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First General Lower Semicontinuity and Relaxation Results for Strong Materials

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We consider the case of strong materials, i.e. the situation where the growth of integrands from below guarantees the lack of discontinuities for deformations with finite energy. We show that, in this case, both lower semicontinuity and relaxation results relay on the a.e. differentiability property of admissible deformations and on the uniform convergence of weakly convergent sequences bounded in energy.

Keywords: Integral functionals, lower semicontinuity, relaxation, mathematical theory of elasticity, strong materials

1. Introduction

In this paper we consider minimization problems of the form

$$J(u) = \int_\Omega L(Du(x))dx \to \min, \quad L \geq 0, \ L \in C(\mathbf{R}^{n\times m} \to \mathbf{R}), \tag{1}$$

$$u\Big|_{\partial\Omega} = f, \quad u \in W^{1,1}(\Omega; \mathbf{R}^m), \tag{2}$$

where Ω is a bounded domain with Lipschitz boundary, and where the notation $u \in W^{1,1}(\Omega; \mathbf{R}^m)$ means that each component of the function $u : \Omega \to \mathbf{R}^m$ belongs to the class $W^{1,1}(\Omega)$. Note that $L(Du)$ is a nonnegative measurable function and, therefore, we assume that $J(u) = \infty$ if $L(Du) \notin L^1$.

A well-known approach to study the existence issue via lower semicontinuity with respect to the weak convergence in $W^{1,1}$ is due to Tonelli [67]. We can guarantee this convergence for a subsequence of any sequence u_k with $J(u_k) \leq c < \infty$ provided

$$L(v) \geq \theta(v), \quad \text{where } \theta(v)/|v| \to \infty \text{ as } |v| \to \infty.$$

However this lower semicontinuity property is difficult to characterize in terms of L. What could be easily characterized is the lower semicontinuity with respect to the weak* convergence in $W^{1,\infty}$. C. Morrey [42] has proved that the functional J in (1) is

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lower semicontinuous with respect to the weak* convergence in $W^{1,\infty}$ if and only if L is quasiconvex at each $A \in \mathbf{R}^{n\times m}$, where quasiconvexity at A means

$$\int_\Omega L(A + D\phi(x))dx \geq L(A) \operatorname{meas} \Omega, \quad \forall \phi \in C_0^\infty(\Omega; \mathbf{R}^m).$$

Quasiconvexity does not depend on the domain, see Proposition 2.3. It is stronger than rank-one convexity, i.e. than the requirement of convexity of all functions $t \to L(A+tB)$ with $\operatorname{rank} B = 1$, (this is still unknown for the case $n = m = 2$) though the properties coincide for quadratic L, see [64].

A well-known class of quasiconvex integrands L is given by polyconvex functions, where L is called polyconvex if it is a convex function $\tilde{L}$ of the minors of Du. If

$$L(v) \geq \alpha|v|^{n+\epsilon} + \beta, \quad \alpha, \epsilon > 0, \tag{3}$$

then sequences bounded in energy are weakly compact in $W^{1,n+\epsilon}$. All the minors turn out to be continuous with respect to the weak convergence in $W^{1,n+\epsilon}$, see [3], [46]. Then the well-known fact that convexity of $\tilde{L}$ implies the lower semicontinuity of the functional

$$\xi \in L^1 \to \int_\Omega \tilde{L}(\xi(x))dx$$

with respect to the weak convergence in L^1, see e.g. [47], implies lower semicontinuity of J. In particular polyconvexity implies quasiconvexity. An advantage is that polyconvexity and (3) guarantee both the weak convergence and lower semicontinuity of J on a subsequence of any sequence bounded in energy. Therefore, an existence result in problem (1), (2) follows. Polyconvexity also fits both (3) and (4), where

$$L(Du) \to \infty \ \text{ as } \det Du \to +0; \qquad L(Du) = \infty, \ \det Du \leq 0, \tag{4}$$

that allows to apply the variational approach to the Mathematical Theory of Elasticity, as it was suggested by J. M. Ball, cf. [3], [15]. Much later L. Székelyhidi [66] showed that if we want to avoid the a priori assumption of smallness of $u - Id$, i.e. if we intend to switch to nonlinearized Elasticity, then we really have to relay on a variational approach instead of using the Euler equations, since their Lipschitz solutions may have everywhere oscillating gradients, contrary to the situation with minimizers, see [22], [37].

The assumption (3) turns out to be crucial for lower semicontinuity in the energetic set, as an example by J. M. Ball shows [4]: for

$$L = \mu|Du|^{n-\epsilon} + |\det Du|, \quad u : B \subset \mathbf{R}^n \to \mathbf{R}^n, \ \epsilon > 0, \ \mu > 0, \tag{5}$$

where B is the unit ball centered at the origin, we have $J(Id) > J(x/|x|)$ provided $\mu = \mu(\epsilon)$ is sufficiently small, since $J(Id) \geq \operatorname{meas} B$ and $x/|x| \in W^{1,n-\epsilon}$ with $\det D(x/|x|) = 0$ a.e. Then we can generate a sequence u_k with $u_k \rightharpoonup Id$ in $W^{1,n-\epsilon}$ for which lower semicontinuity fails, see Proposition 2.3.

As we see, if (3) fails, polyconvexity and, then, quasiconvexity, does not imply the lower semicontinuity in the energetic space. This delicate phenomena was studied for the polyconvex case in many papers, see [1], [10], [13], [21], [25], [38]. A natural explanation of the failure of lower semicontinuity is that the formation of discontinuities require an extra energetic term in addition to the bulk one (1), see [65], [43], [45]. Therefore one can hope that (3) can provide lower semicontinuity also for quasiconvex L. This issue was raised several times, e.g. in [5]–[7], and we devote our paper to this question.

Before stating the results of this paper, we mention some known cases where quasiconvexity implies lower semicontinuity. E. Acerbi and N. Fusco [2], see also P. Marcellini [40], showed that this holds for integrands satisfying the so-called standard growth conditions:

$$c_1|\cdot|^p + c_2 \leq L(\cdot) \leq c_3|\cdot|^p + c_4, \quad p > 1, \ c_3 \geq c_1 > 0. \tag{6}$$

In case L satisfies these estimates with power functions having different and sufficiently close exponents one can show that lower semicontinuity holds in the class associated with the higher exponent with respect to the weak convergence associated with the lower exponent, see [8], [26], [27], [39], [41], [33].

In this paper we assume

$$c_1 G(v) + c_2 \leq L(v) \leq c_3 G(v) + c_4, \quad c_3 \geq c_1 > 0, \tag{7}$$

with a convex and nonnegative function $G : \mathbf{R}^{n\times m} \to \mathbf{R}$. We also assume that there exists $\eta > 0$ such that

$$G(\cdot + v) \leq c_5 G(\cdot) + c_6, \quad |v| \leq \eta, \tag{8}$$

a requirement on the growth of G.

We prove

Theorem 1.1 (Lower semicontinuity). *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be a continuous and nonnegative function satisfying* (3), (7), *and* (8). *Let $u \in W^{1,n+\epsilon}(\Omega; \mathbf{R}^m)$ and let L be quasiconvex at $Du(x)$ for a.e. $x \in \Omega$.*

Then, the convergence $u_k \rightharpoonup u$ in $W^{1,n+\epsilon}(\Omega; \mathbf{R}^m)$ implies

$$\liminf_{k\to\infty} J(u_k) \geq J(u).$$

Here we use $\rightharpoonup$ to denote the weak convergence.

In fact we prove a stronger result, namely Theorem 3.3, when (3) is dropped and $u, u_k \in W^{1,1}$, $k \in \mathbf{N}$, provided u is a.e. differentiable in the classical sense and $||u_k - u||_{L^\infty} \to 0$, $k \to \infty$.

Corollary 1.2 (Existence). *Under assumptions of Theorem* 1.1 *we assume that L is everywhere quasiconvex. Then each problem* (1), (2) *admits a solution provided there exists at least one admissible function u, i.e. $u \in W^{1,n+\epsilon}$ that satisfies both* (2) *and $J(u) < \infty$.*

Theorem 1.3 (Relaxation). *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be a continuous and nonnegative function satisfying* (3), (7), *and* (8). *Let $L^{qc} : \mathbf{R}^{n\times m} \to \mathbf{R}$ be the quasiconvexification of L, i.e.*

$$L^{qc}(A) := \inf\left\{ \int_\Omega L(A + D\phi(x))dx / \operatorname{meas} \Omega : \ \phi \in C_0^\infty(\Omega; \mathbf{R}^m) \right\}. \tag{9}$$

Then L^{qc} is a continuous, nonnegative and quasiconvex function that also satisfies (7). *Moreover the functional*

$$J^{qc}(u) := \int_\Omega L^{qc}(Du(x))dx$$

is the lower semicontinuous envelope of J, i.e. J^{qc} is lower semicontinuous with respect to the weak convergence in $W^{1,n+\epsilon}$ and for each $u \in W^{1,n+\epsilon}$ with $J^{qc}(u) < \infty$ there exists a sequence $u_k \in u + W_0^{1,n+\epsilon}$ such that $u_k \rightharpoonup u$ in $W^{1,n+\epsilon}$ and $J(u_k) \to J^{qc}(u)$ as $k \to \infty$.

Again we prove a stronger result, Theorem 4.2, where the requirement (3) is omitted and the relaxation result holds at those deformations that are a.e. differentiable in the classical sense.

Note that the extra assumptions on strong materials we use in this paper, i.e. the assumptions (7)–(8), allow us to obtain the results via the most obvious properties of admissible deformations: continuity and a.e. differentiability. Further progress could rely on exploiting deeper topological results valid for this case, see e.g. [68], [69]. However, it is difficult to expect to be able to avoid any requirements on the growth of L at infinity, since quasiconvexity is a severely nonlocal property, see [34], [35].

First relaxation results were established long ago by Bogolubov [9] in the one-dimensional case $n = 1$. They were extended to the scalar ($m = 1$) multi-dimensional case by I. Ekeland and R. Temam [24] for integrands with the standard growth (6). The same growth allows to prove the relaxation result for the general case $n, m \geq 1$, see [16]; this result has the form (9). Relaxation obviously fails for L in (5) since in this case $L^{qc} = L$.

The technique we use in this paper does not allow us to consider the nonhomogeneous situation $L = L(x, Du)$ or a more general case $L = L(x, u, Du)$. These cases can be appropriately addressed in the context of Young measure theory, provided one can show that the Young measures generated by sequences bounded in energy can be also generated by sequences with values of the functional converging exactly to the value on the Young measure. The latter class of Young measures is called gradient L-Young measures. In the general case we have only the lower semicontinuity result, i.e., that the value of the functional on a Young measure does not exceed the liminf of the values of the functional along the sequence generating the measure.

D. Kinderlehrer & P. Pedregal [32] and J. Kristensen [36] proved that in the case (6) all Young measures are exhausted by gradient L-Young measures. We showed how to use this result to prove the relaxation theorem for general Caratheodory integrands [48], of course also with the property (6); see [17], [2] for earlier relaxation results in this case. The results of D. Kinderlehrer & P. Pedregal and of J. Kristensen can be extended to other classes of integrands, i.e., when (7) holds with $G(|\cdot|)$ having a fast growth at infinity, i.e. $G'(\cdot)|\cdot|/G(\cdot) \to \infty$ as $|\cdot| \to \infty$ [49], or, in the scalar case $m = 1$, with L satisfying only the inequality (3) [50]. Consequently the relaxation result holds for these cases. One can hope to extend the Young measure approach to other classes of integrands, say to the isotropic case $L(A) = L(QAR)$ for $Q, R \in SO(n)$ (see [61]–[63] for an information on that case), since homogeneous gradient L-Young measures can be characterized for general L [51]. An effective way to deal with Young measures in all problems related to the behaviour of integral functionals on weakly convergent sequences, see [52] for other such problems, turned out to be an approach to consider Young measures as measurable functions with values into a special metric space. For a systematic exposition of this approach to Young measure theory see [53], [54, §4].

We have to mention that lower semicontinuity is an industrial tool to establish existence of solutions in the minimization problems (1), (2). However, in the scalar case $m = 1$ with the property (3), we were able to characterize those integrands $L = L(Du)$ for which each minimization problem (1), (2) is solvable [55], [70]; see [11], [12], [20], [29], [30] for an earlier work in the case of linear boundary data. Again classical differentiability of solutions (of the relaxed problems) turned out to be a crucial property. The result could be extended to other classes of integrands provided this property would hold, see e.g. [50], [14]. In the vector-valued case the only known general solvability result is the situation where $L : \bar{U} \to \mathbf{R}$, $(L|_U)^{qc}$ is a quasiaffine function, here U is a bounded set, see [18], [19], [56] and see [57], [58] for better proofs, [31, p. 218] and [44] for the nonhomogeneous version.

Finally, recently we established that each integral functional coincides with its formal lower semicontinuous extension in a set which is dense in the weak topology and the values taken by the functional on this set completely determine the extension, and moreover, both the functionals are stable in this set, [54]. This result sharpens the standard relaxation theorems. It is also valid in the higher order case.

We prove certain auxiliary propositions in §2. In §3 we prove Theorem 1.1; in fact we establish a bit more general result when (3) is dropped, but we know a priori that $||u_k - u||_{L^\infty} \to 0$ as $k \to 0$ and that u is a.e. differentiable in the classical sense. In §4 we prove Theorem 1.3. A more general Theorem 4.2 asserts that the relaxation holds already under assumptions (7), (8) at those deformations that are a.e. differentiable in the classical sense. An important ingredient of the proof is a possibility to approximate any function with finite energy by nearly piece-wise affine ones, both in a strong norm and in energy, see Lemma 4.1. This approximation is a necessary tool for relaxation results, see also [49]–[51].

Everywhere in this paper we use the following notations: $B(x, \epsilon)$ for a ball in $\mathbf{R}^n$ centered at x with radius $\epsilon > 0$, l_A for any affine function with the gradient equal to A, $\bar{U}$ for the closure of U. Everywhere in this paper we assume that Ω is a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary. Given a subset Ω' of Ω we define

$$J(u; \Omega') := \int_{\Omega'} L(Du(x))dx.$$

2. Auxiliary propositions

In this section we recall and prove a number of auxiliary propositions that are used later in this paper.

The first is the Vitali covering theorem, see [60, p. 109].

Theorem 2.1. *Let U be a bounded subset of $\mathbf{R}^n$ and let F be a family of closed subsets of $\mathbf{R}^n$. Assume that for each $x \in U$ there exists $r(x) > 0$ and a sequence of balls $B(x, \epsilon_k)$*

with $\epsilon_k \to 0$ as $k \to \infty$ such that for certain sets $C_k \in F$, $k \in \mathbf{N}$, we can guarantee that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and $\operatorname{meas} C_k / \operatorname{meas} B(x, \epsilon_k) \geq r(x)$.

Then there exists an at most countable subfamily of disjoint sets $C_j \in F$ such that $\operatorname{meas}(U \setminus \cup_j C_j) = 0$.

Definition 2.2. For $\xi \in L^1$, we set

$$\mathrm{Im}\xi = \{y : \exists x, \text{ a Lebesgue point of } \xi, \text{ such that } y = \xi(x)\}.$$

We use Theorem 2.1 to derive

Proposition 2.3. *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be continuous and nonnegative, let $A \in \mathbf{R}^{n\times m}$, and let $u \in W_0^{1,1}(\Omega; \mathbf{R}^m)$, with $J(l_A + u) < \infty$. Let also $\Omega' \subset \mathbf{R}^n$ be an open bounded set.*

Given $\delta > 0$, there exists a function $u' \in W_0^{1,1}(\Omega'; \mathbf{R}^m)$ such that:

i) $\mathrm{Im}Du = \mathrm{Im}Du'$;

ii) $\frac{\int_{\Omega'} L(A+Du'(x))dx}{\operatorname{meas} \Omega'} = \frac{\int_{\Omega} L(A+Du(x))dx}{\operatorname{meas} \Omega}$, *and*

iii)
$$\frac{||u'||_{L^1(\Omega')}}{\operatorname{meas} \Omega'} \leq \frac{\delta ||u||_{L^1(\Omega)}}{\operatorname{meas} \Omega}.$$

Proof. Without loss of generality we assume that Ω contains the origin.

For each $\epsilon > 0$ we define $u^\epsilon(x) = \epsilon u(x/\epsilon)$. Then $u^\epsilon \in W_0^{1,1}(\epsilon\Omega; \mathbf{R}^m)$, $Du^\epsilon(x) = Du(x/\epsilon)$ and, therefore,

$$\int_{\epsilon\Omega} L(A + Du^\epsilon(x))dx = \epsilon^n \int_{\Omega} L(A + Du(y))dy. \tag{10}$$

We also have

$$\int_{\epsilon\Omega} |u^\epsilon(x)|dx = \epsilon \int_{\epsilon\Omega} |u(x/\epsilon)|dx = \epsilon^{n+1} \int_{\Omega} |u(y)|dy, \text{ i.e.}$$

$$||u^\epsilon||_{L^1(\epsilon\Omega)} / \operatorname{meas}(\epsilon\Omega) = \epsilon ||u||_{L^1(\Omega)} / \operatorname{meas} \Omega. \tag{11}$$

We define F to be the family of all sets of the form $x + \epsilon\bar{\Omega}$, where $\epsilon < \delta$, $x \in \Omega'$, and $(x + \epsilon\bar{\Omega}) \subset \Omega'$. Then F satisfies all the assumptions of Theorem 2.1 for the case $U = \Omega'$. We can apply that theorem to find an at most countable collection $x_j \in \Omega'$, and $\epsilon_j < \delta$, such that the sets $C_j := x_j + \epsilon_j\bar{\Omega}$, $j \in \mathbf{N}$, do not intersect and $C_j \subset \Omega'$, $j \in \mathbf{N}$, $\operatorname{meas}(\Omega' \setminus \cup_j C_j) = 0$.

We define $u' \in W_0^{1,1}(\Omega'; \mathbf{R}^m)$ as follows: set $u'(x) = u^{\epsilon_j}(x - x_j)$ for $x \in C_j$, $j \in \mathbf{N}$, and $u' = 0$ otherwise. Then, from (10), we have

$$\int_{\Omega'} |u'| = \sum \int_{\epsilon_j\Omega} |u^{\epsilon_j}(x)| \leq \frac{\delta}{\operatorname{meas}(\Omega)} \sum \operatorname{meas}(\epsilon_j\Omega) \int_{\Omega} |u|$$

proving the validity of the Proposition. □

Proposition 2.4. *Let L be a nonnegative continuous function that satisfy (7), let $A \in \mathbf{R}^{n\times m}$, and let $u \in W_0^{1,1}(\Omega; \mathbf{R}^m)$ be such that $J(l_A + u) < \infty$.*

Then given $\delta > 0$ there exists a function $\phi \in C_0^\infty(\Omega; \mathbf{R}^m)$ with

$$|J(l_A + \phi) - J(l_A + u)| < \delta. \tag{12}$$

This proposition says that the infimums of the functional (1) in the classes $\{l_A + u : u \in W_0^{1,1}\}$ and in $\{l_A + \phi : \phi \in C_0^\infty\}$ coincide, provided that (7) holds. This fact does not hold in general, as example (5) shows. An open problem is to verify whether Proposition 2.4 remains valid for strong materials, i.e. for L satisfying only (3), see [28] for a relevant information.

Proof of Proposition 2.4. By Proposition 2.3, given an open set $\Omega' \subset\subset \Omega$ with Lipschitz boundary, we can find a function $u' \in W_0^{1,1}(\Omega'; \mathbf{R}^m)$ such that

$$\int_{\Omega'} L(A + Du'(x))dx/\operatorname{meas}\Omega' = \int_{\Omega} L(A + Du(x))dx/\operatorname{meas}\Omega, \tag{13}$$

Set $u' = 0$ in $\Omega \setminus \Omega'$ and consider the ϵ-mollifications w_ϵ of u'. Then $w_\epsilon \in C_0^\infty(\Omega; \mathbf{R}^m)$ for $\epsilon > 0$ sufficiently small, $w_\epsilon \to u'$ in $W^{1,1}(\Omega; \mathbf{R}^m)$ as $\epsilon \to 0$, cf. [23]. Let also g_ϵ be the ϵ-mollifications of the function $g := G(A + Du') : \Omega \to \mathbf{R}$. Then $g_\epsilon \to g$ in L^1. By Jensen inequality (G is convex) we obtain

$$G(A + Dw_\epsilon(x)) \le g_\epsilon(x), \quad x \in \Omega.$$

This, (7), and the convergences $g_\epsilon \to g$, $Dw_\epsilon \to Du'$ in L^1 for $\epsilon \to 0$ allow to assert that $L(A + Dw_\epsilon) \to L(A + Du')$ in L^1 as $\epsilon \to 0$, e.g. via Lemma 4.3 of [59].

Since Ω' can be taken with $\operatorname{meas}(\Omega \setminus \Omega')$ arbitrarily small and since (13) holds we obtain (12).

The proof is complete. □

Lemma 2.5. *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be a continuous and nonnegative function. We define $L^{qc} : \mathbf{R}^{n\times m} \to \mathbf{R}$ as follows*

$$L^{qc}(A) := \inf\{J(l_A + \phi)/\operatorname{meas}\Omega : \phi \in C_0^\infty(\Omega; \mathbf{R}^m)\}.$$

Then L^{qc} is a continuous, nonnegative, and quasiconvex function.

We include proof of this lemma for convenience of the reader. Its versions are well-known in the literature, starting with [16].

Proof. Obviously $L^{qc} \ge 0$.

To prove the continuity of L^{qc} we establish both lower semicontinuity and upper semicontinuity.

Given $\phi \in C_0^\infty(\Omega; \mathbf{R}^m)$ and given a sequence $A_j \to A$, $j \to \infty$, we have $J(l_{A_j} + \phi) \to J(l_A + \phi)$, $j \to \infty$. This gives upper semicontinuity of L^{qc} at A.

To prove lower semicontinuity, given $\epsilon > 0$, we consider $\phi_j \in C_0^\infty(\Omega; \mathbf{R}^m)$, $j \in \mathbf{N}$, such that

$$|L^{qc}(A_j) \operatorname{meas} \Omega - J(l_{A_j} + \phi_j)| < \epsilon. \tag{14}$$

Let $\Omega' \subset\subset \Omega$ be an open set with Lipschitz boundary. By Proposition 2.3 applied to ϕ_j and by (14) we can find $\phi'_j \in W_0^{1,\infty}(\Omega'; \mathbf{R}^m)$ such that

$$|L^{qc}(A_j) \operatorname{meas} \Omega' - J(l_{A_j} + \phi'_j; \Omega')| < \epsilon \operatorname{meas} \Omega' / \operatorname{meas} \Omega, \quad j \in \mathbf{N}. \tag{15}$$

Let $\psi \in C_0^\infty(\Omega)$ be such that $\psi = 1$ in Ω', $0 \le \psi \le 1$. For each $j \in \mathbf{N}$ we define

$$f_j := \psi(l_{A_j} + \phi'_j) + (1 - \psi) l_A; \quad \text{here } \phi'_j = 0 \text{ in } \Omega \setminus \Omega'.$$

Then $f_j \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ and, therefore,

$$L^{qc}(A) \operatorname{meas} \Omega \le J(f_j), \quad j \in \mathbf{N}. \tag{16}$$

For each $j \in \mathbf{N}$ we have

$$Df_j = D\psi \otimes (l_{A_j} - l_A + \phi'_j) + A + \psi(A_j - A + D\phi'_j).$$

In Ω' we have $D\psi = 0$, $\psi = 1$ and, therefore,

$$Df_j = A_j + D\phi'_j, \quad j \in \mathbf{N}; \tag{17}$$

in $\Omega \setminus \Omega'$ we have $\phi'_j = 0$ and, therefore,

$$Df_j = A + \psi(A_j - A) + D\psi \otimes (l_{A_j} - l_A), \quad j \in \mathbf{N}. \tag{18}$$

Then (17), (18) imply

$$\limsup_{i \to \infty} |J(f_j) - J(l_{A_j} + \phi'_j; \Omega')| \le J(l_A; \Omega \setminus \Omega'), \tag{19}$$

and, therefore, we can also use (15), (16) to infer

$$L^{qc}(A) \operatorname{meas} \Omega \le \liminf_{j \to \infty} J(f_j) \le \liminf_{j \to \infty} L^{qc}(A_j) \operatorname{meas} \Omega' + \epsilon + J(l_A; \Omega \setminus \Omega').$$

Since Ω' can be taken with $\operatorname{meas}(\Omega \setminus \Omega')$ arbitrarily small we obtain

$$L^{qc}(A) \le \liminf_{j \to \infty} L^{qc}(A_j) + \epsilon / \operatorname{meas} \Omega.$$

Lower semicontinuity of L^{qc} at A and, then, continuity at A is established.

To prove quasiconvexity of L^{qc} enough to establish the inequality

$$L^{qc}(A) \operatorname{meas} \le \int_\Omega L^{qc}(A + D\phi(x)) dx \tag{20}$$

for piece-wise affine $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$, i.e. for the case when Ω can be decomposed as a countable union of disjoint Lipschitz domains Ω_j, $j \in \mathbf{N}$, with $\phi : \Omega_j \to \mathbf{R}^m$ affine for each $j \in \mathbf{N}$ and a set of zero measure. We assume that $D\phi = A_j$ in Ω_j, $j \in \mathbf{N}$.

Given $\epsilon > 0$ for each $j \in \mathbf{N}$ we can find $\phi_j \in C_0^\infty(\Omega_j; \mathbf{R}^m)$ with

$$L^{qc}(A + A_j)\,\text{meas}\ \Omega_j \geq \int_{\Omega_j} L(A + A_j + D\phi_j(x))dx - \epsilon\,\text{meas}\ \Omega_j, \tag{21}$$

cf. Proposition 2.3. For each $j \in \mathbf{N}$ we define $\psi_j \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ as follows:

$$\psi_j = \phi + \phi_i \ \text{ in } \Omega_i \text{ if } i \leq j,\ \psi_j = \phi \text{ otherwise.}$$

Then (21) implies

$$\begin{aligned} L^{qc}(A)\,\text{meas}\ \Omega \leq \int_\Omega L(A + D\psi_j(x))dx &= \sum_{i=1}^\infty \int_{\Omega_i} L(A + D\psi_j(x))dx \\ &\leq \sum_{i=1}^{j} \{L^{qc}(A + A_i)\,\text{meas}\ \Omega_i + \epsilon\,\text{meas}\ \Omega_i\} \\ &\quad + \sum_{i>j} \int_{\Omega_i} L(A + D\phi(x))dx, \quad j \in \mathbf{N}. \end{aligned} \tag{22}$$

The right-hand side in (22) converges to

$$\sum_{i=1}^\infty L^{qc}(A + A_i)\,\text{meas}\ \Omega_i + \epsilon\,\text{meas}\ \Omega = \int_\Omega L^{qc}(A + D\phi(x))dx + \epsilon\,\text{meas}\ \Omega.$$

This way we obtain (20), i.e. L^{qc} is quasiconvex at A.

Proof of Lemma 2.5 is complete. □

3. Proof of Theorem 1.1

We will exploit certain advantages that are guaranteed by the situation of strong materials, i.e. by the inequality (3).

Lemma 3.1. *Let $u \in W^{1,n+\epsilon}(\Omega; \mathbf{R}^m)$. Then u is continuous and a.e. differentiable in the classical sense.*

In case $u_k \in W^{1,n+\epsilon}(\Omega; \mathbf{R}^m)$ and $u_k \rightharpoonup u$ in $W^{1,n+\epsilon}$, $k \to \infty$, we also have $||u_k - u||_C \to 0$.

See e.g. [23, Ch. 4,6] for a proof.

Lemma 3.2. *Let $f_k : \Omega \to [0, \infty[$ satisfy $||f_k||_{L^1(\Omega)} < const < \infty$, $k \in \mathbf{N}$.*

Then there exists a subsequence (not relabeled) and a Radon measure μ supported in $\bar{\Omega}$ such that $f_k \rightharpoonup^* \mu$ in the sense of measures. Moreover $\mu = \mu_s + \mu_r$, where μ_s, μ_r are Radon measures such that μ_s is singular, μ_r is regular with respect to Lebesque measure; in particular $\mu_r = f dx$ with $f \in L^1(\Omega)$.

Again see [23, Ch. 1] for these results.

In this section we establish a more general, than Theorem 1.1, result which is

Theorem 3.3. *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be continuous and nonnegative. Assume that L satisfies both* (7) *and* (8).

Let $u \in W^{1,1}(\Omega;\mathbf{R}^m)$ satisfy $J(u) < \infty$ and assume that u is a.e. differentiable in the classical sense, L is quasiconvex at $Du(x)$ for a.e. $x \in \Omega$.

If $u_k \in W^{1,1}(\Omega;\mathbf{R}^m)$, $k \in \mathbf{N}$, and if $||u_k - u||_{L^\infty} \to 0$, $k \to \infty$, then

$$\liminf_{k\to\infty} J(u_k) \geq J(u).$$

Proof. The result follows in case $J(u_k) \to \infty$, $k \to \infty$. Therefore we may assume without loss of generality that

$$J(u_k) \to I < \infty, \quad k \to \infty. \tag{23}$$

Moreover by Lemma 3.2 we may also assume that the functions $f_k := L(Du_k)$, $k \in \mathbf{N}$, generate a Radon measure μ supported in $\bar{\Omega}$:

$$f_k \rightharpoonup^* \mu, \quad k \to \infty. \tag{24}$$

Let $x_0 \in \Omega$ be a point of classical differentiability of u, a Lebesque point of the functions $x \in \Omega \to Du(x)$, $x \in \Omega \to G(Du(x))$, and a point of quasiconvexity of L at $Du(x)$.

Our aim is to prove the following main estimate: given $\delta > 0$ there exists $r(x_0,\delta) > 0$ such that for each $r \leq r(x_0,\delta)$ there exists $k(r) \in \mathbf{N}$ such that for $k \geq k(r)$ we have

$$\begin{aligned}&\int_{B(x_0,r)} L(Du(x))dx \\ &\leq \int_{B(x_0,r)} L(Du_k(x))dx + \delta \,\mathrm{meas}\; B(x_0,r) + c\mu\left\{\bar{B}(x_0,r) \setminus B(x_0,(1-\delta)r)\right\}.\end{aligned} \tag{25}$$

Classical differentiability of u at x_0 means

$$u(x) = u(x_0) + \langle Du(x_0), x - x_0\rangle + o(x-x_0) = l_A + o(x-x_0), \quad \text{here } A = Du(x_0),$$

where $|o(x-x_0)| \leq w(|x-x_0|)$ with w being a nonnegative nonincreasing function such that $w(r)/r \to 0$ as $r \to +0$.

We may consider another nonnegative nonincreasing function w_1 such that $w_1(r) \to 0$ and $w(r)/w_1(r)r \to 0$ as $r \to +0$.

Given $r > 0$ we define $u_k^r \in l_A + W_0^{1,1}(B(x_0,r);\mathbf{R}^m)$ by

$$u_k^r := \phi u_k + (1-\phi) l_A,$$

where $\phi \in C_0^\infty(B(x_0,r))$, $0 \leq \phi \leq 1$, $\phi = 1$ in $B(x_0, r(1-w_1(r)))$; we can choose ϕ to satisfy the inequality

$$|D\phi| \leq \frac{2}{rw_1(r)}. \tag{26}$$

In $B(x_0, r(1-w_1(r))$ we have

$$Du_k^r = Du_k. \tag{27}$$

In $B(x_0,r)\setminus B(x_0,r(1-w_1(r))$ we have

$$Du_k^r=\phi(Du_k)+(1-\phi)Du(x_0)+D\phi\otimes(u_k-l_A), \tag{28}$$

where due to (26)

$$|D\phi\otimes(u_k-l_A)|\le\frac{2}{rw_1(r)}2w(r)\le\eta \tag{29}$$

provided r is sufficiently small and $||u_k-u||_{L^\infty(B(x_0,r);\mathbf{R}^m)}\le w(r)$ that holds for all $k\ge k(r)$, see (8) for the definition of $\eta>0$. Then (8), (28), (29) and convexity of G imply

$$\begin{aligned}G(Du_k^r(x))&\le c_5G(\phi(Du_k(x))+(1-\phi)Du(x_0))+c_6\\&\le c_5\phi G(Du_k(x))+(1-\phi)G(Du(x_0))+c_6.\end{aligned}$$

Therefore the inequality (7) implies

$$L(Du_k^r(x))\le c_7\{L(Du_k(x))+L(Du(x_0))\}+c_8. \tag{30}$$

By Proposition 2.4 and by quasiconvexity of L at $Du(x_0))$ we infer (see also (27), (30))

$$\begin{aligned}&\int_{B(x_0,r)}L(Du(x_0))dx\le\int_{B(x_0,r)}L(Du_k^r(x))dx\\&\le\int_{B(x_0,r(1-w_1(r)))}L(Du_k(x))dx\\&\quad+c_7\int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))}\{L(Du_k(x))+L(Du(x_0))\}dx\\&\quad+c_8\,\text{meas}\,\{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))\}.\end{aligned} \tag{31}$$

Moreover

$$\left|\int_{B(x_0,r)}\{L(Du(x))-L(Du(x_0))\}dx\right|\le w_2(r)\,\text{meas}\;B(x_0,r), \tag{32}$$

where $w_2(r)\to0$ as $r\to+0$.

Then (31), (32) result in

$$\begin{aligned}&\int_{B(x_0,r)}L(Du(x))dx\\&\le w_2(r)\,\text{meas}\;B(x_0,r)+\int_{B(x_0,r)}L(Du_k(x))dx\\&\quad+c_7\int_{\bar B(x_0,r)\setminus B(x_0,r(1-w_1(r)))}L(Du_k(x))dx\\&\quad+(c_7L(Du(x_0))+c_8)\,\text{meas}\,\{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))\}\\&\le\int_{B(x_0,r)}L(Du_k(x))dx+\{w_2(r)+(c_7L(Du(x_0))+c_8)c_9w_1(r)\}\,\text{meas}\;B(x_0,r)\\&\quad+c_7\int_{\bar B(x_0,r)\setminus B(x_0,r(1-w_1(r)))}L(Du_k(x))dx.\end{aligned} \tag{33}$$

In case $r > 0$ is sufficiently small and $k \geq k(r)$ we can guarantee that the second term in the right-hand side does not exceed $\delta \,\text{meas}\, B(x_0, r)/2$ and that $c_7 c_9 w_1(r) \leq \delta/2$. The third term does not exceed

$$c_7 \mu(\bar{B}(x_0, r) \setminus B(x_0, r(1 - w_1(r)))) + c_7 \,\text{meas}\, \{\bar{B}(x_0, r) \setminus B(x_0, r(1 - w_1(r)))\}$$

for k sufficiently large, see (24). Since the second term in the last expression is less than $c_7 c_9 w_1(r) \,\text{meas}\, B(x_0, r)$ we infer (25) with $c = c_7$.

Now we can use (25) to prove the theorem.

By Lemma 3.2 $\mu = \mu_s + f dx$ with $f \in L^1(\Omega)$. Then we can find a compact set $K_\delta \subset \bar{\Omega}$ of zero measure such that $\mu_s(\Omega \setminus K_\delta) \leq \delta$. Consider the set $U \subset (\Omega \setminus K_\delta)$ as union of all those points $x_0 \in (\Omega \setminus K_\delta)$ where (25) holds; then $\text{meas}\, U = \text{meas}\, \Omega$. By (25) we can define a Vitali cover of U. Then Theorem 2.1 allows to find a finite collection of disjoint balls $\bar{B}_1, \ldots, \bar{B}_M \subset (\Omega \setminus K_\delta)$, with $\text{meas}\, (\Omega \setminus \cup_{i=1}^M B_i) \leq \delta$, for each of which (25) holds if $k \geq \max\{k(i) : i = 1, \ldots, M\}$. For these k we have

$$\begin{aligned} \int_{\cup_{i=1}^M B_i} L(Du(x))dx \leq & \int_{\cup_{i=1}^M B_i} L(Du_k(x))dx + \delta \left(\text{meas}\, \cup_{i=1}^M B_i\right) \\ & + c\mu_s(\Omega \setminus K_\delta) + c||f||_{L^1(\cup_{i=1}^M (B_i \setminus (1-\delta)B_i))} \\ \leq & \; J(u_k) + \delta(\text{meas}\, \Omega + c) + c||f||_{L^1(\Omega_\delta)}, \end{aligned} \tag{34}$$

where $\text{meas}\, \Omega_\delta \leq c_9 \delta \,\text{meas}\, \Omega$. Therefore

$$\liminf_{k \to \infty} J(u_k) \geq J(u).$$

□

Proof. Proof of Theorem 1.1 can be reduced to the situation considered in Theorem 3.3. In case (23) holds we can use (7) and lower semicontinuity with respect to the weak convergence in L^1 of integral functionals with convex integrands, see e.g. [47], to derive $J(u) < \infty$. Then Lemma 3.1 asserts that the assumptions of Theorem 1.1 imply the assumptions of Theorem 3.3. This proves the result. □

The result of Corollary 1.2 immediately follows from Theorem 1.1.

4. Proof of Theorem 1.3

A crucial ingredient of the proof of Theorem 1.3 is an approximation of admissible functions by nearly piece-wise affine ones both in $W^{1,1}$-norm and in energy.

Let Ω' be an open bounded set of $\mathbf{R}^n$. We say that a function $f : \Omega' \to \mathbf{R}^m$ is *finitely piece-wise affine* in Ω' provided there is a decomposition of Ω' into a finite collection of open disjoint sets, in each of which f is affine, and a set of zero measure.

Lemma 4.1. *Let L be a continuous and nonnegative function and let L satisfy* (7), (8). *Let also $u \in W^{1,1}(\Omega; \mathbf{R}^m)$ be a.e. differentiable in the classical sense with $J(u) < \infty$.*

Then there exist open sets $\Omega_k \subset\subset \Omega$ with Lipschitz boundary and functions $u_k \in u + W_0^{1,1}(\Omega; \mathbf{R}^m)$ such that $u_k : \Omega_k \to \mathbf{R}^m$ is finitely piece-wise affine in Ω_k, $k \in \mathbf{N}$, and

$$\text{meas}\, (\Omega \setminus \Omega_k) \to 0, \quad ||u_k - u||_{W^{1,1} \cap L^\infty(\Omega)} \to 0, \quad J(u_k) \to J(u) \quad \textit{as } k \to \infty.$$

Proof. Let $x_0 \in \Omega$ be a Lebesque point of the functions $x \to Du(x)$, $x \to L(Du(x))$ and a point of classical differentiability of u. Then

$$u(x) = u(x_0) + \langle Du(x_0), x - x_0 \rangle + o(x - x_0) = l_A + o(x - x_0), \quad A := Du(x_0),$$

where $|o(x-x_0)| \le w(|x-x_0|)$ with a nonnegative nonincreasing function w such that $w(r)/r \to 0$ as $r \to +0$.

We can find another nonnegative nonincreasing function w_1 such that $w_1(r) \to 0$, $w(r)/rw_1(r) \to 0$ as $r \to +0$.

Given $r > 0$ we define $u_r : B(x_0, r) \to \mathbf{R}^m$ as follows:

$$u_r = \phi l_A + (1 - \phi)u,$$

where $\phi \in C_0^\infty(B(x_0, r))$, $\phi = 1$ in $B(x_0, r(1 - w_1(r)))$, $0 \le \phi \le 1$. The function ϕ can be chosen satisfying the inequality

$$|D\phi| \le 2/rw_1(r).$$

Then $u_r \in u + W_0^{1,1}(B(x_0, r); \mathbf{R}^m)$ and

$$Du_r(x) = \phi(Du(x_0) - Du(x)) + Du(x) + D\phi(x) \otimes (l_A - u(x)). \tag{35}$$

In $B(x_0, r(1 - w_1(r)))$ we have

$$Du_r(x) = Du(x_0). \tag{36}$$

In $B(x_0, r) \setminus B(x_0, r(1 - w_1(r)))$ we have

$$Du_r(x) = \phi Du(x_0) + (1 - \phi)Du(x) + D\phi(x) \otimes (l_A - u(x)). \tag{37}$$

If $r > 0$ is sufficiently small then the last term in (37) does not exceed by modulus $2w(r)/rw_1(r) \le \eta$, see (8), and we can apply (8) to infer

$$G(Du_r(x)) \le c_5 \phi G(Du(x_0)) + c_5(1 - \phi)G(Du(x)) + c_6.$$

Due to (7)

$$\begin{aligned}
&\int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} L(Du_r(x))dx \\
&\le c_7 \int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} \{L(Du(x_0)) + L(Du(x))\}dx \\
&\qquad + c_8 \,\mathrm{meas}\, (B(x_0, r) \setminus B(x_0, r(1 - w_1(r))) \\
&\le \{c_7 L(Du(x_0)) + c_8\} c_9 w_1(r) \,\mathrm{meas}\, B(x_0, r) \\
&\qquad + c_7 \int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} L(Du(x))dx \\
&\le \delta \,\mathrm{meas}\, B(x_0, r) + c_7 \int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} L(Du(x))dx
\end{aligned} \tag{38}$$

if r is sufficiently small and $\delta > 0$ is beforehand fixed.

For sufficiently small $r > 0$ we also have

$$||u_r - u||_{L^\infty(B(x_0,r))} \le ||\phi(l_A - u)||_{L^\infty} \le w(r) \le \delta, \tag{39}$$

$$\begin{aligned}
&||Du_r - Du||_{L^1(B(x_0,r))} \\
\le &\int_{B(x_0,r(1-w_1(r)))} |Du(x_0) - Du(x)| dx \\
&\quad + \int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} |\phi(Du(x_0) - Du(x)) + D\phi \otimes (l_A - u)| dx \\
\le &\int_{B(x_0,r)} |Du(x_0) - Du(x)| dx + (2w(r)/rw_1(r))c_9 w_1(r) \operatorname{meas} B(x_0,r) \\
\le &\, w_2(r) \operatorname{meas} B(x_0,r) + 2c_9(w(r)/r) \operatorname{meas} B(x_0,r) \le \delta \operatorname{meas} B(x_0,r)
\end{aligned} \tag{40}$$

since $w(r)/r, w_2(r) \to 0$ as $r \to +0$ (recall that x_0 is a Lebesque point for the function $x \to Du(x)$).

Finally, due to (36), (38) we can take $r > 0$ so small that $w_1(r) \le \delta$ and

$$\begin{aligned}
&\left| \int_{B(x_0,r)} \{L(Du(x)) - L(Du_r(x))\} dx \right| \\
\le &\left| \int_{B(x_0,r(1-w_1(r)))} \{L(Du(x)) - L(Du(x_0))\} dx \right| \\
&\quad + \left| \int_{B(x_0,r)\setminus B(x_0,r(1-w_1(r)))} \{L(Du(x)) - L(Du_r(x))\} dx \right| \\
\le &\, 2\delta \operatorname{meas} B(x_0,r) + (c_7 + 1) \int_{B(x_0,r)\setminus B(x_0,r(1-\delta))} L(Du(x)) dx,
\end{aligned} \tag{41}$$

here we exploited the assumption that x_0 is a Lebesque point for the function $x \to L(Du(x))$.

Given $k \in \mathbf{N}$ we can apply (39)–(41) and the Vitali covering theorem (Theorem 2.1) to find a finite collection of disjoint balls $\bar{B}_1, \ldots, \bar{B}_{M(k)} \subset \Omega$ such that

$$\operatorname{meas} \left(\Omega \setminus \cup_{i=1}^{M(k)} B_i \right) \le 1/k \tag{42}$$

and for each $B_i = B(x_i, r_i)$, $i = 1, \ldots, M(k)$, the estimates (39)–(41) holds for $\delta = 1/k$ with $u_{r_i} \in u + W_0^{1,1}(B(x_i, r_i) : \mathbf{R}^m)$ such that u_{r_i} is affine in $B(x_i, (1 - 1/k)r_i)$.

Then we define $u_k = u_{r_i}$ in B_i, $i = 1, \ldots, M(k)$, $u_k = u$ otherwise.

The inequality (39) implies

$$||u_k - u||_{L^\infty} \le 1/k. \tag{43}$$

The inequality (40) implies

$$||Du_k - Du||_{L^1} \le (1/k) \operatorname{meas} \Omega. \tag{44}$$

Finally the inequality (41) implies

$$|J(u_k) - J(u)| \leq (2/k)\,\mathrm{meas}\;\Omega + (c_7+1)\int_{\cup_{i=1}^{M(k)}(B(x_i,r_i)\setminus B(x_i,r_i(1-1/k))} L(Du(x))dx$$
$$= 2\,\mathrm{meas}\;\Omega/k + (c_7+1)J(u;\tilde{\Omega}_k)$$

with meas $\tilde{\Omega}_k \to 0$ as $k \to \infty$. Therefore

$$J(u_k) \to J(u),\;\; k \to \infty. \tag{45}$$

The inequalities (43)–(45) imply the convergences

$$||u_k - u||_{L^\infty} \to 0, \quad ||u_k - u||_{W^{1,1}} \to 0, \quad J(u_k) \to J(u) \;\text{ as } k \to \infty.$$

Moreover u_k is affine in $B(x_i, r_i(1-1/k))$, $i = 1, \ldots, M(k)$. Since meas $(\Omega \setminus \cup_{i=1}^{M(k)} B(x_i, r_i(1-1/k))) \to 0$ as $k \to \infty$ the proof is complete. □

As we already mentioned in Introduction in this section we establish a bit stronger, than Theorem 1.3, result – we prove the relaxation theorem in the case (7), (8) at deformations which are a.e. differentiable in the classical sense.

Theorem 4.2. *Let $L : \mathbf{R}^{n\times m} \to \mathbf{R}$ be a continuous and nonnegative function that satisfies* (7), (8).

Then L^{qc} is also continuous and nonnegative, moreover it is quasiconvex and satisfies (7). *In case $u \in W^{1,1}(\Omega;\mathbf{R}^m)$ with $J^{qc}(u) < \infty$ and u is a.e. differentiable in the classical sense there exists a sequence $u_k \in u + W_0^{1,1}(\Omega;\mathbf{R}^m)$ such that $||u_k - u||_{L^\infty} \to 0$ and $J(u_k) \to J^{qc}(u)$ as $k \to \infty$.*

Proof. By Lemma 2.5 the integrand L^{qc} is nonnegative, continuous and quasiconvex. It also satisfies the same estimates (7) as L. In fact, the right-hand side inequality holds since $L^{qc} \leq L$; the left-hand side inequality is valid since the function $c_1 G(\cdot) + c_2$ is convex and we may apply the Jensen inequality to derive

$$c_1 G(A) + c_2 \leq c_1 \int_\Omega G(A + D\phi(x))dx + c_2$$
$$\leq \int_\Omega L(A + D\phi(x))dx, \quad \forall \phi \in C_0^\infty(\Omega;\mathbf{R}^m),$$

i.e. $c_1 G(A) + c_2 \leq L^{qc}(A)$.

Let $u \in W^{1,1}(\Omega;\mathbf{R}^m)$ be a.e. differentiable in the classical sense and let $J^{qc}(u) < \infty$. Be Lemma 4.1 we can isolate functions $u_k \in u + W_0^{1,1}(\Omega;\mathbf{R}^m)$ and open sets $\Omega_k \subset\subset \Omega$, $k \in \mathbf{N}$, such that each Ω_k has Lipschitz boundary and $u_k : \Omega_k \to \mathbf{R}^m$ is finitely piece-wise affine in Ω_k, $k \in \mathbf{N}$; moreover

$$\mathrm{meas}\,(\Omega \setminus \Omega_k) \to 0, \quad ||u_k - u||_{W^{1,1}\cap L^\infty(\Omega)} \to 0, \quad J^{qc}(u_k) \to J^{qc}(u) \;\text{ as } k \to \infty. \tag{46}$$

Then

$$L^{qc}(Du_k(\cdot)) \to L^{qc}(Du(\cdot)) \;\text{ in } L^1,\; k \to \infty, \tag{47}$$

cf. e.g. Lemma 4.3 of [59].

Given $k \in \mathbf{N}$ we decompose Ω_k into a finite collection of disjoint open sets Ω_k^j, $j \in \{1, \dots, M(k)\}$, such that $u_k : \Omega_k^j \to \mathbf{R}^m$ is affine for each $j \in \mathbf{N}$ and a set of zero measure. For each $j \in \{1, \dots, M(k)\}$ we may isolate $\phi_j^k \in W_0^{1,\infty}(\Omega_k^j; \mathbf{R}^m)$ with

$$\left| J^{qc}\left(u_k; \Omega_k^j\right) - J\left(u_k + \phi_j^k; \Omega_k^j\right) \right| \leq \frac{1}{k} \operatorname{meas} \Omega_k^j, \tag{48}$$

$$||\phi_j^k||_{L^\infty(\Omega_k^j)} \leq 1/k, \tag{49}$$

cf. Proposition 2.3.

We define $w_k : \Omega \to \mathbf{R}^m$ as follows

$$w_k = u_k + \phi_j^k \ \text{ in } \Omega_k^j, \ j \in \{1, \dots, M(k)\},$$

$$w_k = u_k \ \text{ in } \Omega \setminus \Omega_k.$$

Then (46), (49) imply

$$||w_k - u||_{L^\infty} \to 0, \quad k \to \infty.$$

Moreover $w_k \in u + W_0^{1,1}(\Omega; \mathbf{R}^m)$ and

$$|J(w_k) - J^{qc}(u)| \leq |J(w_k; \Omega_k) - J^{qc}(u; \Omega_k)| + |J(u_k; \Omega \setminus \Omega_k) - J^{qc}(u; \Omega \setminus \Omega_k)|. \tag{50}$$

The first term in the right-hand side does not exceed

$$|J(w_k; \Omega_k) - J^{qc}(u_k; \Omega_k)| + |J^{qc}(u_k; \Omega_k) - J^{qc}(u; \Omega_k)|, \tag{51}$$

where the first term in (51) converges to zero by (48) and the second term in (51) converges to zero by (47). The second term in (50) converges to zero since $\operatorname{meas}(\Omega \setminus \Omega_k) \to 0$ (and, then, $J^{qc}(u; \Omega \setminus \Omega_k) \to 0$ as $k \to \infty$) and since

$$\int_{\Omega \setminus \Omega_k} G(Du_k(x))dx \to 0, \quad k \to 0,$$

because of (47) and (7).

Therefore $J(w_k) \to J^{qc}(u)$ as $k \to \infty$.

This completes the proof of the theorem. □

Proof of Theorem 1.3. By Theorem 4.2 and by Lemma 3.1 given $u \in W^{1,1}(\Omega; \mathbf{R}^m)$ with $J^{qc}(u) < \infty$ we can isolate a sequence $u_k \in u + W_0^{1,1}(\Omega; \mathbf{R}^m)$ such that

$$||u_k - u||_{L^\infty} \to 0, \quad J(u_k) \to J^{qc}(u) \ \text{ as } k \to \infty.$$

Then (3) implies $u_k \rightharpoonup u$ in $W^{1,n+\epsilon}$. This completes the proof of the theorem. □

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