).

Finally we prove a new uniqueness result for *strong* local minimizers of $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$ to the effect that subject to (H1), $(\mathrm{H3})_{\bar{\xi}}$ alone *any* such $u \in \mathcal{A}_p(\Omega)$ satisfies $\mathbb{F}[u;\Omega] = \mathbb{F}[\bar{\xi}x;\Omega]$ and therefore subject to the additional *strictly* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$ it must be that $u = \bar{\xi}x$ on $\bar{\Omega}$!

2. The main result

Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded *starshaped* domain (with respect to the origin). Without loss of generality we assume in the sequel that there exists a *strictly* positive function $d : \mathbb{S}^{n-1} \to \mathbb{R}$ such that

$$\partial\Omega = \left\{ \omega \neq 0 : |\omega| = d\left(\frac{\omega}{|\omega|}\right) \right\}.$$

It is then clear that $\Omega = \{0\} \cup \{x \neq 0 : |x| < d(x/|x|)\}$. Moreover the *unit* outward normal to the boundary at a point $\omega \in \partial\Omega$ is given by

$$\nu = \frac{1}{\alpha(\theta)} \left[\theta - (\mathbf{I}_n - \theta \otimes \theta) \frac{\nabla d(\theta)}{d(\theta)} \right]$$

where $\alpha(\theta) = d^{-1}(\theta)\sqrt{d^2(\theta) + |\nabla d(\theta)|^2 - \langle \theta, \nabla d(\theta) \rangle^2}$ and $\theta = \omega/|\omega|$.

Definition 2.1 (Classical solution). A pair $(u, \mathfrak{p})$ is said to be a *classical* solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$ if and only if the following hold.

(1) $\quad u \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n)$,

[4]For technical reasons one needs to restrict to $p \in [n, \infty[$ for the multiplicity result relating to *strong* local minimizers and to $p \in]1, \infty[$ for the one relating to smooth solutions.

(2) $\mathfrak{p} \in \mathrm{C}^1(\Omega) \cap \mathrm{C}^0(\bar{\Omega})$,

(3) $(u, \mathfrak{p})$ satisfy the *system* of equations[5]

$$\begin{cases} \operatorname{div}\{\mathbf{F}_\xi(\nabla u) - \mathfrak{p}[\operatorname{cof} \nabla u]\} = 0 & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega. \end{cases}$$

Now suppose that $(u, \mathfrak{p})$ is a classical solution as described in Definition 2.1. We set

$$\mathbf{G}(x, z, \xi) = \mathbf{G}(x, z, \xi; \mathfrak{p}) := \mathbf{F}(\xi) - \mathfrak{p}(x)(\det \xi - 1), \tag{5}$$

for all $x \in \Omega$, $z \in \mathbb{R}^n$ and $\xi \in \mathbb{M}_{n\times n}$. Next with the aid of $\mathbf{G}$ we introduce the *Hamilton* [or the *energy-momentum*] tensor

$$\mathbf{T}^\beta_\alpha(x, z, \xi) := \xi^i_\alpha \mathbf{G}_{\xi^i_\beta}(x, z, \xi) - \delta^\beta_\alpha \mathbf{G}(x, z, \xi). \tag{6}$$

Theorem 2.2. *Let $(u, \mathfrak{p})$ be a classical solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Let $\mathbf{F}$ be of class C^2. Then with $\mathbf{G}$ and $\mathbf{T}$ as in* (5) *and* (6) *we have that*

$$div\,\{\mathbf{T}(x, u, \nabla u)\} + \mathbf{G}_x(x, u, \nabla u) = 0 \tag{7}$$

in Ω.[6]

Proof. (By direct *verification*) Indeed expanding the above identity *componentwise* we have that

$$\begin{aligned} \mathbf{L}_\alpha :=\ & [div\,\{\mathbf{T}(x, u, \nabla u)\} + \mathbf{G}_x(x, u, \nabla u)]_\alpha \\ =\ & \frac{\partial \mathbf{T}^\beta_\alpha}{\partial x_\beta}(x, u, \nabla u) + \mathbf{G}_{x_\alpha}(x, u, \nabla u) \\ =\ & \frac{\partial}{\partial x_\beta}\left\{u^i_{,\alpha}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof} \nabla u]_{i\beta}\right)\right\} \\ & \qquad - \frac{\partial}{\partial x_\alpha}\{\mathbf{F} - \mathfrak{p}(x)[\det \nabla u - 1]\} - \frac{\partial \mathfrak{p}}{\partial x_\alpha}(x)[\det \nabla u - 1]. \end{aligned}$$

Therefore taking advantage of $\det \nabla u = 1$ and by direct *differentiation* we can write

$$\begin{aligned} \mathbf{L}_\alpha =\ & u^i_{,\alpha\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof} \nabla u]_{i\beta}\right) \\ & \qquad + u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof} \nabla u]_{i\beta}\right) - \mathbf{F}_{\xi^i_\beta} u^i_{,\alpha\beta} \\ =\ & -\mathfrak{p}(x)\frac{\partial}{\partial x_\alpha}\det \nabla u + u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof} \nabla u]_{i\beta}\right) \\ =\ & u^i_{,\alpha}\frac{\partial}{\partial x_\beta}\left(\mathbf{F}_{\xi^i_\beta} - \mathfrak{p}(x)[\operatorname{cof} \nabla u]_{i\beta}\right) = 0, \end{aligned}$$

which is the required conclusion. □

[5] Although we are primarily concerned with the case $v = \bar{\xi}x$, for reasons that will become clear later, we allow $v \in \mathrm{C}^1(\bar{\Omega}, \mathbb{R}^n)$ to be arbitrary.

[6] This is the so-called *stationarity* condition in its *strong* form as opposed to its *weak* form given by (8) below.

For the sake of future reference we next introduce the *unconstrained* energy functional

$$\begin{aligned}\mathbb{G}[u,\mathfrak{p};\Omega] &:= \int_\Omega \mathbf{G}(x,u,\nabla u)\,dx\\ &= \int_\Omega \left(\mathbf{F}(\nabla u)-\mathfrak{p}(x)[\det\nabla u-1]\right)\,dx.\end{aligned}$$

Then setting $u_\varepsilon(x) := u(x+\varepsilon\varphi)$ with $\varphi\in \mathrm{C}_c^\infty(\Omega,\mathbb{R}^n)$ an application of Theorem 2.2 and the *divergence* theorem along with a straight-forward calculation gives

$$\begin{aligned}\frac{d}{d\varepsilon}\mathbb{G}[u_\varepsilon,\mathfrak{p};\Omega]\bigg|_{\varepsilon=0} &= \int_\Omega \left(\mathbf{T}^\beta_\alpha\varphi^\alpha_{,\beta}-\mathbf{G}_{x_\alpha}\varphi^\alpha\right)\,dx\\ &= \int_\Omega \Big(u^i_{,\alpha}\mathbf{G}_{\xi^i_\beta}\varphi^\alpha_{,\beta}-\delta^\beta_\alpha\mathbf{G}\varphi^\alpha_{,\beta}-\mathbf{G}_{x_\alpha}\varphi^\alpha\Big)\,dx=0. \end{aligned}\tag{8}$$

Theorem 2.3. *Let $(u,\mathfrak{p})$ be a classical solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

(*1*) $\mathbf{F}$ *is of class* C^2,

(*2*) $\mathbf{F}$ *satisfies* $(\mathrm{H4})_\xi$ *for all* $\xi\in\{\nabla u(\omega):\omega\in\partial\Omega\}$.

Then with $\mathbf{G}$ *and* $\mathbf{T}$ *as in* (5) *and* (6) *we have that*

$$\mathbb{G}[u,\mathfrak{p};\Omega]\le\mathbb{G}[\bar u,\bar{\mathfrak{p}};\Omega]\tag{9}$$

where $\bar u,\bar{\mathfrak{p}}$ denote the homogeneous degree-one and degree-zero extensions of $u,\mathfrak{p}$ to Ω respectively, that is,

$$\bar u(x):=\frac{r}{d(\theta)}u(\theta d(\theta))$$

and

$$\bar{\mathfrak{p}}(x):=\mathfrak{p}(\theta d(\theta))$$

for $x\in\bar\Omega$ where $r=|x|$ and $\theta=x/|x|$.[7]

Proof. For the sake of clarity and convenience we present this in the following *two* steps.

Step 1. ($\mathbb{G}[u,\mathfrak{p};\Omega]$ as a *boundary* integral) For $t\in[0,1]$ and $\varepsilon>0$ put

$$\mathbf{s}_\varepsilon(t)=\begin{cases}1 & \text{for } 0\le t\le 1-\varepsilon\\ 1-\dfrac{t-(1-\varepsilon)}{\varepsilon} & \text{for } 1-\varepsilon\le t\le 1,\end{cases}$$

[7]In the course of the proof of this theorem we make repeated use of the following integration formula. For every $f\in L^1(\Omega)$ we have that

$$\int_\Omega f(x)\,dx=\int_0^1\int_{\partial\Omega}\rho^{n-1}\frac{d(\theta)}{\alpha(\theta)}f(\rho\omega)\,d\mathcal{H}^{n-1}(\omega)d\rho.$$

(See [6] for a proof.)

and set

$$\varphi(x) = \mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right) x. \tag{10}$$

Then one can easily verify that

$$\begin{aligned}\nabla\varphi(x) &= \mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{I}_n + |x|\frac{1}{d(\theta)}\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\theta\otimes\left(\theta - (\mathbf{I}_n - \theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\\ &= \mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{I}_n + |x|\frac{\alpha(\theta)}{d(\theta)}\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\theta\otimes\nu,\end{aligned}$$

where $\theta = x/|x|$ and $\nu = \nu(\theta d(\theta))$ is the unit *outward* normal to $\partial\Omega$. Moreover it is evident that

$$\mathbf{1}_\Omega = \lim_{\varepsilon\downarrow 0}\mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right) \tag{11}$$

where the *limit* is being understood both as $\mathcal{L}^n$-a.e. in Ω and strongly in $L^1(\Omega)$. Now upon substituting φ as given by (10) into (8) and re-arranging terms it follows after taking into account (11) that

$$\begin{aligned}n\mathbb{G}[u,\mathfrak{p};\Omega] =& \lim_{\varepsilon\downarrow 0}\int_\Omega n\mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{G}(x,u,\nabla u)\,dx\\ =& \lim_{\varepsilon\downarrow 0}\int_\Omega\Big\{-\frac{\alpha(\theta)}{d(\theta)}|x|(\theta\cdot\nu)\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{G}(x,u,\nabla u)\\ &+\mathbf{s}_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\langle\mathbf{G}_\xi(x,u,\nabla u),\nabla u\rangle\\ &+\frac{\alpha(\theta)}{d(\theta)}|x|\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\langle\mathbf{G}_\xi(x,u,\nabla u),\nabla u\,\theta\otimes\nu\rangle\Big\}\,dx\\ =& \lim_{\varepsilon\downarrow 0}\{\mathbf{I}+\mathbf{II}+\mathbf{III}\}.\end{aligned} \tag{12}$$

We now proceed by considering each term separately. Indeed, with regards to the *first* term we have that

$$\begin{aligned}\mathbf{I} = \mathbf{I}(\varepsilon) =& \int_\Omega -\frac{\alpha(\theta)}{d(\theta)}|x|(\theta\cdot\nu)\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{G}(x,u,\nabla u)\,dx\\ =& \int_\Omega -\frac{1}{d(\theta)}|x|\mathbf{s}'_\varepsilon\left(\frac{|x|}{d(\theta)}\right)\mathbf{F}(\nabla u(x))\,dx\\ =& \int_{1-\varepsilon}^1\int_{\partial\Omega}\frac{1}{\varepsilon}\rho^n\frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\rho\omega))\,d\mathcal{H}^{n-1}(\omega)d\rho.\end{aligned}$$

Thus by passing to the limit $\varepsilon\downarrow 0$ we have that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\mathbf{I} =& \lim_{\varepsilon\downarrow 0}\int_{1-\varepsilon}^1\int_{\partial\Omega}\frac{1}{\varepsilon}\rho^n\frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\rho\omega))\,d\mathcal{H}^{n-1}(\omega)d\rho\\ =& \int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\omega))\,d\mathcal{H}^{n-1}(\omega).\end{aligned}$$

In a similar way with regards to the *second* term we have that

$$
\begin{aligned}
\mathbf{II} = \mathbf{II}(\varepsilon) = & \int_\Omega \mathbf{s}_\varepsilon \left(\frac{|x|}{d(\theta)}\right) \langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u \rangle \, dx \\
= & \int_\Omega \mathbf{s}_\varepsilon \left(\frac{|x|}{d(\theta)}\right) \langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u \rangle \, dx.
\end{aligned}
$$

Utilising (11) and Lebesgue's theorem on dominated converge, passing to the limit $\varepsilon \downarrow 0$ yields

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \mathbf{II} = & \lim_{\varepsilon \downarrow 0} \int_\Omega \mathbf{s}_\varepsilon \left(\frac{|x|}{d(\theta)}\right) \langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u \rangle \, dx \\
= & \int_\Omega \langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u \rangle \, dx \\
= & \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], u(\omega) \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega),
\end{aligned}
$$

where in the *second* identity we have appealed to the *divergence* theorem along with the fact that $(u, \mathfrak{p})$ is a solution to the Euler-Lagrange equations associated with $\mathbb{F}$ over $\mathcal{A}_p$.

Finally with regards to the *third* term we can write

$$
\begin{aligned}
& \mathbf{III} = \mathbf{III}(\varepsilon) \\
= & \int_\Omega \frac{\alpha(\theta)}{d(\theta)} |x| \mathbf{s}'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) \langle \mathbf{G}_\xi(x, u, \nabla u), \nabla u \, \theta \otimes \nu \rangle \, dx \\
= & \int_\Omega \frac{\alpha(\theta)}{d(\theta)} |x| \mathbf{s}'_\varepsilon \left(\frac{|x|}{d(\theta)}\right) \langle \mathbf{F}_\xi(\nabla u) - \mathfrak{p}(x)[\mathrm{cof}\ \nabla u], \nabla u \, \theta \otimes \nu \rangle \, dx \\
= & \int_{1-\varepsilon}^1 \int_{\partial\Omega} -\frac{1}{\varepsilon} \rho^n d(\theta) \\
& \qquad \times \{ \langle \mathbf{F}_\xi(\nabla u(\rho\omega)) - \mathfrak{p}(\rho\omega)[\mathrm{cof}\ \nabla u(\rho\omega)], \nabla u(\rho\omega) \, \theta \otimes \nu \rangle \} \, d\mathcal{H}^{n-1}(\omega) d\rho.
\end{aligned}
$$

Thus by passing to the limit $\varepsilon \downarrow 0$ we have that

$$
\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \mathbf{III} \\
= & \lim_{\varepsilon \downarrow 0} \int_{1-\varepsilon}^1 \int_{\partial\Omega} -\frac{1}{\varepsilon} \rho^n d(\theta) \\
& \qquad \times \{ \langle \mathbf{F}_\xi(\nabla u(\rho\omega)) - \mathfrak{p}(\rho\omega)[\mathrm{cof}\ \nabla u(\rho\omega)], \nabla u(\rho\omega) \, \theta \otimes \nu \rangle \} \, d\mathcal{H}^{n-1}(\omega) d\rho \\
= & \int_{\partial\Omega} -d(\theta) \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], \nabla u(\omega) \, \theta \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega).
\end{aligned}
$$

Hence referring to (12) and *summarising* the above conclusions we have that

$$\begin{aligned}
&n\mathbb{G}[u,\mathfrak{p};\Omega]\\
&= \int_\Omega n\mathbf{G}(x,u,\nabla u)\,dx\\
&= \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)}\mathbf{F}(\nabla u(\omega))\,d\mathcal{H}^{n-1}(\omega)\\
&\qquad + \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\;\nabla u(\omega)], u(\omega)\otimes\nu\rangle\,d\mathcal{H}^{n-1}(\omega)\\
&\qquad - \int_{\partial\Omega} d(\theta)\langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\;\nabla u(\omega)], \nabla u(\omega)\theta\otimes\nu\rangle\,d\mathcal{H}^{n-1}(\omega). \qquad (13)
\end{aligned}$$

Step 2. (A *lower* bound on $\mathbb{G}[\bar{u},\bar{\mathfrak{p}};\Omega]$) Recall that the *homogeneous* degree-one extension of u to Ω is given by

$$\bar{u}(x) = \frac{|x|}{d(\theta)}u(\theta d(\theta))$$

for $x\in\bar{\Omega}$ with $\theta = x/|x|$. It can therefore be easily checked that

$$\begin{aligned}
\nabla\bar{u}(x) &= \nabla u(\theta d(\theta)) + \left\{\left(\frac{u(\theta d(\theta))}{d(\theta)} - \nabla u(\theta d(\theta))\theta\right)\otimes\left(\theta - (\mathbf{I}_n - \theta\otimes\theta)\frac{\nabla d(\theta)}{d(\theta)}\right)\right\}\\
&= \nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)}\left\{[u(\omega) - d(\theta)\nabla u(\omega)\theta]\otimes\nu\right\} \qquad (14)
\end{aligned}$$

for $x\in\bar{\Omega}$ where $\omega = \theta d(\theta)\in\partial\Omega$. In particular we have that

$$\begin{aligned}
\det\nabla\bar{u}(x) &= \det\nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)}\langle[\nabla u(\omega)]^{-1}[u(\omega) - d(\theta)\nabla u(\omega)\theta],\nu\rangle\\
&= 1 + \frac{\alpha(\theta)}{d(\theta)}\langle[\mathrm{cof}\;\nabla u(\omega)]^t[u(\omega) - d(\theta)\nabla u(\omega)\theta],\nu\rangle. \qquad (15)
\end{aligned}$$

Thus we can write

$$\begin{aligned}
&n\mathbb{G}[\bar{u},\bar{\mathfrak{p}};\Omega]\\
&= n\int_\Omega \mathbf{G}(x,\bar{u},\nabla\bar{u};\bar{\mathfrak{p}})\,dx\\
&= n\int_\Omega \mathbf{F}(\nabla\bar{u}) - \bar{\mathfrak{p}}(x)[\det\nabla\bar{u} - 1]\,dx\\
&= n\int_0^1\int_{\partial\Omega} \rho^{n-1}\frac{d(\theta)}{\alpha(\theta)}\times\{\mathbf{F}(\nabla\bar{u}(\rho\omega)) - \bar{\mathfrak{p}}(\rho\omega)[\det\nabla\bar{u}(\rho\omega) - 1]\}\;d\mathcal{H}^{n-1}(\omega)d\rho\\
&= \int_{\partial\Omega}\frac{d(\theta)}{\alpha(\theta)}\{\mathbf{F}(\nabla\bar{u}(\omega)) - \bar{\mathfrak{p}}(\omega)[\det\nabla\bar{u}(\omega) - 1]\}\;d\mathcal{H}^{n-1}(\omega) \qquad (16)
\end{aligned}$$

where in concluding the *last* line we have used the identities $\nabla\bar{u}(\rho\omega) = \nabla\bar{u}(\omega)$ and $\bar{\mathfrak{p}}(\rho\omega) = \bar{\mathfrak{p}}(\omega)$ for $\rho\in[0,1]$ and $\omega\in\partial\Omega$ as a consequence of *homogeneity*.

Now anticipating on the integral on the *right* in (16) we *first* note that in view of the *rank-one* convexity of $\mathbf{F}$ at the points $\nabla u(\omega)$ using (14) [with $x = \omega$] we have that

$$\begin{aligned} \mathbf{F}(\nabla \bar{u}(\omega)) &= \mathbf{F}(\nabla u(\omega) + \frac{\alpha(\theta)}{d(\theta)}[u(\omega) - d(\theta)\nabla u(\omega)\theta] \otimes \nu) \\ &\geq \mathbf{F}(\nabla u(\omega)) + \frac{\alpha(\theta)}{d(\theta)}\langle \mathbf{F}_\xi(\nabla u(\omega)), [u(\omega) - d(\theta)\nabla u(\omega)\theta] \otimes \nu \rangle. \end{aligned} \tag{17}$$

Hence substituting from (15) and (17) into (16) and making note of the inequality $d/\alpha > 0$ we can write

$$\begin{aligned} n\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] &= \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \{\mathbf{F}(\nabla \bar{u}(\omega)) - \bar{\mathfrak{p}}(\omega)[\det \nabla \bar{u}(\omega) - 1]\} \, d\mathcal{H}^{n-1}(\omega) \\ &\geq \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \mathbf{F}(\nabla u(\omega)) \, d\mathcal{H}^{n-1}(\omega) \\ &\quad + \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)), [u(\omega) - d(\theta)\nabla u(\omega)\theta] \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) \\ &\quad - \int_{\partial\Omega} \mathfrak{p}(\omega) \langle [\mathrm{cof}\ \nabla u(\omega)]^t [u(\omega) - d(\theta)\nabla u(\omega)\theta], \nu \rangle \, d\mathcal{H}^{n-1}(\omega). \end{aligned}$$

Finally, re-arranging terms and comparing the expression on the *right* in the above with (13) immediately yields

$$\begin{aligned} & n\mathbb{G}[\bar{u}, \bar{\mathfrak{p}}; \Omega] \\ \geq & \int_{\partial\Omega} \frac{d(\theta)}{\alpha(\theta)} \mathbf{F}(\nabla u(\omega)) \, d\mathcal{H}^{n-1}(\omega) \\ & \quad + \int_{\partial\Omega} \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], u(\omega) \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) \\ & \quad - \int_{\partial\Omega} d(\theta) \langle \mathbf{F}_\xi(\nabla u(\omega)) - \mathfrak{p}(\omega)[\mathrm{cof}\ \nabla u(\omega)], \nabla u(\omega)\theta \otimes \nu \rangle \, d\mathcal{H}^{n-1}(\omega) \\ \geq & \ n\mathbb{G}[u, \mathfrak{p}; \Omega] \end{aligned}$$

which is the required conclusion. □

In the remainder of this section we confine oursleves to the case of *linear* boundary conditions, that is, $v = \bar{\xi}x$ where $\xi \in \mathbb{M}_{n\times n}$ and $\det \bar{\xi} = 1$.

Theorem 2.4 (Uniqueness I). *Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded starshaped domain and consider the energy functional $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

(*1*) $\mathbf{F}$ *is of class* C^2,
(*2*) $\mathbf{F}$ *satisfies* (H1) *and* $(\mathrm{H3})_{\bar{\xi}}$,
(*3*) $(u, \mathfrak{p})$ *a classical solution* (*see Definition* 2.1),
(*4*) $\mathbf{F}$ *satisfies* $(\mathrm{H4})_\xi$ *for all* $\xi \in \{\nabla u(\omega) : \omega \in \partial\Omega\}$.

Then,

$$\mathbb{F}[u; \Omega] = \mathbb{F}[\bar{\xi}x; \Omega] = \inf_{\mathcal{A}_p(\Omega)} \mathbb{F}[\cdot; \Omega].$$

If, additionally, $\mathbf{F}$ *is strictly quasiconvex at* $\bar{\xi}$ *then* $u = \bar{\xi}x$ *on* $\bar{\Omega}$.

Proof. Evidently $\bar{u} = \bar{\xi}x$ and therefore $\det \nabla \bar{u} = 1$ in Ω.[8] Hence referring to the estimate (9) in Theorem 2.3 and the *quasiconvexity* of $\mathbf{F}$ at $\bar{\xi}$ we can write

$$\mathbb{F}[\bar{u};\Omega] \le \mathbb{F}[u;\Omega] = \mathbb{G}[u,\mathfrak{p};\Omega] \le \mathbb{G}[\bar{u},\bar{\mathfrak{p}};\Omega] = \mathbb{F}[\bar{u};\Omega].$$

The remaining assertion is now a *trivial* consequence of the latter and the *strict* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$. □

Remark 2.5. The proof of Theorem 2.3 and Theorem 2.4 remain unchanged if $\mathbf{F}$ is of class C^1 and in Definition 2.1 (1) is replaced by $u \in \mathrm{C}^1(\bar{\Omega}, \mathbb{R}^n)$, (2) by $\mathfrak{p} \in \mathrm{C}^0(\bar{\Omega})$ and (3) by $(u, \mathfrak{p})$ being a *weak* solution to the corresponding system of Euler-Lagrange equation provided that *additionally* (8) holds.

Theorem 2.6 (Uniqueness II). *Let $\Omega \subset \mathbb{R}^n$ be a bounded starshaped domain and consider the energy functional $\mathbb{F}$ over $\mathcal{A}_p(\Omega)$. Assume that*

(1) $\mathbf{F}$ *is of class* C^0,

(2) $\mathbf{F}$ *satisfies* (H1) *and* $(\mathrm{H3})_{\bar{\xi}}$,

(3) *$u \in \mathcal{A}_p(\Omega)$ is a strong local minimizer of $\mathbb{F}$, i.e., that there exists $\rho = \rho(u) > 0$ such that $\mathbb{F}[u;\Omega] \le \mathbb{F}[w;\Omega]$ for all $w \in \mathcal{A}_p(\Omega)$ with $||u - w||_{W^{1,p}} \le \rho$.*

Then,

$$\mathbb{F}[u;\Omega] = \mathbb{F}[\bar{\xi}x;\Omega] = \inf_{\mathcal{A}_p(\Omega)} \mathbb{F}[\cdot;\Omega]. \tag{18}$$

If, additionally, $\mathbf{F}$ is strictly quasiconvex at $\bar{\xi}$ then $u = \bar{\xi}x$ on $\bar{\Omega}$.

Proof. The *second* identity in (18) is a result of (*1*), (*2*) and a straight-forward *approximation* and so it suffices to justify only the *first* equality. Indeed for the sake of a contradiction assume $\mathbb{F}[u;\Omega] > \mathbb{F}[\bar{\xi}x;\Omega]$ and for $\delta \in (0,1]$ and $x \in \Omega$ set

$$u_\delta(x) := \begin{cases} \delta u(\frac{x}{\delta}) & x \in \bar{\Omega}_\delta, \\ \bar{\xi}x & x \in \Omega \backslash \bar{\Omega}_\delta, \end{cases}$$

where $\Omega_\delta = \delta\Omega$. Then $\det \nabla u_\delta = 1$ $\mathcal{L}^n$-a.e. in Ω and so $u_\delta \in \mathcal{A}_p(\Omega)$. Moreover a straight-forward calculation gives

$$\begin{aligned} \mathbb{F}[u_\delta;\Omega] &= \mathbb{F}[u;\Omega] + (1 - \delta^n)\left\{\mathbb{F}[\bar{\xi}x;\Omega] - \mathbb{F}[u;\Omega]\right\} \\ &< \mathbb{F}[u;\Omega] \end{aligned}$$

whilst $u_\delta \to u$ in $W^{1,p}$ as $\delta \uparrow 1$. This contradicts (*3*) and so the assertion is justified. The final part is now a *trivial* consequence of the latter and the *strict* quasiconvexity of $\mathbf{F}$ at $\bar{\xi}$. □

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[8] Note that in general $\det \nabla \bar{u} = 1$ is *false*! [See (15)] However, interestingly, subject to $u = \bar{\xi}x$ on $\partial\Omega$ as described the latter identity holds throughout Ω.

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