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Homogenization of Non-Linear Functionals with Laminate-Type Growth

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The limit energy density, by Γ-convergence, of sequences of functionals whose densities satisfy a growth condition of order $p_j(x)$, depending on the laminate structure of the domain, is computed explicitly through a finite dimensional minimization problem.

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1. Introduction

Γ-convergence is a variational convergence for sequences of functionals I_j defined in an appropriate space X_j, which is intimately related to the asymptotic behaviour of minimum problems $\min\{I_j(u) : u \in X_j\}$. We say that the sequence $\{I_j\}$ is Γ-convergent, with respect to an appropriate topology, to the functional I if, for any function u, such functional evaluated at u is a lower bound for the lower limits of sequences $\{I_j(u_j)\}$, given any sequence $\{u_j\}$ converging to u. Besides, such lower bound should be attained for at least one sequence $\{u_j\}$. See [10, 11]. Given a sequence of integral functionals $\{I_j\}$ Γ-converging to a functional I, the main aim is to prove that the Γ-limit I is an integral functional and its integrand may be represented explicitly.

Many authors have studied Γ-convergence of sequences of functionals of the form

$$I_j(u) = \int_\Omega W(jx, \nabla u(x))\, dx$$

defined in some Sobolev space, where Ω is an open bounded subset of $\mathbb{R}^n$, and $\{jx\}$ is the oscillating sequence (with j tending to ∞), under the hypothesis that the function $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is periodic in the first variable, and satisfies a standard growth condition. See [6, 13, 14]. A great interest in studying Γ-convergence of functionals with periodic integrands comes from the description of the macroscopic behaviour of periodic structures in composite materials, and, in this way, is related to the homogenization theory. See [2, 5, 7, 27, 22].

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Nevertheless, we could ask about the Γ-convergence in the non-periodic setting: how could we represent explicitly the integrand of the Γ-limit of sequences of functionals of the type

$$I_j(u) = \int_\Omega W(a_j(x), \nabla u(x))\, dx$$

for general sequences of functions $a_j : \Omega \to \mathbb{R}^m$ and with non-periodic integrand $W : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$? In [18], an explicit characterization of the integrand of the Γ-limit (with respect to the weak topology in $W^{1,p}(\Omega)$) is obtained through a minimization problem depending only on W, and the Young measure associated with the sequence $\{a_j\}$. Such characterization of the integrand makes sense if the sequence $\{a_j\}$ satisfies the local condition called Average Gradient Property (AGP), see [18, Definition 4.1]. The sequence $\{a_j\}$ is said to satisfy the AGP if, in a neighbourhood of a.e. $x \in \Omega$, any sequence of piecewise constant functions, equal to the average of $\{a_j(x + r_j\cdot)\}$ on subsets of some partition of the unit ball $B \subset \mathbb{R}^n$, can be approximated by a sequence of gradients. However, in practise this condition is not so easy to handle. In [21] a sufficient condition on the sequence $\{a_j\}$, called Composition Gradient Property (CGP), is introduced. Namely, it is proved that if the sequence $\{a_j\}$ satisfies the CGP, then it also satisfies the AGP. We say that the sequence $\{a_j\}$ satisfies the CGP if its composition with some continuous, one-to-one field is "essentially a sequence of gradients" in the sense that may be approximated (in the strong topology of a Lebesgue space) by a sequence of gradients. In some sense, Γ-limits of integral functionals can be understood and written down in terms of new integral functionals provided the sequence of functionals depend on a sequence of functions which, even though they may not be gradients, "have a gradient structure".

In this work our aim is to study the Γ-convergence of sequences of functionals satisfying a non-standard growth condition depending on the periodic laminate structure of the domain Ω. Precisely, we would like to understand the behaviour of the sequence of functionals

$$F_j(u) = \int_\Omega f(p_j(x), \nabla u(x))\, dx,$$

when $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, convex in the second variable, and satisfies the nonstandard growth condition

$$\alpha|\rho|^{|\lambda|} \le f(\lambda, \rho) \le \beta\left(1 + |\rho|^{|\lambda|}\right) \quad \text{for all } (\lambda, \rho) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1}$$

for some $\beta \ge \alpha > 0$. Here the sequence of functions $p_j : \Omega \to \mathbb{R}^m$ stands for a single laminate (see [4, 15, 16]) defined by

$$p_j(x) = A_1\chi_{(0,t)}(jx \cdot \vec{n}) + A_2(1 - \chi_{(0,t)}(jx \cdot \vec{n})),$$

for vectors $A_1, A_2 \in \mathbb{R}^m$ such that $|A_1| = p \le q = |A_2|$, and unit vector $\vec{n} \in \mathbb{R}^n$, where $\chi_{(0,t)}(y \cdot \vec{n})$ is the characteristic function of the interval $(0, t)$ over $(0, 1)$, extended by periodicity to $\mathbb{R}$. Thus the sequence $\{f(p_j(x), \cdot)\}$ behaves like the power $|\cdot|^p$ in the layers orthogonal to $\vec{n}$ with proportion t, and like the power $|\cdot|^q$ otherwise, i.e.

$$\alpha|\rho|^p \le f(p_j(x), \rho) \le \beta(1 + |\rho|^p), \quad \text{if } \langle jx \cdot \vec{n}\rangle \in (0, t)$$
$$\alpha|\rho|^q \le f(p_j(x), \rho) \le \beta(1 + |\rho|^q), \quad \text{if } \langle jx \cdot \vec{n}\rangle \in (t, 1),$$

for every $j \in \mathbb{N}$ and $\rho \in \mathbb{R}^n$, where $\langle y \rangle$ stands for the fractional part of $y \in \mathbb{R}$. Notice that the nonstandard growth condition (1) is different from assuming either

$$\alpha|\rho|^p \leq f(\lambda, \rho) \leq \beta(1 + |\rho|^q), \quad \text{for all } (\lambda, \rho) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{2}$$

or

$$\alpha|\rho|^{p(x)} \leq f(p_j(x), \rho) \leq \beta(1 + |\rho|^{p(x)}), \quad \text{for a.e. } x \in \Omega \text{ and all } \rho \in \mathbb{R}^n, \tag{3}$$

taking some continuous function $p : \Omega \to (1, \infty)$. When the integrand f satisfies the previous $p(x)$-growth condition, the associated sequence of functionals F_j is defined in the generalized Sobolev space $W^{1,p(x)}(\Omega)$ given by

$$\begin{aligned} &W^{1,p(x)}(\Omega) \\ &= \left\{ u \in L^1(\Omega) : \int_\Omega |\eta u(x)|^{p(x)} dx < \infty, \int_\Omega |\eta \nabla u(x)|^{p(x)} dx < \infty, \text{ for some } \eta > 0 \right\}. \end{aligned}$$

See [1, 8, 12]. A great interest in studying this generalized Sobolev spaces comes from the modelling of the so called electroheological fluids, i.e. special non-Newtonian fluids which change their mechanical properties in the presence of electromagnetic fluids. See [23, 24]. Such fluids may be modelled by the homogeneous $p(x)$-Laplacean

$$-\text{div}\left[\, p(x)\, |\nabla u(x)|^{p(x)-2} \nabla u(x) \,\right] = 0 \ \text{ in } \Omega,$$

whose weak solutions are in $W^{1,p(x)}(\Omega)$.

Here we want to go further considering a laminate-type growth condition depending on the parameter j, so that each functional F_j is defined in the generalized Sobolev space $W^{1,|p_j(x)|}(\Omega)$ given by

$$\begin{aligned} &W^{1,|p_j(x)|}(\Omega) \\ &= \left\{ u \in L^1(\Omega) : \int_\Omega |\eta u(x)|^{|p_j(x)|} dx < \infty, \int_\Omega |\eta \nabla u(x)|^{|p_j(x)|} dx < \infty, \text{ for some } \eta > 0 \right\}, \end{aligned}$$

since it holds

$$\alpha|\rho|^{|p_j(x)|} \leq f(p_j(x), \rho) \leq \beta(1 + |\rho|^{|p_j(x)|}) \ \text{ for every } \rho \in \mathbb{R}^n, \text{ a.e. } x \in \Omega.$$

In this way our aim is to study the Γ-convergence of this sequence of functionals F_j defined in the special generalized space $W^{1,|p_j(x)|}(\Omega)$. One of the reasons for this study comes from the homogenization of the $|p_j(x)|$-Laplacean given by

$$-\text{div}\left[\, |p_j(x)|\, |\nabla u_j(x)|^{|p_j(x)|-2} \nabla u_j(x) \,\right] = 0 \ \text{ in } \Omega,$$

whose associated energies F_j defined by putting

$$F_j(u) \ = \ \int_\Omega |\nabla u(x)|^{|p_j(x)|} \, dx \tag{4}$$

are combination, depending on j, of different powers, and are defined in intermediate classes of functions between Sobolev spaces $W^{1,q}(\Omega)$ and $W^{1,p}(\Omega)$. See [19]. It is known that if the exponents are the same, e.g. $p = 2 = q$, then the resulting homogenized density will be also a power-law with the same exponent. See [27, 20, 25]. Moreover Γ-convergence of functionals with periodic integrands satisfying (p, q)-growth conditions, with $1 < p \leq q < p^\star$ ($p^\star$ is the Sobolev exponent of p), was already studied, see [6]; and the integral representation of the Γ-limit for general sequences of functionals assuming growth of order $p(x)$ was proved in [9]. In this work, our main contribution is the explicit characterization of the limit energy density of the sequence $\{F_j\}$ through a finite dimensional minimization problem, under no restrictions on the upper exponent q.

On the other hand, if one considers the Γ-convergence in different topologies and with different structures on Ω, interesting and surprising phenomena may occur. For instance, the Lavrentiev phenomenon may appear when we take the chess-board structure on the plane and power-laws, with different powers. Indeed, the asymptotic behaviour of sequences of functionals given in (4) was already studied in the case of chess-board structures, i.e. when the integrands behave like the power $|\cdot|^p$ on the black squares, and like the power $|\cdot|^q$ on the white squares. In this case, the homogenized integrand depends on the exponent of the Sobolev space where we are minimizing. See [28, 26].

2. Main Results

Our main result is the following characterization of the limit energy density ψ, and the main ingredient in the proof is the laminate (gradient) structure of the domain.

Theorem 2.1. *Let Ω be an open bounded set in $\mathbb{R}^n$, with Lipschitz boundary. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, convex in the second variable, satisfying the nonstandard growth condition*

$$\alpha|\rho|^{|\lambda|} \leq f(\lambda, \rho) \leq \beta(1 + |\rho|^{|\lambda|}) \quad \text{for all } (\lambda, \rho) \in \mathbb{R}^m \times \mathbb{R}^n,$$

for some $\beta \geq \alpha > 0$. Consider the sequence of functions $p_j : \Omega \to \mathbb{R}^m$ defined by

$$p_j(x) = A_1\chi_{(0,t)}(jx \cdot \vec{n}) + A_2(1 - \chi_{(0,t)}(jx \cdot \vec{n})),$$

for vectors $A_1, A_2 \in \mathbb{R}^m$ such that $|A_1| = p \leq q = |A_2|$, and unit vector $\vec{n} \in \mathbb{R}^n$. The sequence of functionals F_j given by

$$F_j(u) \ = \ \int_\Omega f(p_j(x), \nabla u(x))\, dx$$

is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional

$$F(u) \ = \ \int_\Omega \psi(\nabla u(x))\, dx,$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is defined by putting

$$\psi(\rho) = \min_{\Lambda_1, \Lambda_2 \in \mathbb{R}^n} \Big\{ t\, f(A_1, \Lambda_1) + (1-t)\, f(A_2, \Lambda_2) \ : \ \rho = t\Lambda_1 + (1-t)\Lambda_2, \ \Lambda_1 - \Lambda_2 \parallel \vec{n} \Big\} \tag{5}$$

or, equivalently,

$$\psi(\rho) \;=\; \min_{c\in\mathbb{R}} \;\Big\{ t\, f(A_1, \rho - (1-t)\, c\vec{n}) \;+\; (1-t)\, f(A_2, \rho + t\, c\vec{n}) \Big\}. \tag{6}$$

We achieve an explicit characterization for the limit energy density ψ through a finite minimization problem, depending only on $t \in (0,1)$ and $\vec{n}$. The function ψ is well defined by (6) provided, for any $\rho \in \mathbb{R}^n$, the function $g(\rho,\cdot) : \mathbb{R} \to [0,+\infty)$ defined by putting

$$g(\rho,c) = t\, f(A_1, \rho - (1-t)\, c\vec{n}) + (1-t)\, f(A_2, \rho + t\, c\vec{n})$$

is continuous, convex and bounded from below, so that there exists $c(\rho) \in \mathbb{R}$ satisfying

$$\psi(\rho) = g(\rho, c(\rho)).$$

In other words the minimization problem attains its minimum due to the convexity imposed on the integrand f. From expression (5), we conclude that the Γ-limit F is a functional defined in the intermediate space denoted by

$$\Psi(\Omega) = \left\{ u = v + w \;:\; v \in W^{1,p}(\Omega),\; w \in W^{1,q}(\Omega) \right\},$$

which is a Banach space with respect to the norm

$$\|u\|_{\Psi} = \inf\left\{ \|v\|_{W^{1,p}} + \|w\|_{W^{1,q}} \;:\; u = v + w,\; v \in W^{1,p}(\Omega),\; w \in W^{1,q}(\Omega) \right\}.$$

The condition $\Lambda_1 - \Lambda_2 \parallel \vec{n}$ in (5) plays an interesting role in the asymptotic behaviour. Indeed, for any $\rho \in \mathbb{R}^n$ parallel to the unit vector $\vec{n}$, it holds

$$\psi(\rho) \le t f(A_1, t\rho) + (1-t) f(A_2, 0) \le k(1 + |\rho|^p),$$

for some constant $k \in \mathbb{R}$. This is clearly seen in the following example.

Let us consider the density $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by

$$f(\lambda,\rho) = |\rho - g(\lambda)|^{\lambda}$$

for some continuous function $g : \mathbb{R} \to \mathbb{R}$, and $A_1 = 2$, $A_2 = 3$. Thus the associated limit energy density $\psi : \mathbb{R} \to \mathbb{R}$ is defined as the inf-convolution

$$\psi(\rho) = \inf_{a,b\in\mathbb{R}} \left\{ t|a - g(2)|^2 + (1-t)|b - g(3)|^3 \;:\; \rho = ta + (1-t)b \right\}.$$

If we take $g(2) = 2$ and $g(3) = 0$ then, after some calculus, the optimal pair $(a,b) \in \mathbb{R}\times\mathbb{R}$ of the previous minimization problem

$$\text{for } \rho \ge 2t \text{ is } \begin{cases} a = \frac{1}{t}\rho + \left(\frac{1-t}{3t^2}\right)\left((1-t) - \sqrt{(1-t)^2 + 6t\rho - 12t^2}\right) \\ b = -\frac{1}{3t}\left((1-t) - \sqrt{(1-t)^2 + 6t\rho - 12t^2}\right), \end{cases}$$

and

$$\text{for } \rho < 2t \text{ is } \begin{cases} a = \frac{1}{t}\rho - \frac{(1-t)}{3t^2}\left((1-t) - \sqrt{(1-t)^2 + 12t^2 - 6t\rho}\right) \\ b = \frac{1}{3t}\left((1-t) - \sqrt{(1-t)^2 + 12t^2 - 6t\rho}\right), \end{cases}$$

with $t \in (0,1)$. Therefore we conclude that for every $\rho \in \mathbb{R}$ it holds

$$\psi(\rho) = \begin{cases} \frac{(\rho-2t)^2}{t} - \frac{2(1-t)}{27t^3}\left((1-t)^2 + 6t\rho - 12t^2\right)^{3/2} \\ \qquad\qquad + \frac{2(1-t)^2}{27t^3}\left((1-t)^2 + 9t\rho - 18t^2\right) & \text{if } \rho \geq 2t \\ \frac{(\rho-2t)^2}{t} - \frac{2(1-t)}{27t^3}\left((1-t)^2 + 12t^2 - 6t\rho\right)^{3/2} \\ \qquad\qquad + \frac{2(1-t)^2}{27t^3}\left((1-t)^2 + 18t^2 - 9t\rho\right) & \text{if } \rho < 2t. \end{cases}$$

Notice that ψ has quadratic growth depending on the parameter $t \in (0,1)$ in the sense that

$$\lim_{|\rho|\to\infty} \frac{\psi(\rho)}{\rho^2} = \frac{1}{t}.$$

So we conclude that the sequence of energies

$$F_j(u) = \int_\Omega |u'(x) - g(p_j(x))|^{p_j(x)}\, dx,$$

with $p = 2$ and $q = 3$, Γ-converges to the quadratic functional F defined by

$$F(u) = \int_\Omega \psi(u'(x))\, dx.$$

More general, we may state the following corollary.

Corollary 2.2. *Let $h : \mathbb{R} \to (0, +\infty)$ and $g : \mathbb{R} \to \mathbb{R}^n$ be continuous functions such that $h(\lambda) \geq \gamma > 0$. Let $p_j : \Omega \to \{p, q\}$ be defined by*

$$p_j(x) = p\ \chi_{(0,t)}(jx \cdot \vec{n}) + q(1 - \chi_{(0,t)}(jx \cdot \vec{n})),$$

with $1 < p \leq q < \infty$, $t \in (0,1)$, and $\vec{n} \in \mathbb{R}^n$ such that $|\vec{n}| = 1$. Then the sequence of functionals

$$F_j(u) = \int_\Omega h(p_j(x))\ |\nabla u(x) - g(p_j(x))|^{p_j(x)}\, dx \tag{7}$$

is Γ-convergent (in the weak topology of $W^{1,p}(\Omega)$) to the functional F whose density $\psi : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi(\rho) = \min_{c\in\mathbb{R}} \left\{ t\, h(p)\, |\rho - (1-t)c\vec{n} - g(p)|^p + (1-t)\, h(q)\, |\rho + tc\vec{n} - g(q)|^q \right\}.$$

It is worthwhile to stress how the limit energy density is defined explicitly through a finite dimensional minimization problem. On the other hand, notice that functionals F_j in (7) are associated to equations of type

$$-\mathrm{div}\left[h(p_j(x))p_j(x)|\nabla u(x) - g(p_j(x))|^{p_j(x)-2}(\nabla u(x) - g(p_j(x)))\right] = 0 \ \text{ in } \Omega.$$

Therefore we may say that the sequence of solutions u_j of the previous family of $p_j(x)$-Laplaceans weak converges, as j tends to ∞, to the solution of the homogenized equation

$$-\mathrm{div}\,[\nabla\psi(\nabla u(x))] = 0 \ \text{ in } \Omega.$$

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is divided into two parts.

Step 1: We want to prove for any weak convergent sequence $\{u_j\}$ to u in $W^{1,p}(\Omega)$, it holds

$$\liminf_{j\to\infty} \int_\Omega f(p_j(x), \nabla u_j(x))\, dx \geq \int_\Omega \psi(\nabla u(x))\, dx. \tag{8}$$

Let $\{u_j\}$ be a weak convergent sequence to u in $W^{1,p}(\Omega)$, and $\nu = \{\nu_x\}_{x\in\Omega}$ be the joint Young measure associated with the sequence of pairs $\{(p_j, \nabla u_j)\}$ supported on $\mathbb{R}^m \times \mathbb{R}^n$. Thus we have

$$\liminf_{j\to\infty} \int_\Omega f(p_j(x), \nabla u_j(x))\, dx \geq \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^n} f(\lambda,\rho)\, d\nu_x(\lambda,\rho)\, dx.$$

From the slicing decomposition of the joint Young measure ν, it follows that, for a.e. $x \in \Omega$,

$$\begin{aligned}\nu_x(\lambda,\rho) &= \mu_{\lambda,x}(\rho)\otimes\sigma(\lambda)\\ &= t\,\mu_{A_1,x}(\rho)\otimes\delta_{A_1}(\lambda) + (1-t)\,\mu_{A_2,x}(\rho)\otimes\delta_{A_2}(\lambda)\end{aligned}$$

for some probability measures $\mu_{A_1,x}$ and $\mu_{A_2,x}$, provided the sequence $\{p_j\}$ generates the homogeneous Young measure $\sigma = t\,\delta_{A_1} + (1-t)\,\delta_{A_2}$. Then

$$\begin{aligned}&\int_\Omega \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(\lambda,\rho)\, d\mu_{\lambda,x}(\rho)\, d\sigma(\lambda)\, dx\\ =& \int_\Omega \left[t\int_{\mathbb{R}^n} f(A_1,\rho)\, d\mu_{A_1,x}(\rho) + (1-t)\int_{\mathbb{R}^n} f(A_2,\rho)\, d\mu_{A_2,x}(\rho)\right] dx\\ \geq& \int_\Omega \left[t\, f(A_1, \phi(x,A_1)) + (1-t) f(A_2,\phi(x,A_2))\right]\, dx,\end{aligned}$$

applying Jensen's inequality and considering the map $\phi : \Omega\times\mathbb{R}^m \to \mathbb{R}^n$ defined by putting

$$\phi(x,\lambda) \;=\; \int_{\mathbb{R}^n} \rho\; d\mu_{\lambda,x}(\rho).$$

Notice that the measure $\theta = \{\theta_x\}_{x\in\Omega}$ given by $\theta_x = t\,\mu_{A_1,x} + (1-t)\,\mu_{A_2,x}$ is the gradient Young measure associated with the sequence $\{\nabla u_j\}$, so that its weak limit ∇u may be represented as

$$\begin{aligned}\nabla u(x) &= t\int_{\mathbb{R}^n} \rho\, d\mu_{A_1,x}(\rho) + (1-t)\int_{\mathbb{R}^n} \rho\, d\mu_{A_2,x}(\rho)\\ &= t\,\phi(x,A_1) + (1-t)\,\phi(x,A_2) \quad \text{for a.e. } x\in\Omega.\end{aligned}$$

Moreover, for fixed $x \in \Omega$,

$$\begin{aligned}& t\, f(A_1,\phi(x,A_1)) + (1-t)\, f(A_2,\phi(x,A_2))\\ \geq& \min_{\Lambda_1,\Lambda_2\in\mathbb{R}^n} \left[t\, f(A_1,\Lambda_1) + (1-t)\, f(A_2,\Lambda_2)\right]\end{aligned}$$

whenever $\nabla u(x) = t\Lambda_1 + (1-t)\Lambda_2$ and $\Lambda_1 - \Lambda_2$ is parallel to $\vec{n}$. In this way we reach inequality (8).

Step 2: We will prove there exists a weak convergent sequence $\{u_j\}$ to u in $W^{1,p}(\Omega)$ such that

$$\limsup_{j\to\infty} \int_\Omega f(p_j(x), \nabla u_j(x))\, dx \le \int_\Omega \psi(\nabla u(x))\, dx.$$

For each $\rho \in \mathbb{R}^n$ there exists $c(\rho) \in \mathbb{R}$ such that

$$\psi(\rho) = tf(A_1, \rho - (1-t)c(\rho)\vec{n}) + (1-t)f(A_2, \rho + tc(\rho)\vec{n}).$$

So let $u \in W^{1,p}(\Omega)$ be such that

$$\begin{aligned}\int_\Omega \psi(\nabla u(x))\, dx = \int_\Omega &\Big[tf(A_1, \nabla u(x) - (1-t)c(\nabla u(x))\vec{n}) \\ &+(1-t)f(A_2, \nabla u(x) + tc(\nabla u(x))\vec{n})\Big]\, dx \;<\; \infty,\end{aligned}$$

where $c(\nabla u(\cdot))$ is the minimizer of the problem for $\psi(\nabla u(\cdot))$.

Consider the increasing sequence $\{\Omega_l\}_{l\in\mathbb{N}}$ of subsets of Ω defined by

$$\Omega_l = \{x \in \Omega \,:\, |\nabla u(x)| < l\}$$

such that $|\Omega \setminus \Omega_l|$ tends to 0 as l goes to ∞. Since u is a Lipschitz function in Ω_l, then there exists an extension $u_l : \mathbb{R}^n \to \mathbb{R}$, such that $u_l(x) = u(x)$ for any $x \in \Omega_l$, preserving the Lipschitz constant.

For a fixed $l \in \mathbb{N}$, let us take the Lipschitz function $v = u_l$ such that $\int_{\Omega_l} \psi(\nabla v(x))\, dx = \int_{\Omega_l} \psi(\nabla u(x))\, dx < \infty$. From [17, Lemma 7.9], there exists a set of points $\{x_k^{(j)}\} \subset \Omega_l \setminus N$, $|N| = 0$, and positive numbers $\{r_k^{(j)}\}$, $r_k^{(j)} < 1/j$, such that, for each $j \in \mathbb{N}$, the family of pairwise disjoint balls $\{x_k^{(j)} + r_k^{(j)}B\}_k$ (B is the unit ball in $\mathbb{R}^n$) satisfies

$$\Omega_l = \bigcup_k \left(x_k^{(j)} + r_k^{(j)}B\right) \bigcup N,$$

and

$$\begin{aligned}\int_{\Omega_l} \psi(\nabla u(x))\, dx &= \int_{\Omega_l} \Big[tf(A_1, \nabla u(x) - (1-t)c(\nabla u(x))\vec{n}) \\ &\qquad +(1-t)f(A_2, \nabla u(x) + tc(\nabla u(x))\vec{n})\Big]\, dx \\ &= \lim_{j\to\infty} \sum_k \Big[tf(A_1, \nabla u(x_k^{(j)}) - (1-t)c(\nabla u(x_k^{(j)}))\vec{n}) \\ &\qquad +(1-t)f(A_2, \nabla u(x_k^{(j)}) + tc(\nabla u(x_k^{(j)}))\vec{n})\Big]\, |x_k^{(j)} + r_k^{(j)}B|.\end{aligned}$$

For a.e. $x_k^{(j)} \in \Omega_l$, let us define the sequence of functions $w_i^{(j,k)} : B \to \mathbb{R}$ by putting

$$w_i^{(j,k)}(y) = \left(\nabla u(x_k^{(j)}) + tc(\nabla u(x_k^{(j)}))\vec{n}\right) \cdot y - \frac{c(\nabla u(x_k^{(j)}))}{i} \int_0^{iy\cdot\vec{n}} \chi_{(0,t)}(s)\, ds,$$

with

$$\nabla w_i^{(j,k)}(y) = \left(\nabla u(x_k^{(j)}) - (1-t)c(\nabla u(x_k^{(j)}))\vec{n}\right)\chi_{(0,t)}(iy\cdot\vec{n}) + \left(\nabla u(x_k^{(j)}) + tc(\nabla u(x_k^{(j)}))\vec{n}\right)(1-\chi_{(0,t)}(iy\cdot\vec{n})),$$

such that $\{w_i^{(j,k)}\}$ converges weakly* to $\nabla u(x_k^{(j)})y$ in $W^{1,\infty}(B)$. Then, the sequence of pairs $\{(p_i(x_k^{(j)} + r_k^{(j)}\cdot), \nabla w_i^{(j,k)})\}$ generates the homogeneous Young measure $\nu_{x_k^{(j)}}$, with compact support in $\mathbb{R}^m \times \mathbb{R}^n$, given by

$$\nu_{x_k^{(j)}} = t\,\delta_{(A_1,\nabla u(x_k^{(j)})-(1-t)c(\nabla u(x_k^{(j)}))\vec{n})} + (1-t)\,\delta_{(A_2,\nabla u(x_k^{(j)})+tc(\nabla u(x_k^{(j)}))\vec{n})}.$$

In particular, we have

$$\lim_{i\to\infty}\frac{1}{|B|}\int_B f(p_i(x_k^{(j)} + r_k^{(j)}y), \nabla w_i^{(j,k)}(y))\,dy$$
$$= \left[tf(A_1, \nabla u(x_k^{(j)}) - (1-t)c(\nabla u(x_k^{(j)}))\vec{n}) + (1-t)f(A_2, \nabla u(x_k^{(j)}) + tc(\nabla u(x_k^{(j)}))\vec{n})\right].$$

Now, for each $i \in \mathbb{N}$, consider smooth cut-off functions $g_i : B \to [0,1]$ for which

$$g_i(y) = 1 \text{ if } y \in D_i = \left\{\operatorname{dist}(y,\partial B) \geq \frac{2}{i^2}\right\},$$
$$g_i(y) = 0 \text{ if } y \in \left\{\operatorname{dist}(y,\partial B) \leq \frac{1}{i^2}\right\},$$

so that we may define the sequence of functions $u_i^{(j)} : \Omega_l \to \mathbb{R}$ by putting

$$u_i^{(j)}(x) = \left(r_k^{(j)} w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) + u(x_k^{(j)})\right) g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) + u(x)\left(1 - g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right)$$

if $x \in x_k^{(j)} + r_k^{(j)}B$. Thus we have

$$\begin{aligned}
\nabla u_i^{(j)}(x) &= \nabla w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) + \nabla u(x)\left(1 - g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right)\\
&\quad + \frac{1}{r_k^{(j)}}\left(r_k^{(j)} w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) + u(x_k^{(j)}) - u(x)\right)\nabla g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\\
&= \nabla w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) + \nabla u(x)\left(1 - g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right)\\
&\quad - \left(\frac{u(x) - u(x_k^{(j)})}{r_k^{(j)}} - \nabla u(x_k^{(j)})\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right)\nabla g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\\
&\quad - \left(\nabla u(x_k^{(j)})\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) - w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right)\nabla g_i\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)
\end{aligned}$$

if $x \in x_k^{(j)} + r_k^{(j)}B$, where $\left\{\nabla u(x_k^{(j)})\left(\frac{\cdot - x_k^{(j)}}{r_k^{(j)}}\right) - w_i^{(j,k)}\left(\frac{\cdot - x_k^{(j)}}{r_k^{(j)}}\right)\right\}$ converges strongly, as $i \to \infty$, to 0 in $L^\infty(x_k^{(j)} + r_k^{(j)}B)$, and

$$\begin{aligned}
&\lim_{j\to\infty} \sup_k \int_{x_k^{(j)}+\varepsilon_k^{(j)}B} \left| \frac{u(x) - u(x_k^{(j)})}{r_k^{(j)}} - \nabla u(x_k^{(j)}) \left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right) \right|^q dx \\
&= \lim_{j\to\infty} \sup_k (r_k^{(j)})^n \int_B \left| \frac{u(x_k^{(j)} + r_k^{(j)}y) - u(x_k^{(j)})}{r_k^{(j)}} - \nabla u(x_k^{(j)})y \right|^q dy \\
&\leq \lim_{j\to\infty} \sup_k (r_k^{(j)})^n \int_B \left| \nabla u(x_k^{(j)} + r_k^{(j)}y) - \nabla u(x_k^{(j)}) \right|^q dy \;=\; 0,
\end{aligned}$$

for any $q \geq p$. Therefore

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{i\to\infty} \int_{\Omega_l} f(p_i(x), \nabla u_i^{(j)}(x))\, dx \\
&= \lim_{j\to\infty}\lim_{i\to\infty} \sum_k \left[\int_{x_k^{(j)}+r_k^{(j)}D_i} f\left(p_i(x), \nabla w_i^{(j,k)}\left(\frac{x - x_k^{(j)}}{r_k^{(j)}}\right)\right) dx \right. \\
&\qquad \left. + \int_{x_k^{(j)}+r_k^{(j)}B\setminus D_i} f(p_i(x), \nabla u(x))\, dx \right] \\
&= \lim_{j\to\infty}\lim_{i\to\infty} \sum_k (r_k^{(j)})^n \int_{D_i} f(p_i(x_k^{(j)} + r_k^{(j)}y), \nabla w_i^{(j,k)}(y))\, dy \\
&= \lim_{j\to\infty} \sum_k (r_k^{(j)})^n |B| \Big[t f(p, \nabla u(x_k^{(j)}) - (1-t)c(\nabla u(x_k^{(j)}))\vec{n}) \\
&\qquad + (1-t) f(q, \nabla u(x_k^{(j)}) + tc(\nabla u(x_k^{(j)}))\vec{n}) \Big] \\
&= \int_{\Omega_l} \Big[t f(A_1, \nabla u(x) - (1-t)c(\nabla u(x))\vec{n}) \\
&\qquad + (1-t) f(A_2, \nabla u(x) + tc(\nabla u(x))\vec{n}) \Big]\, dx \\
&= \int_{\Omega_l} \psi(\nabla u(x))\, dx.
\end{aligned}$$

In this way, for each $l \in \mathbb{N}$, there exists a subsequence $j(i) \to \infty$, as $i \to \infty$, such that $\{u_i^{(j(i))}\} \subset W^{1,\infty}(\Omega_l)$ satisfies

$$\begin{aligned}
&\limsup_{i\to\infty} \int_{\Omega_l} f(p_i(x), \nabla u_i^{(j(i))}(x))\, dx \\
&\leq \limsup_{i\to\infty} \limsup_{j\to\infty} \int_{\Omega_l} f(p_i(x), \nabla u_i^{(j)}(x))\, dx = \int_{\Omega_l} \psi(\nabla u(x))\, dx.
\end{aligned}$$

By dominated convergence it follows that

$$\lim_{l\to\infty} \int_{\Omega_l} \psi(\nabla u(x))\, dx = \lim_{l\to\infty} \int_{\Omega} \chi_{\Omega_l}(x)\psi(\nabla u(x))\, dx = \int_{\Omega} \psi(\nabla u(x))\, dx < \infty.$$

Thus

$$\limsup_{l\to\infty}\limsup_{i\to\infty}\int_{\Omega_l} f(p_i(x), \nabla u_i^{(j(i))}(x))\ dx \le \int_\Omega \psi(\nabla u(x))\ dx.$$

$\square$

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