Journal of Convex Analysis
Volume 17 (2010), No. 2, 441–449

On Moduli of Smoothness and Squareness

A. J. Guirao*
Departamento de Matemática Aplicada, Escuela Técnica Superior de Arquitectura, Universidad Politécnica de Valencia, Camino de Vera s/n, 46020 Valencia, Spain
anguisa2@mat.upv.es

M. Ivanov†
Faculty of Mathematics and Informatics, Sofia University, 5, J. Bourchier blvd., 1164 Sofia, Bulgaria
milen@fmi.uni-sofia.bg

S. Lajara‡
Departamento de Matemáticas, Escuela de Ingenieros Industriales, Universidad de Castilla-La Mancha, Campus Universitario, 02071 Albacete, Spain
Sebastian.Lajara@uclm.es

Received: December 19, 2008

We study the possible equivalence classes of the moduli of smoothness of finite-dimensional Banach spaces. We show that these can be arbitrary subject to Figiel condition and, moreover, they can be realised via Orlicz spaces. Some dimension dependencies are also studied. As an application we answer in negative an open question concerning the modulus of squareness.

Keywords: Modulus of smoothness, modulus of squareness

1991 Mathematics Subject Classification: 46B03, 46B20

1. Introduction

Recall one of the standard notions of equivalence of two functions $f, g : \mathbb{R} \to \mathbb{R}$ at zero:

$$f \asymp g \iff cf(t) \leq g(t) \leq Cf(t)$$

for some constants $c, C > 0$ and all t in a neighbourhood of zero. A slightly weaker relation

$$f \approx g \iff cf(at) \leq g(t) \leq Cf(bt)$$

for some strictly positive constants and t in a neighbourhood of zero, appears in many contexts, notable for us being [3, 6, Ch. 4]. The value of this notion is partially in the fact that if $f \approx g$ then $f^* \approx g^*$, where f^* is the *Fenchel conjugate* of f.

Therefore, thanks to Lindenstrauss duality formula [7, p. 61] (which essentially says the modulus of smoothness ρ_{X^*} of the dual X^* is Fenchel conjugate to the modulus

*Partially supported by MICINN MTM 2008-05396.
†Partially supported by MM-1401; and by Grant 105/2008 NIS, Sofia University; and by Bulgarian National Fund for Scientific Research, contract DO 02-360/2008.
‡Partially supported by MICINN MTM 2008-05396; and by JCCM PCI-08-0023.

ISSN 0944-6532 / $ 2.50

of convexity δ_X of the Banach space X) $\approx$ equivalence proved useful in studying the geometry of Banach spaces.

Of course, for functions like t^p both notions coincide. More generally, we call the function f *pretty* if

$$f(t) \asymp f(kt), \quad \forall k > 0. \tag{1}$$

It is immediately seen that if f is pretty then $f \asymp g$ if and only if $f \approx g$ in which case g is also pretty. Note that an Orlicz function f is pretty if and only if it satisfies Δ_2 condition at zero (that is, $f(2t) \leq cf(t)$ for all small enough $t > 0$), cf. [6, p. 138].

In [5] the functions which can be modulus of convexity of some two-dimensional normed space are characterised up to $\approx$ equivalence. The method used is direct geometrical construction of the unit sphere. By duality, the main result of [5] implies characterisation of the functions which can be modulus of smoothness.

In the present work we elaborate on some analytical techniques from [8] in order to obtain the characterisation of modulus of smoothness functions. We show that any $\asymp$ equivalence class can be realised in arbitrary finite dimension by Orlicz space. The presented analytical method provides more explicit (if compared to [5]) formula of the norm for each modulus of smoothness function and each dimension.

As an application we show that the modulus of squareness, e.g. [1], of a uniformly convex space needs not be integrable function. In this way we answer negatively a question posed in [1]. It should be mentioned that we are not able to obtain this counterexample directly from [5] because the known correspondence between the modulus of convexity and the modulus of squareness is significantly less tight than that between the modulus of smoothness and the modulus of squareness [1, Theorem 2.4.i]. It remains open if the former can be tightened accordingly.

The paper is organised as follows. In Section 2 we recall the definitions and previous results we use, and state our main results. Section 3 is devoted to the details of the characterisation of the modulus of smoothness functions. In the final Section 4 we present our counterexample concerning the modulus of squareness.

2. Preliminaries and main results

Recall that the modulus of convexity, resp. smoothness, of a Banach space X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\}, \quad 0 \leq \varepsilon \leq 2,$$

resp.

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\| - 2}{2} : \|x\| = \|y\| = 1\right\}, \quad \tau \geq 0.$$

Lindenstrausss duality formula, e.g. [7, p. 261], reads

$$\rho_{X^*}(\tau) = \sup\left\{\tau\varepsilon/2 - \delta_X(\varepsilon) : 0 \leq \varepsilon \leq 2\right\}.$$

As Fenchel conjugate ρ_{X^*} is convex function. This is true for general ρ_X, see [3]. It is easy to see that ρ_X is Lipschitz, increasing and $\rho_X(0) = 0$ (indeed, $\rho_X(\tau) \leq \tau$ by triangle inequality). That is to say, ρ_X is an *Orlicz function.* A much deeper property of ρ_X is revealed by Nordlander Theorem [9]: $\rho_X(\tau) \geq \rho_H(\tau) \asymp \tau^2$, where H stands for Hilbert space. It was shown by Figiel [3] that any modulus of smoothness satisfies (2).

We will show that – up to $\asymp$ equivalence – the latter, to which we refer as *Figiel condition*, is also sufficient.

Definition 2.1. An Orlicz function N will be called a *modulus of smoothness function* provided it satisfies the inequality

$$\frac{N(s)}{s^2} \leq C\frac{N(t)}{t^2}, \quad \forall\, 0 < t \leq s, \tag{2}$$

for some $C \geq 1$. We denote the smallest such constant by C_N; and the set of all modulus of smoothness functions by $\mathcal{S}$.

It is immediate that all functions in $\mathcal{S}$ are pretty. Also, if $N \in \mathcal{S}$ and $N_1(t) = t^2 \sup_{s\geq t} N(s)/s^2$ then $N \asymp N_1$ and $C_{N_1} = 1$.

Given an Orlicz function N, the n-dimensional Orlicz space $l_N^{(n)}$ is $\mathbb{R}^n$ with unit sphere $S_{l_N^{(n)}} = \{x :\ \varphi(x) = 1\}$, where

$$\varphi(x) = \sum_{i=1}^{n} N\left(|x_i|\right), \quad x = (x_i)_{i=1}^n,\, x_i \in \mathbb{R}. \tag{3}$$

We refer e.g. to [6, p. 137] for a definition of Orlicz sequence space l_N.

We introduce an analogue of Maleev-Troyanski function G_N [8, p. 133].

Definition 2.2. Let $N \in \mathcal{S}$ and $n \in \mathbb{N}$. The function $G_{N,n} : [0,1] \to \mathbb{R}$ is defined by

$$G_{N,n}(\tau) = \tau^2 \sup\left\{\sum_{i=1}^{n} \frac{N(u_i v_i)}{u_i^2} :\ (v_i)_{i=1}^n \in S_{l_N^{(n)}}, v_i \geq 0,\, u_i \in [\tau, 1]\right\}. \tag{4}$$

Lemma 3.2 shows in particular that $N \asymp G_{N,n}$.

The following regularisation of an Orlicz function N is frequently used, e.g. [7, Ch. 4]:

$$M(t) = \int_0^t \frac{N(s)}{s}\, ds. \tag{5}$$

For the properties of this regularisation, see Lemma 3.1. For now it is important that $M \asymp N$ and $C_M \leq C_N$.

We are ready to present the characterisation of modulus of smoothness.

Theorem 2.3. *Let $N \in \mathcal{S}$ and M be given by* (5)*. There exist constants $k > 0$ and $K > 0$ such that for all $n \in \mathbb{N}$ and all $\tau \in [0,1]$*

$$\frac{k}{nM^{-1}(1/n)} G_{M,n}(\tau) \leq \rho_{l_M^{(n)}}(\tau) \leq K G_{M,n}(\tau).$$

In particular $\mathbb{R}^n$ can be renormed in such a way that its modulus of smoothness is $\asymp$ equivalent to N.

Using that $\delta_{X^*} \approx \rho^*_X$, see [3], the above characterisation can be dualised to modulus of convexity.

In Example 3.3 we show that there exists a function $N \in \mathcal{S}$ such that $\frac{1}{nM^{-1}(1/n)} \to 0$ as $n \to \infty$, and $G_M \not\asymp N$. Therefore, our result is not implied by the infinite-dimensional results found in [8, 3].

As an application of the above Theorem, we get the following result which, roughly speaking, says that the modulus of smoothness can be arbitrarily worsened through renorming.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a Banach space of dimension at least 2. Let $N \in \mathcal{S}$ be arbitrary. There exists equivalent norm $|\cdot|$ such that*

$$cN(\tau) \leq \rho_{(X,|\cdot|)}(\tau) \leq C\max\{N(\tau), \rho_{(X,\|\cdot\|)}(\tau)\}$$

for some $c, C > 0$.

Recall the definition of modulus of squareness $\xi_X : [0,1) \to [1,\infty)$ from [10], see also [1], of a normed space X:

$$\xi_X(\beta) = \sup\{\omega(x,y) : \|y\| \leq \beta < 1 < \|x\|\},$$

where

$$\omega(x,y) = \frac{\|x - z(x,y)\|}{\|x\| - 1},$$

and $z(x,y)$ is the intersection of the segment $[x,y]$ and the unit sphere S_X.

It is remarkable that the modulus of squareness characterises *both* uniform convexity and uniform smoothness. Recently a localisation of this modulus was defined in [4]. (Localisation in this context means a notion which characterises Fréchet (resp. Gâteaux) smoothness and local uniform (resp. strict) convexity and which, when uniformity is imposed, gives rise to modulus of squareness.) This notion answers [1, Problem 4.8].

We summarise in Proposition 4.1 the known properties of the modulus of squareness that we use. For some other important properties of this modulus we refer to Theorem O from [1].

Problem 4.5 in [1] asks if ξ_X is integrable for any uniformly convex X. The following Theorem answers this question.

Theorem 2.5. *Let*

$$\mu(t) = \begin{cases} \frac{t}{1-\log t}, & t \in [0,1], \\ t^2, & t \geq 1, \end{cases} \tag{6}$$

and $M(t) = \int_0^t \mu(s)/s\, ds$. Then for $X = l_M^{(n)}$

$$\xi_{X^*}(\beta) \asymp \frac{1}{(\beta - 1)\log(1-\beta)}$$

as $\beta \to 1$. In particular, X^* is uniformly convex but ξ_{X^*} is not integrable.

Note that l_M^* could not serve as counterexample, because by Theorem 4.a.9 from [6, p. 143] $l_1 \subset l_M$ and thus the latter is not reflexive.

3. Modulus of smoothness functions

We start with the following well-known

Lemma 3.1. *Let $N \in \mathcal{S}$ and M be given by (5). Then $M \asymp N$, $C_M \le C_N$ and for all $u \in (0,1]$ and v such that $|v| < u$*

$$M(|u+v|) + M(|u-v|) - 2M(|u|) \le B\frac{M(|u|)}{u^2}v^2, \tag{7}$$

where B is some positive constant.

Proof. Lemma 20 from [3] contains all but the estimate $C_M \le C_N$. For the latter take $0<t<s$ and compute $M(t) = \int_0^t N(u)/u\,du = \int_0^1 N(tu)/u\,du \le C_N\,(t^2/s^2)\int_0^1 N(su)/u\,du$ $= C_N t^2 M(s)/s^2$, that is, $C_M \le C_N$. □

We now turn to the equivalence between $M \in S$ and $G_{M,n}$, see Definition 2.2.

Lemma 3.2. *Let $N \in \mathcal{S}$ and $N(1) = 1$. Then for all $n \in \mathbb{N}$ and $\tau \in [0,1]$*

$$N(\tau) \le G_{N,n}(\tau) \le C_N n N^{-1}(1/n) N(\tau). \tag{8}$$

Proof. Left hand side inequality is trivial: take $v = (1,0,\ldots)$ and $u = (\tau,\tau,\ldots)$ in (4).

Now, we proceed with the proof of the other inequality. It will be useful to break $G_{N,n}$ into more simple terms. Let

$$h_c(t) = \max_{s\in[t,1]} \frac{N(cs)}{s^2}; \qquad h(t) = h_1(t). \tag{9}$$

In other words, $h(t)$ is a decreasing function to which $N(t)/t^2$ is equivalent on $(0,1]$. Indeed $N(t) \le t^2 h(t) \le C_N N(t)$. It is also obvious that

$$h_c(t) \le c^2 h(ct).$$

Since the variables separate,

$$G_{N,n}(\tau) = \tau^2 \sup\left\{\sum_{i=1}^n h_{v_i}(\tau) :\ (v_i)_1^n \in S_{l_N^{(n)}},\ v_i \ge 0\right\}. \tag{10}$$

But $h_{v_i}(\tau) \le v_i^2 h(\tau v_i) \le C_M N(\tau v_i)/\tau^2$. Since $v_i \in [0,1]$ (recall that $N(1) = 1$) and N is convex, $N(0) = 0$, we have $N(\tau v_i) \le v_i N(\tau)$. Therefore, $h_{v_i}(\tau) \le C_N v_i N(\tau)/\tau^2$. Substituting in (10) we get

$$G_{N,n}(\tau) \le C_N N(\tau) \sup\left\{\sum_{i=1}^n v_i :\ (v_i)_1^n \in S_{l_N^{(n)}},\ v_i \ge 0\right\}.$$

Since the inverse function N^{-1} is concave, $N^{-1}(1/n) = N^{-1}\left(n^{-1}\sum_1^n N(v_i)\right) \geq n^{-1}\sum_1^n N^{-1}\left(N(v_i)\right) = n^{-1}\sum_1^n v_i$. That is, the above supremum is less than $nN^{-1}(1/n)$. $\square$

Note that the value $C_N nN^{-1}(1/n)$ is optimal, as explained in Example 3.3.

Proof of Theorem 2.3. Multiplying N by a constant changes nothing, so we assume that $M(1) = 1$.

We show first that for fixed u, v, such that $|u| \leq 1$ and $|v| \leq \tau$,

$$M(|u+v|) + M(|u-v|) - 2M(|u|) \leq k\tau^2 h_{\frac{|v|}{\tau}}(\tau), \tag{11}$$

see (9). Indeed, we may assume that $u, v \geq 0$. There are two possibilities.

If $u \leq v$ then $M(v \pm u) \leq M(2v) \leq k_1 M(v) \leq k_1 \tau^2 h_{\frac{v}{\tau}}(\tau)$.

If $v < u$ then by (7) the left hand side of (11) is estimated by $BM(u)v^2/u^2 \leq BC_M M(v) \leq BC_M \tau^2 h_{\frac{v}{\tau}}(\tau)$.

Take now $x, y \in l_M^{(n)}$ such that $\|x\| = 1$ and $\|y\| = \tau$. Then $|x_i| \leq 1$ and $|y_i| \leq \tau$ and we may sum (11) coordinate-wise to obtain

$$\varphi(x+y) + \varphi(x-y) - 2 \leq k\tau^2 \sum_{i=1}^n h_{\frac{|v_i|}{\tau}}(\tau) \leq kG_{M,n}(\tau)$$

by (10). (φ was defined by (3))

Suppose that $\|x+y\| \geq 1$ and $\|x-y\| \geq 1$. By convexity we have $\varphi(x \pm y) \geq \|x \pm y\|$ and the above estimate implies $\|x+y\| + \|x-y\| - 2 \leq kG_{M,n}(\tau)$. Thus, using [3, Lemma 12], we get $\rho_X(\tau) \leq 16kG_{M,n}(\tau)$.

Since $l_M^{(2)}$ embeds isometrically in $l_M^{(n)}$, we have that $\rho_{l_M^{(n)}} \geq \rho_{l_M^{(2)}}$ and it is enough to prove the left hand side inequality for the latter. Let $x = (1,0), y = (0,1) \in S_{l_M^{(2)}}$ and define $n(t) = \|x + ty\|$. It is obvious that $\rho_{l_M^{(2)}}(\tau) \geq n(\tau) - 1$. It is intuitively clear that $n(t) - 1$ is alike $M(t)$ as $t \to 0$. In order to prove this, we need to compute the derivative of $n(t)$. Since the latter satisfies (by the definition of Orlicz norm)

$$M(1/n(t)) + M(t/n(t)) = 1,$$

after differentiation and rearrangement we get

$$\frac{n'(t)}{n(t)} = m(t) := \frac{M'(t/n(t))}{M'(1/n(t)) + tM'(1/n(t))}.$$

Therefore,

$$\liminf_{t\to 0} \frac{n(t)-1}{M(t)} = \liminf_{t\to 0} \frac{n'(t)}{M'(t)} = \liminf_{t\to 0} \frac{n(t)m(t)}{M'(t)}.$$

Replacing the multipliers which tend to non-zero limit by the latter we get

$$\liminf_{t\to 0} \frac{n(t)-1}{M(t)} = \frac{1}{M'(1)} \liminf_{t\to 0} \frac{M'(t/n(t))}{M'(t)}.$$

By definition $M'(t) = N(t)/t$ and so

$$\liminf_{t\to 0} \frac{n(t)-1}{M(t)} = \frac{1}{N(1)} \liminf_{t\to 0} \frac{N(t/n(t))}{N(t)} \geq \frac{1}{N(1)} . \frac{1}{C_N},$$

because $N(t) \leq C_N n^2(t) N(t/n(t))$, as $n(t) \geq 1$; and $n(t) \to 1$. We see that there is a constant $K' > 0$ such that $n(t) - 1 \geq K' M(t)$ for $t \in (0,1]$. We apply (8) and complete the proof. □

Proof of Corollary 2.4. Let Y be a two-dimensional subspace of X and let P be a bounded projection of X onto Y. By Theorem 2.3 there is equivalent norm $\|\cdot\|_1$ on Y such that $\rho_{(Y,\|\cdot\|_1)} \asymp N$. Then by [3, Proposition 19] the equivalent norm $|\cdot|$, defined by the formula

$$|x|^2 = \|Px\|_1^2 + \|x - Px\|^2$$

satisfies the claim. □

Example 3.3. For the function μ defined by (6) we have $C_\mu = 1$ and

$$\lim_{\tau\to 0} \frac{G_{\mu,n}(\tau)}{\mu(\tau)} = n\mu^{-1}(1/n), \tag{12}$$

meaning that the estimate (8) is optimal.

As $\mu'(t) = \frac{1}{(1-\log t)^2} + \frac{1}{1-\log t}$ on $[0,1]$ and the latter is increasing function, μ is Orlicz. It is obvious that $\mu(t)/t^2$ is decreasing. So, $\mu \in \mathcal{S}$ and $C_\mu = 1$.

Note first of all that the inverse of μ can be expressed through Lambert function W, that is the inverse of $t \to te^t$ on $[0,\infty)$, e.g. [2, p. 27]:

$$\mu^{-1}(t) = tW(e^t/t).$$

By differentiating $W(t)e^{W(t)} = t$, we get $tW'(t) = W(t)/(1+W(t))$. Since $W(t)/(1+W(t)) \to 1$ as $t \to \infty$, $W'(t) \asymp 1/t$ and thus $W(t) \asymp \log t$ as $t \to \infty$. This implies

$$\mu^{-1}(t) \asymp -t\log t, \quad t \to 0, \tag{13}$$

so (12) goes to infinity like $\log n$.

Next, we calculate explicitly $G_{\mu,n}$. Since $t \to \mu(tv_i)/t^2$ is decreasing, (4) reduces to

$$G_{\mu,n}(\tau) = \max\left\{\sum_{i=1}^n \mu(\tau v_i) : (v_i) \in S_{l_\mu^{(n)}}, v_i \geq 0\right\}.$$

Considering Lagrange function $L(v,\lambda) = \sum_1^n \mu(\tau v_i) - \lambda\left(\sum_1^n \mu(v_i) - 1\right)$ we would find that the extremal points are either where some of the v_i's is zero (but that case reduces to lower dimension), or at the unit vector collinear to $(1,1,\dots,1)$, because the function $f(x) = \mu(\tau x)/\mu(x)$ is strictly decreasing on $[0,1]$.

Of course, the latter needs some explanation. It is enough to show that $f'(x) < 0$ for $x \in (0,1)$. That is, $\tau\mu'(\tau x)\mu(x) - \mu(\tau x)\mu'(x) < 0$, or

$$\tau\frac{\mu'(\tau x)}{\mu(\tau x)} < \frac{\mu'(x)}{\mu(x)} \iff \tau g'(\tau x) < g'(x),$$

where $g(x) = \log\mu(x) = \log x - \log(1 - \log x)$ for $x \in (0,1)$ by (6). So, $g'(x) = \frac{1}{x}\left(1 + \frac{1}{1-\log x}\right)$ and the above reduces to $\log\tau x < \log x$.

If then the maximum is attained at vector of equal co-ordinates, the former will be equal to $\mu^{-1}(1/n)$. Therefore,

$$G_{\mu,n}(\tau) = n\mu\left(\tau\mu^{-1}(1/n)\right). \tag{14}$$

Since $\lim_{\tau\to 0}\mu(c\tau)/\mu(\tau) = c$, we have (12).

From (14) and (13) we can deduce also that for fixed $\tau \in (0,1)$ the value of $G_{\mu,n}(\tau)$ goes to infinity. Therefore, the equivalence in Theorem 2.3 can not be uniform on $n \in \mathbb{N}$ if N is equal μ. Also, in this case Maleev-Troyanski function $G_M(\tau)$ (see [8]) is equal τ and thus not equivalent to $M \asymp \mu \asymp G_{M,n}$.

4. A counterexample concerning modulus of squareness

First, we recall the known properties of the modulus of squareness which we will use.

Proposition 4.1 ([1]). *Let X be a normed space. Then X is uniformly convex if and only if* $\lim_{\beta\to 1}(1-\beta)\xi_X(\beta) = 0$. *Also,*

$$\xi_{X^*}(\beta) = \frac{1}{\xi^{-1}(1/\beta)}; \tag{15}$$

and

$$\rho_X(\beta) \le \xi_X(\beta) - 1 \le \frac{2\rho_X(\beta)}{1-\beta}, \quad \beta \in (0,1/2), \tag{16}$$

that is, $\rho_X(\beta) \asymp \xi(\beta) - 1$.

Proof. These are the statements of Theorem O(f,h) and Theorem 2.4(i) from [1]. □

Proof of Theorem 2.5. As $\mu \in \mathcal{S}$, see Example 3.3, Theorem 2.3 applies with $N = \mu$ and M. Also, $M \asymp \mu$ by Lemma 3.1.

Since $\rho_X \asymp \mu$ by Theorem 2.3, and $\rho_X \asymp \xi_X - 1$ by (16), we have $\mu \asymp \xi_X - 1$. That is,

$$1 + c\mu(t) \le \xi_X(t) \le 1 + C\mu(t)$$

for some $c, C > 0$ and t in a neighbourhood of zero. Let

$$\xi_X(t) = \frac{1}{\beta}. \tag{17}$$

We have that $\beta = \beta(t) \to 1$ as $t \to 0$ and, of course, $\beta < 1$. Obviously, $1 - c_1\mu(t) \leq \beta(t) \leq 1 - c_2\mu(t)$ for some strictly positive c_i and t small enough. Obviously, for increasing f and g

$$f \geq g \iff f^{-1} \leq g^{-1}.$$

Therefore,

$$\mu^{-1}(c_1\mu(t)) \leq \mu^{-1}(1 - \beta(t)) \leq \mu^{-1}(c_2\mu(t)).$$

It is easily checked that the function $t \to -t\log t$ is pretty. Thus (13) implies that μ^{-1} is also pretty. So, the above inequalities reduce to

$$\mu^{-1}(1 - \beta(t)) \asymp t.$$

From (17) it follows that

$$\mu^{-1}(1-\beta) \asymp \xi_X^{-1}\left(\frac{1}{\beta}\right), \quad \beta \to 1.$$

From (13) we know that $\mu^{-1}(1-\beta) \asymp (\beta - 1)\log(1-\beta)$ as $\beta \to 1$, so (15) implies

$$\xi_{X^*}(\beta) = 1/\xi_X^{-1}\left(\frac{1}{\beta}\right) \asymp \frac{1}{(\beta-1)\log(1-\beta)}.$$

$\square$

Acknowledgements. We would like to thank Prof. S. Troyanski from Murcia University and Prof. R. Deville from University Bordeaux I for helpful and stimulating discussions.

References

[1] C. Benítez, K. Przesławski, D. Yost: A universal modulus for normed spaces, Stud. Math. 127 (1998) 21–46.

[2] N. G. de Bruijn: Asymptotic Methods in Analysis, North-Holland, Amsterdam (1961).

[3] T. Figiel: On the moduli of convexity and smoothness, Stud. Math. 56 (1976) 121–155.

[4] A. J. Guirao: On the local moduli of squareness, Studia Math. 184 (2008) 175–189.

[5] A. J. Guirao, P. Hájek: On the moduli of convexity, Proc. Amer. Math. Soc. 135 (2007) 3233–3240.

[6] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces I. Sequence Spaces, Springer, Berlin (1977).

[7] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces II. Function Spaces, Springer, Berlin (1979).

[8] R. Maleev, S. Troyanski: On the moduli of convexity and smoothness in Orlicz spaces, Studia Math. 54 (1975) 131–141.

[9] G. Nordlander: The modulus of convexity in normed linear spaces, Ark. Mat. 4 (1960) 15–17.

[10] K. Przesławski, D. Yost: Lipschitz retracts, selectors, and extensions, Michigan Math. J. 42 (1995) 555–571.