Journal of Convex Analysis
Volume 17 (2010), No. 3&4, 945–959

Hidden Convexity in some Nonlinear PDEs from Geomety and Physics

Yann Brenier
CNRS, Université de Nice, Dép. de Mathématiques,
Parc Valrose, 06108 Nice, France
brenier@math.unice.fr

Dedicated to Hedy Attouch on the occasion of his 60th birthday.

Received: March 9, 2009
Revised manuscript received: March 16, 2010

1. Introduction

There is a prejudice among some specialists of non linear partial differential equations and differential geometry: convex analysis is an elegant theory but too rigid to address some of the most interesting and challenging problems in their field. Convex analysis is mostly attached to elliptic and parabolic equations of variational origin, for which a suitable convex potential can be exhibited and shown to be minimized (either statically or dynamically). The Dirichlet principle for linear elliptic equation is archetypal.
Hyperbolic PDEs, for example, seem to be inaccessible to convex analysis, since they are usually derived from variational principles that are definitely not convex. (However, convexity plays an important role in the so-called entropy conditions.) Also, elliptic systems with variational formulations (such as in elasticity theory) often involve structural conditions quite far from convexity (such as Hadamard's "rank one" conditions). (However, convexity can be often restored, for example through the concept of polyconvexity [8], or by various kinds of "relaxation" methods [55, 6].) The purpose of the present paper is to show few examples of nonlinear PDEs (mostly with strong geometric features) for which there is a hidden convex structure. This is not only a matter of curiosity. Once the convex structure is unrevealed, robust existence and uniqueness results can be unexpectedly obtained for very general data. Of course, as usual, regularity issues are left over as a hard post-process, but, at least, existence and uniqueness results are obtained in a large framework. The paper will address:

1. THE MONGE-AMPERE EQUATION (solving the Minkowski problem and strongly related to the so-called optimal transport theory since the 1990's)
2. THE EULER EQUATION (describing the motion of inviscid and incompressible fluids, interpreted by Arnold as geodesic curves on infinite dimensional groups of volume preserving diffeomorphisms)
3. MULTIDIMENSIONAL HYPERBOLIC SCALAR CONSERVATION LAWS (a simplified model for multidimensional systems of hyperbolic conservation laws)

ISSN 0944-6532 / $ 2.50

4. THE BORN-INFELD SYSTEM (a non-linear electromagnetic model introduced in 1934, playing an important role in high energy Physics since the 1990's)

Finally, let us mention that we borrowed the expression "hidden convexity" from a lecture by L. C. Evans about various models where the same phenomena occur (such as growing sandpiles [5] and weak KAM theory).

2. Monge-Ampère equation and optimal transportation maps

Given two positive functions α and β of same integral over $\mathbf{R^d}$, we look for a convex solution Φ of the the Monge-Ampère equation:

$$\beta(\nabla\Phi(x))\det(D^2\Phi(x)) = \alpha(x), \quad x \in \mathbf{R^d}. \tag{1}$$

This nonlinear PDE is usually related to the Minkowski problem, which amounts to find hypersurfaces of prescribed Gaussian curvature.

2.1. A weak formulation

Assuming *a priori* that $x \in \mathbf{R^d} \to \nabla\Phi(x)$ is a diffeomorphism (with a jacobian matrix $D^2\Phi(x)$ everywhere symmetric positive), we immediately see, using the change of variable $y = \nabla\Phi(x)$, that (1) is equivalent to the following "weak formulation":

$$\int f(y)\beta(y)dy = \int f(\nabla\Phi(x))\alpha(x)dx \tag{2}$$

for all suitable test function f on $\mathbf{R^d}$. In the words of measure theory, this just means that $\beta(y)dy$ as a Borel measure on $\mathbf{R^d}$ is the image of the measure $\alpha(x)dx$ by the map $x \to \nabla\Phi(x)$. Notice that such a weak formulation has nothing to do with the usual definition of weak solutions in the sense of distribution (that does not make sense for a fully non-linear equation such as (1)). It is also weaker than the concept of "viscosity solution", as discussed in [26].

2.2. A convex variational principle

Proposition 2.1. *Let us consider all smooth convex functions* Ψ *on* $\mathbf{R^d}$ *with a smooth Legendre-Fenchel transform*

$$\Psi^*(y) = \sup_{x\in\mathbf{R^d}} x\cdot y - \Psi(x). \tag{3}$$

Then, in this family, a solution Φ *to the Monge-Ampère equation* (1) *is a minimizer of the convex functional*

$$J[\Psi] = \int \Psi(x)\alpha(x)dx + \int \Psi^*(y)\beta(y)dy. \tag{4}$$

Proof. For any suitable convex function Ψ, we have:

$$J[\Psi] = \int \Psi(x)\alpha(x)dx + \int \Psi^*(y)\beta(y)dy = \int (\Psi(x) + \Psi^*(\nabla\Phi(x)))\alpha(x)dx$$

(since $\nabla\Phi$ transports α toward β)

$$\geq \int x \cdot \nabla\Phi(x)\alpha(x)dx$$

(by definition of the Legendre transform (3))

$$= \int (\Phi(x) + \Phi^*(\nabla\Phi(x)))\alpha(x)dx$$

(indeed, in the definition of $\Phi^*(y) = \sup\ x \cdot y - \Phi(x)$, the supremum is achieved whenever $y = \nabla\Phi(x)$, which implies $\Phi^*(\nabla\Phi(x)) = x \cdot \nabla\Phi(x) - \Phi(x)$)

$$= \int \Phi(x)\alpha(x)dx + \int \Phi^*(y)\beta(y)dy = J[\Phi],$$

which shows that, indeed, Φ is a minimizer for (4))

2.3. Existence and uniqueness for the weak Monge-Ampère problem

Based on the previous observation, using the tools of convex analysis, one can solve the Monge-Ampère problem in its weak formulation, for a quite large class of data, with both existence and uniqueness of a solution:

Theorem 2.2 ([15], see also [54]). *Whenever α and β are nonnegative Lebesgue integrable functions on $\mathbf{R^d}$, with same integral, and bounded second order moments,*

$$\int |x|^2\alpha(x)dx < +\infty, \qquad \int |y|^2\beta(y)dy < +\infty,$$

there is a unique L^2 map T with convex potential $T = \nabla\Phi$ that solves the Monge-Ampère problem in its weak formulation (2)*, for all continuous function f such that: $|f(x)| \leq 1 + |x|^2$.*
This map is called the optimal transport map between $\alpha(x)dx$ and $\beta(y)dy$.

By map with convex potential, we exactly mean a Borel map T with the following property: there is a lsc convex function Φ defined on $\mathbf{R^d}$, valued in $]-\infty, +\infty]$, such that, for $\alpha(x)dx$ almost everywhere $x \in \mathbf{R^d}$, Φ is differentiable at x and $\nabla\Phi(x) = T(x)$.

Comments. The usual proof [16, 51, 46, 18] is based on the duality method introduced by Kantorovich to solve the so-called Monge-Kantorovich problem, based on the concept of joint measure (or coupling measure) [40]. [In the special case when α and β are both compactly supported, we can use Rockafellar's duality theorem (as quoted in [24]), just by working on the Banach space E (and its dual space E') of all continuous functions defined on a fixed closed ball B_R, with a large enough radius $R > 0$, so that the supports of α and β are contained in B_R. See more details in [15].] A proof by the direct method of calculus of variations is also possible, as done by Gangbo [35]. This theorem can be seen as the starting point of the so-called "optimal transport theory" which has turned out to be a very important and active field of research in the recent years, with a lot of interactions between calculus of variations, convex analysis, differential geometry, PDEs, functional analysis and probability theory and several

applications outside of mathematics (see [54] for a review). A typical (and striking) application to the isoperimetric inequality is given in the appendix of the present paper.
This theoretical result has also practical applications, for instance in medical imaging [2, 3] and astronomy [34] and it is important to have good numerical methods. From this viewpoint, the state of the art is not so satisfactory. In particular, there is no algorithm, to the best of our knowledge, that can match the efficiency of the best known algorithms for *linear* elliptic equations (multigrid methods, fast Poisson solvers etc...). Several methods can be used, such as linear programming [7, 34], continuum mechanics methods [9, 25], time continuation methods [2], direct Monge-Ampère solvers [29, 31] for instance, but none of them can be considered as very efficient: this is a challenging problem in numerical analysis.

3. The Euler equations

3.1. Geometric definition of the Euler equations

The Euler equations were introduced in 1755 [32] to describe the motion of inviscid fluids. In the special case of an incompressible fluid moving inside a bounded convex domain D in $\mathbf{R^d}$, a natural configuration space is the set SDiff(D) of all orientation and volume preserving diffeomorphisms of D. Then, a solution of the Euler equations can be defined as a curve $t \to g_t$ along SDiff(D) subject to:

$$\frac{d^2 g_t}{dt^2} \circ g_t^{-1} + \nabla p_t = 0, \tag{5}$$

where p_t is a time dependent scalar field defined on D (called the "pressure field").
As shown by Arnold [4, 30], these equations have a very simple geometric interpretation. Indeed, g_t is just a geodesic curve (with constant speed) along SDiff(D), with respect to the L^2 metric inherited from the Euclidean space $L^2(D, \mathbf{R^d})$, and $-\nabla p_t$ is the acceleration term, taking into account the curvature of SDiff(D). From this interpretation in terms of geodesics, we immediately deduce a variational principle for the Euler equations. However, this principle cannot be convex due to the non convexity of the configuration space. (Observe that SDiff(D) is contained in a sphere of the space $L^2(D, \mathbf{R^d})$ and cannot be convex, except in the trivial case $d = 1$ where it reduces to the identity map.)

3.2. A concave maximization principle for the pressure

Surprisingly enough, the pressure field obeys (at least on short time intervals) a concave maximization principle. More precisely,

Theorem 3.1. *Let* (g_t, p_t) *be a smooth solution to the Euler equations* (5) *on a time interval* $[t_0, t_1]$ *small enough so that*

$$(t_1 - t_0)^2 D^2 p_t(x) \leq \pi^2, \quad \forall x \in D \tag{6}$$

(in the sense of symmetric matrices). Then p_t *is a maximizer of the CONCAVE functional*

$$q \quad \to \quad \int_{t_0}^{t_1} \int_D q_t(x) dt dx + \int_D J_q[g_{t_0}(x), g_{t_1}(x)] dx, \tag{7}$$

among all t dependent scalar field q_t defined on D. Here

$$J_q[x,y] = \inf \int_{t_0}^{t_1} \left(-q_t(z(t)) + \frac{|z'(t)|^2}{2} \right) dt, \tag{8}$$

where the infimum is taken over all curves $t \to z(t) \in D$ such that $z(t_0) = x \in D$, $z(t_1) = y \in D$, is defined for all pair of points (x,y) in D.

Proof. The proof is very elementary and does not essentially differ from the one we used for the Monge-Ampère equation in the previous section (which is somewhat surprising since the Euler equations and the MA equation look quite different). The main difference is the smallness condition we need on the size of the time interval. Let us consider a time dependent scalar field q_t defined on D. By definition of J_q:

$$\int_D J_q\left[g_{t_0}(x), g_{t_1}(x)\right] dx \leq \int_{t_0}^{t_1} \int_D \left(\frac{1}{2} \left| \frac{dg_t}{dt} \right|^2 - q_t(g_t(x)) \right) dt dx.$$

Using a standard variational argument, we see that, under the smallness condition (6), the Euler equation (5) asserts that, for all $x \in D$

$$J_p[g_{t_0}(x), g_{t_1}(x)] = \int_{t_0}^{t_1} \left(\frac{1}{2} \left| \frac{dg_t}{dt} \right|^2 - p_t(g_t(x)) \right) dt.$$

Integrating in $x \in D$, we get:

$$\int_D J_p[g_{t_0}(x), g_{t_1}(x)] dx = \int_{t_0}^{t_1} \int_D \left(\frac{1}{2} \left| \frac{dg_t}{dt} \right|^2 - p_t(g_t(x)) \right) dt dx.$$

Since $g_t \in \mathrm{SDiff}(D)$ is volume preserving, we have:

$$\int_D (q_t(x) - q_t(g_t(x)) dx = \int_D (p_t(x) - p_t(g_t(x)) dx = 0.$$

Finally,

$$\begin{aligned} &\int_{t_0}^{t_1} \int_D q_t(x) dt dx + \int_D J_q[g_{t_0}(x), g_{t_1}(x)] dx \\ \leq &\int_{t_0}^{t_1} \int_D p_t(x) dt dx + \int_D J_p[g_{t_0}(x), g_{t_1}(x)] dx \end{aligned}$$

which shows that, indeed, (p_t) is a maximizer.

3.3. Global convex analysis of the Euler equations

The maximization principle is the starting point for a global analysis of the Euler equations. Of course, there is no attempt here to solve the Cauchy problem in the large for $d \geq 3$, which is one of the most outstanding problems in nonlinear PDEs theory.

(This would more or less amount to prove the geodesic completeness of SDiff(D).) We rather address the existence of minimizing geodesics between arbitrarily given points of the configuration space SDiff(D). This problem may have no classical solution, as shown by Shnirelman [47]. Combining various contributions by Shnirelman, Ambrosio-Figalli and the author [17, 48, 19, 1], we get the global existence and uniqueness result:

Theorem 3.2 ([17], see also [1]). *Let g_0 and g_1 be given volume preserving Borel maps of D (not necessarily diffeomorphisms) and $t_0 < t_1$. Then*

1) *There is a* unique *t dependent pressure field p_t, with zero mean on D, that solves (in a suitable weak sense) the maximization problem stated in Theorem* 3.1.

2) *There is a sequence g_t^n valued in* SDiff(D) *such that*

$$\frac{d^2 g_t^n}{dt^2} \circ (g_t^n)^{-1} + \nabla p_t \to 0,$$

in the sense of distributions and $g_0^n \to g_0$, $g_1^n \to g_1$ in L^2.

3) *Any sequence of approximate minimizing geodesics (g_t^n) (in a suitable sense) betwween g_0 and g_1 has the previous behaviour.*

4) *The pressure field is well defined in the space $L^2(]t_0, t_1[, BV_{loc}(D))$.*

Of course, these results are not as straightforward as Theorem 3.1 and requires a lot of technicalities (generalized flows, etc...). However, they still rely on convex analysis which is very surprising in this infinite dimensional differential geometric setting. Notice that the uniqueness result is also surprising. Indeed, between two given points, minimizing geodesics are not necessarily unique (as can be easily checked). However the corresponding acceleration field $-\nabla p_t$ is unique! It is unlikely that such a property could be proven using classical differential geometric tools. It is probably an output of the hidden convex structure. Let us finally notice that the improved regularity obtained by Ambrosio and Figalli [1] (they show that p belongs to $L^2(]t_0, t_1[, BV_{loc}(D))$ instead of ∇p a locally bounded measure, as previously obtained in [19]) is just sufficient to give a full meaning to the maximization problem. (A different formulation, involving a kind of Kantorovich duality is used in [17, 19] and requires less regularity.)

4. Convex formulation of multidimensional scalar conservation laws

4.1. Hyperbolic systems of conservation laws

The general form of multidimensional nonlinear conservation laws is:

$$\partial_t u_t + \sum_{i=1}^{d} \partial_i (F_i(u_t)) = 0,$$

where $u_t(x) \in V \subset R^m$ is a time dependent vector-valued field defined on a $d-$ dimensional manifold (say the flat torus $\mathbf{T^d} = \mathbf{R^d}/\mathbf{Z^d}$ for simplicity) and each $F_i : V \subset R^m \to R^m$ is a given nonlinear function. This general form includes systems of paramount importance in Mechanics and Physics, such as the gas dynamics and the Magnetohydrodynamics equations, for example. A simple necessary (and nearly sufficient) condition for the Cauchy problem to be well-posed for short times is the hyperbolicity condition which requires, for all $\xi \in \mathbf{R^d}$ and all $v \in V$ the $m \times m$ real matrix

$$\sum_{i=1}^{d} \xi_i F_i'(v)$$

to be diagonalizable with real eigenvalues. For many systems of physical origin, with a variational origin, there is an additional conservation law:

$$\partial_t(U(u_t)) + \sum_{i=1}^{d} \partial_i(G_i(u_t)) = 0, \tag{9}$$

where U and G_i are scalar functions (depending on F). (This usually follows from Noether's invariance theorem.) Whenever U is a strictly convex function, the system automatically gets hyperbolic. For most hyperbolic systems, solutions are expected to become discontinuous in finite time, even for smooth initial conditions. There is no theory available to solve the initial value problem in the large (see [28] for a modern review), except in two extreme situations. First, for a single space variable ($d = 1$) and small initial conditions (in total variation), global existence and uniqueness of "entropy solutions" have been established through the celebrated results of J. Glimm (existence) and A. Bressan and collaborators (well posedness) [38, 10]. (Note that some special systems can also be treated with the help of compensated compactness methods [52], without restriction on the size of the initial conditions.) Next, in the multidimensional case, global existence and uniqueness of "entropy solutions" have been obtained by Kruzhkov [41] in the case of a single (scalar) conservation law ($m = 1$).

Theorem 4.1 (Kruzhkov [41]). *Assume F to be Lipschitz continuous. Then, for all $u_0 \in L^1(\mathbf{T^d})$, there is a unique (u_t), in the space $C^0(R_+, L^1(\mathbf{T^d}))$ with initial value u_0, such that:*

$$\partial_t u_t + \nabla \cdot (F(u_t)) = 0, \tag{10}$$

is satisfied in the distributional sense and, for all Lipschitz convex function U defined on R, the "entropy" inequality

$$\partial_t(U(u_t)) + \nabla \cdot (Z(u_t)) \leq 0, \tag{11}$$

holds true in the distributional sense, where

$$Z(v) = \int_0^v F'(w)U'(w)dw.$$

In addition, for all pairs of such "entropy" solutions $(u, \tilde{u})$,

$$\int_{\mathbf{T^d}} |u_t(x) - \tilde{u}_t(x)|\, dx \ \leq \ \int_{\mathbf{T^d}} |u_s(x) - \tilde{u}_s(x)|\, dx, \quad \forall t \geq s \geq 0. \tag{12}$$

This result is often quoted as a typical example of maximal monotone operator theory in L^1. (For the concept of maximal monotone operator, we refer to [24, 6].) The use of the non hilbertian space L^1 is crucial. Indeed (except in the trivial linear case $F(v) = v$), the entropy solutions do not depend on their initial values in a Lipschitz continuous way in any space L^p except fot $p = 1$. This is due to the fact that, even for a smooth initial condition, the corresponding entropy solution u_t may become discontinuous for some $t > 0$ and, therefore, cannot belong to any Sobolev space $W^{1,p}(\mathbf{T^d})$ for $p > 1$.

4.2. A purely convex formulation of multidimensional scalar conservation laws

Clearly, convexity is already involved in Kruzhkov's formulation (10,11) of scalar conservation laws, through the concept of "entropy inequality". However, a deeper, hidden, convex structure can be exhibited, as observed recently by the author [23]. As a matter of fact, the Kruzhkov entropy solutions can be fully recovered just by solving a rather straightforward convex sudifferential inequality in the Hilbert space L^2. For notational simplicity, we limit ourself to the case when the initial condition u_0 is valued in the unit interval.

Theorem 4.2 (Brenier, 2006, [23]). *Assume $u_0(x)$ to be valued in $[0,1]$, for $x \in \mathbf{T^d}$. Let $Y_0(x,a)$ be any bounded function of $x \in \mathbf{T^d}$ and $a \in [0,1]$, non decreasing in a, such that*

$$u_0(x) = \int_0^1 1\{Y_0(x,a) < 0\} da, \tag{13}$$

for instance: $Y_0(a) = a - u_0(x)$. Then, the unique Kruzhkov solution to (10) is given by

$$u_t(x) = \int_0^1 1\{Y_t(x,a) < 0\} da, \tag{14}$$

where Y_t solves the convex subdifferential inequality in $L^2(\mathbf{T^d} \times [\mathbf{0},\mathbf{1}])$:

$$0 \in \partial_t Y_t + F'(a) \cdot \nabla_x Y_t + \partial\eta[Y_t], \tag{15}$$

where $\eta[Y] = 0$ if $\partial_a Y \geq 0$, and $\eta[Y] = +\infty$ otherwise.

Observe that $Y \to F'(a) \cdot \nabla_x Y + \partial\eta[Y]$ defines a maximal monotone operator and generates a semi-group of contractions in $L^2(\mathbf{T^d} \times [\mathbf{0},\mathbf{1}])$ [24].

Sketch of Proof. Multidimensional scalar conservation laws enjoy a comparison principle (this is why they are so simple with respect to general systems of conservation laws). In other words, if a family of initial conditions $u_0(x,y)$ is non decreasing with respect to a real parameter y, the corresponding Kruzhkov solutions $u_t(x,y)$ will satisfy the same property. This key observation enables us to use a kind of level set method, in the spirit of Sethian and Osher [45, 44], and even more closely, in the spirit of the paper by Tsai, Giga and Osher [53]. Assume, for a while, that $u_t(x,y)$ is *a priori* smooth and strictly increasing with respect to y. Thus, we can write

$$u_t(x, Y_t(x,a)) = a, \qquad Y_t(x, u_t(x,y)) = y$$

where $Y_t(x,a)$ is smooth and strictly increasing in $a \in [0,1]$. Then, a straightforward calculation shows that Y must solve the simple linear equation

$$\partial_t Y_t + F'(a) \cdot \nabla_x Y_t = 0 \tag{16}$$

(which has $Y_t(x,a) = Y_0(x - tF'(a), a)$ as exact solution). Unfortunately, this linear equation is not able to preserve the monotonicity condition $\partial_a Y \geq 0$ in the large. Subdifferential inequality (15) is, therefore, a natural substitute for it. The remarkable fact is that this rather straighforward modification exactly matches the Kruzhkov entropy inequalities. More precisely, as Y solves (15) , then

$$u_t(x,y) = \int_0^1 1\{Y_t(x,a) < y\} da$$

can be shown to be the right entropy solutions with initial conditions $u_0(x,y)$. For more details, we refer to [23].

Remark. Our approach is reminiscent of both the "kinetic method" and the "level set" method. The kinetic approach amounts to linearize the scalar conservation laws as (16) by adding an extra variable (here a). This idea (that has obvious roots in the kinetic theory of Maxwell and Boltzmann) was independently introduced for scalar conservation laws by Giga-Miyakawa and the author [14, 15, 37]. Using this approach, Lions, Perthame and Tadmor [43] later introduced the so-called kinetic formulation of scalar conservation laws and, using the averaging lemma of Golse, Perthame and Sentis [39], established the remarkable result that multidimensional scalar conservation laws enjoy a regularizing effect when they are genuinely nonlinear. (In other words, due to shock waves, entropy solutions automatically get a fractional amount of differentiability!). On the other side, the level set method by Osher and Sethian [45, 44] describes functions according to their level sets (here $Y(t,x,a) = y$). This is a very general and powerful approach to all kinds of numerical and analytic issues in pure and applied mathematics. An application of the level set method to scalar conservation laws was made by Tsai, Giga and Osher in [53] and more or less amounts to introduce a viscous (parabolic) approximation of subdifferential inequality (15). Finally, let us mention that some very special systems of conservation laws can be treated in a similar way [21].

5. The Born-Infeld system

Using convential notations of classical electromagnetism, the Born-Infeld system reads:

$$\partial_t B + \nabla \times \left(\frac{B \times (D \times B) + D}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) = 0, \quad \nabla \cdot B = 0,$$

$$\partial_t D + \nabla \times \left(\frac{D \times (D \times B) - B}{\sqrt{1 + D^2 + B^2 + (D \times B)^2}} \right) = 0, \quad \nabla \cdot D = 0,$$

This system is a nonlinear correction to the Maxwell equations, which can describe strings and branes in high energy Physics [12, 13, 11, 36]. Concerning the initial value problem, global smooth solutions have been proven to exist for small localized initial conditions by Chae and Huh [27] (using Klainerman's null forms and following a related work by Lindblad [42]). The additional conservation law

$$\partial_t h + \nabla \cdot Q = 0,$$

where

$$h = \sqrt{1 + D^2 + B^2 + (D \times B)^2}, \quad Q = D \times B.$$

provides an "entropy function" h which is a convex function of the unknown (D, B) only in a neighborhood of $(0, 0)$. However, h is clearly a convex function of B, D, and $B \times D$. Thus, there is a hope to restore convexity by considering $B \times D$ as an independent variable, which will be done subsequently by "augmenting" the Born-Infeld system.

5.1. The augmented Born-Infeld (ABI) system

Using Noether's invariance theorem, we get from the BI system 4 additional ("momentum-energy") conservation laws:

$$\partial_t Q + \nabla \cdot \left(\frac{Q \otimes Q - B \otimes B - D \otimes D}{h}\right) = \nabla \left(\frac{1}{h}\right), \qquad \partial_t h + \nabla \cdot Q = 0. \tag{17}$$

We call augmented Born-Infeld system (ABI) the 10×10 system of equations made of the 6 original BI evolution equations

$$\partial_t B + \nabla \times \left(\frac{B \times Q + D}{h}\right) = \partial_t D + \nabla \times \left(\frac{D \times Q - B}{h}\right) = 0, \tag{18}$$

with the differential constraints

$$\nabla \cdot B = 0, \qquad \nabla \cdot D = 0, \tag{19}$$

together with the 4 additional conservation laws (17) but WITHOUT the algebraic constraints

$$h = \sqrt{1 + D^2 + B^2 + (D \times B)^2}, \quad Q = D \times B. \tag{20}$$

These algebraic constraints define a 6 manifold in the space $(h, Q, D, B) \in R^{10}$ that we call "BI manifold". We have the following consistency result:

Proposition 5.1 (Brenier, 2004 [20]). *Smooth solutions of the ABI system* ((17), (18), (19)) *preserve the BI manifold* (20). *Therefore, any smooth solution of the original BI system can be seen as a smooth solution to the ABI system* ((17), (18), (19)) *with an initial condition valued on the BI manifold.*

5.2. First appearance of convexity in the ABI system

Surprisingly enough, the 10×10 augmented ABI system ((17), (18), (19)) admits an extra conservation law:

$$\partial_t U + \nabla \cdot Z = 0,$$

where

$$U(h, Q, D, B) = \frac{1 + D^2 + B^2 + Q^2}{h}$$

is convex (and Z is a rational function of h, Q, D, B). This leads to the GLOBAL hyperbolicity of the system.
Notice that the ABI system looks like Magnetohydrodynamics equations and enjoys *classical* Galilean invariance:

$$(t, x) \to (t, x + u\,t), \qquad (h, Q, D, B) \to (h, Q - hu, D, B),$$

for any constant speed $u \in R^3$!
For a large class of nonlinear Maxwell equations, a similar extension can be done (with 9 equations instead of 10) as in [49]. It should be mentioned that a similar method was introduced earlier in the framework of nonlinear elastodynamics with polyconvex energy (see [28]).

5.3. Second appearance of convexity in the ABI system

The 10×10 ABI (augmented Born-Infeld) system is *linearly degenerate* [28] and stable under weak-* convergence: weak limits of uniformly bounded sequences in L^∞ of smooth solutions depending on one space variable only are still solutions. (This can be proven by using the Murat-Tartar "div-curl" lemma.) Thus, we may conjecture that the convex-hull of the BI manifold is a natural configuration space for the (extended) BI theory. (As a matter of fact, the differential constraints $\nabla \cdot D = \nabla \cdot B = 0$ must be carefully taken into account, as pointed out to us by Felix Otto.) The convex hull has full dimension. More precisely, as shown by D. Serre [50], the convexified BI manifold is just defined by the following inequality:

$$h \geq \sqrt{1 + D^2 + B^2 + Q^2 + 2|D \times B - Q|}. \tag{21}$$

Observe that, on this convexified BI manifold (21):

1) The electromagnetic field (D, B) and the "density and momentum" fields (h, Q) can be chosen *independently* of each other, as long as they satisfy the required *inequality* (21). Thus, in some sense, the ABI system describes a coupling between field and matter, although the original Born-Infeld model is purely electromagnetic.
2) "Matter" may exist without electromagnetic field: $B = D = 0$, which leads to the Chaplygin gas (a possible model for 'dark energy' or 'vacuum energy')

$$\partial_t Q + \nabla \cdot \left(\frac{Q \otimes Q}{h} \right) = \nabla \left(\frac{1}{h} \right), \qquad \partial_t h + \nabla \cdot Q = 0,$$

3) 'Moderate' Galilean transforms are allowed

$$(t, x) \to (t, x + U\, t), \qquad (h, Q, D, B) \to (h, Q - hU, D, B),$$

which is impossible on the original BI manifold (consistently with special relativity) but becomes possible under weak completion (see the related discussion on "subrelativistic" conditions in [22]).

6. Appendix: A proof of the isoperimetric inequality using an optimal transport map

In this appendix, we describe a typical and striking application of optimal transport map methods. Let Ω be a smooth bounded open set and B_1 the unit ball in $\mathbf{R^d}$. The isoperimetric inequality reads (with obvious notations):

$$|\Omega|^{1-1/d} |B_1|^{1/d} \leq \frac{1}{d} |\partial \Omega|.$$

Let $\nabla\Phi$ the optimal transportation map between

$$\alpha(x) = \frac{1}{|\Omega|} 1\{x \in \Omega\}, \qquad \beta(y) = \frac{1}{|B_1|} 1\{x \in B_1\}.$$

In such a situation, according to Caffarelli's regularity result [26], $\nabla\Phi$ is a diffeomorphism between Ω and B_1 (up to their boundaries) with C^2 internal regularity (which is not a trivial fact) and

$$\det(D^2\Phi(x)) = \frac{|B_1|}{|\Omega|}, \quad x \in \Omega.$$

holds true in the classical sense. Then the proof (adaptated from Gromov) of the isoperimetric inequality is straightforward and sharp. Indeed, since $\nabla\Phi$ maps Ω to the unit ball, we have:

$$|\partial\Omega| = \int_{\partial\Omega} d\sigma(x) \geq \int_{\partial\Omega} \nabla\Phi(x) \cdot n(x) d\sigma(x)$$

(denoting by $d\sigma$ and $n(x)$ respectively the Hausdorff measure and the unit normal along the boundary of Ω)

$$= \int_\Omega \Delta\Phi(x) dx$$

(using Green's formula)

$$\geq d \int_\Omega (\det(D^2\Phi(x)))^{1/d} dx$$

(using that $(\det A)^{1/d} \leq 1/d \ \mathrm{Trace}(A)$ for any nonnegative symmetric matrix A)

$$= d|\Omega|^{1-1/d}|B_1|^{1/d}$$

since $\det(D^2\Phi(x)) = \frac{|B_1|}{|\Omega|}, \ x \in \Omega$.
So, the isoperimetric inequality

$$|\Omega|^{1-1/d}|B_1|^{1/d} \leq \frac{1}{d}|\partial\Omega|$$

follows, with equality *only* when Ω is a ball, as can be easily checked by tracing back the previous inequalities. Notice that Gromov's original proof does not require the map T to be optimal (it is enough that its jacobian matrix has positive eigenvalues). However, the optimal map plays a crucial role for various refinements of the isoperimetric inequality (in particular its quantitative versions by Figalli-Maggi-Pratelli [33], for example).

References

[1] L. Ambrosio, A. Figalli: On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, Calc. Var. Partial Differ. Equ. 31 (2008) 497–509.

[2] S. Angenent, S. Haker, A. Tannenbaum: Minimizing flows for the Monge-Kantorovich problem, SIAM J. Math. Anal. 35 (2003) 61–97.

[3] S. Angenent, E. Pichon, A. Tannenbaum: Mathematical methods in medical image processing, Bull. Amer. Math. Soc., New Ser. 43 (2006) 365–396.

[4] V. I. Arnold, B. Khesin: Topological Methods in Hydrodynamics, Springer, New York (1998).

[5] G. Aronsson, L. Evans, Y. Wu: Fast/slow diffusion and growing sandpiles, J. Differ. Equations 131 (1996) 304–335.

[6] H. Attouch, G. Buttazzo, G. Michaille: Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization, MPS/SIAM Series on Optimization 6, SIAM, Philadelphia (2006).

[7] M. Balinski: A competitive (dual) simplex method for the assignment problem, Math. Program. 34 (1986) 125–141.

[8] J. Ball: Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977) 337–403.

[9] J.-D. Benamou, Y. Brenier: A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, Numer. Math. 84 (2000) 375–393.

[10] S. Bianchini, A. Bressan: Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. Math. (2) 161 (2005) 223–342.

[11] G. Boillat. C. Dafermos, P. Lax, T.P. Liu: Recent Mathematical Methods in Nonlinear Wave Propagation (Montecatini Terme, 1994), Lecture Notes in Math. 1640, Springer, Berlin (1996).

[12] M. Born: Théorie non-linéire du champ electromagnetique, Ann. Inst. H. Poincaré 7 (1937) 155–265.

[13] M. Born, L. Infeld: Foundations of the new field theory, Proc. R. Soc. Lond., Ser. A 144 (1934) 425–451.

[14] Y. Brenier: Une application de la symétrisation de Steiner aux équations hyperboliques: la méthode de transport et écroulement, C. R. Acad. Sci., Paris, Sér. I 292 (1981) 563–566.

[15] Y. Brenier: Résolution d'équations d'évolution quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension $N + 1$, J. Differ. Equations 50 (1983) 375–390.

[16] Y. Brenier: Décomposition polaire et réarrangement monotone des champs de vecteurs, C. R. Acad. Sci., Paris, Sér. I 305 (1987) 805–808.

[17] Y. Brenier: The least action principle and the related concept of generalized flows for incompressible perfect fluids, J. Amer. Math. Soc. 2 (1989) 225–255.

[18] Y. Brenier: Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44 (1991) 375–417.

[19] Y. Brenier: Minimal geodesics on groups of volume-preserving maps, Commun. Pure Appl. Math. 52 (1999) 411–452.

[20] Y. Brenier: Hydrodynamic structure of the augmented Born-Infeld equations, Arch. Ration. Mech. Anal. 172 (2004) 65–91.

[21] Y. Brenier: Order preserving vibrating strings and applications to electrodynamics and magnetohydrodynamics, Methods Appl. Anal. 11 (2004) 515–532.

[22] Y. Brenier: Non relativistic strings may be approximated by relativistic strings, Methods Appl. Anal. 12 (2005) 153–167.

[23] Y. Brenier: L2 formulation of multidimensional scalar conservation laws, Arch. Rational Mech. Anal., to appear, arXiv:math/0609761.

[24] H. Brezis: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematics Studies 5, North-Holland, Amsterdam (1973).

[25] G. Buttazzo, C. Jimenez, E. Oudet: An optimization problem for mass transportation with congested dynamics, SIAM J. Control Optim. 48 (2009) 1961–1976.

[26] L. Caffarelli: Boundary regularity of maps with convex potentials, Commun. Pure Appl. Math. 45 (1992) 1141–1151.

[27] D. Chae, H. Huh: Global existence for small initial data in the Born-Infeld equations, J. Math. Phys. 44 (2003) 6132–6139.

[28] C. Dafermos: Hyperbolic Conservation Laws in Continuum Physics, Springer, Berlin (2000).

[29] E. Dean, R. Glowinski: On the numerical solution of the elliptic Monge-Ampère equation in dimension two: a least-squares approach, in: Partial Differential Equations. Modelling and Numerical Simulation, R. Glowinski, P. Neittaanmäki (eds.), Comput. Methods Appl. Sci. 16, Springer, Dordrecht (2008) 43–63.

[30] D. Ebin, J. Marsden: Groups of diffeomorphisms ano the motion of an incompressible fluid, Ann. Math. (2) 92 (1970) 102–163.

[31] G. Loeper, F. Rapetti: Numerical solution of the Monge-Ampère equation by a Newton algorithm, C. R., Math., Acad. Sci. Paris 340 (2005) 319–324.

[32] L. Euler: Opera Omnia, Series Secunda 12 (1755) 274–361.

[33] A. Figalli, F. Maggi, A. Pratelli: A mass transportation approach to quantitative isoperimetric inequalities, CVGMT preprint, Pisa (2008).

[34] U. Frisch, S. Matarrese, R. Mohayaee, A. Sobolevski: A reconstruction of the initial conditions of the universe by optimal mass transportation, Nature 417 (2002) 260–262.

[35] W. Gangbo: An elementary proof of the polar factorization of vector-valued functions, Arch. Ration. Mech. Anal. 128 (1994) 381–399.

[36] G. W. Gibbons: Aspects of Born-Infeld Theory and String/M-Theory, arXiv:hep-th/0106059.

[37] Y. Giga, T. Miyakawa: A kinetic construction of global solutions of first order quasilinear equations, Duke Math. J. 50 (1983) 505–515.

[38] J. Glimm: Solutions in the large for nonlinear hyperbolic systems of equations, Commun. Pure Appl. Math. 18 (1965) 697–715.

[39] F. Golse, B. Perthame, R. Sentis: Un résultat de compacité pour les équations de transport, C. R. Acad. Sci., Paris, Sér. I 301 (1985) 341–344.

[40] L. V. Kantorovich: On a problem of Monge, Usp. Mat. Nauk 3 (1948) 225–226.

[41] S. N. Kruzhkov: First order quasilinear equations with several independent variables, Mat. Sb., N. Ser. 81 (123) (1970) 228–255.

[42] H. Lindblad: A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time, Proc. Amer. Math. Soc. 132 (2004) 1095–110.

[43] P.-L. Lions, B. Perthame, E. Tadmor: A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. 7 (1994) 169–191.

[44] S. Osher, R. Fedkiw: Level Set Methods. Geometric Level Set Methods in Imaging, Vision, and Graphics, Springer, New York (2003).

[45] S. Osher, J. Sethian: Fronts propagating with curvature-dependent speed, J. Comput. Phys. 79 (1988) 12–49.

[46] L. Rüschendorf, S. T. Rachev: A characterization of random variables with minimum L^2-distance, J. Multivariate Anal. 32 (1990) 48–54.

[47] A. Shnirelman: On the geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid, Math. USSR, Sb. 56 (1987) 79–105.

[48] A. I. Shnirelman: Generalized fluid flows, their approximation and applications, Geom. Funct. Anal. 4 (1994) 586–620.

[49] D. Serre: Hyperbolicity of the nonlinear models of Maxwell's equations, Arch. Ration. Mech. Anal. 172 (2004) 309–331.

[50] D. Serre: A remark on Y. Brenier's approach to Born-Infeld electro-magnetic fields, in: Nonlinear Partial Differential Equations and Related Analysis (Evanston, 2002–2003), G.-Q. Chen et al. (ed.), Contemp. Math. 371, AMS, Providence (2005) 265–270.

[51] C. Smith, M. Knott: Note on the optimal transportation of distributions, J. Optimization Theory Appl. 52 (1987) 323–329.

[52] L. Tartar: Compacité par compensation: résultats et perspectives, Nonlinear Partial Differential Equations and Their Applications (Paris, 1981–1982), H. Brezis et al. (ed.), Res. Notes in Math. 84, Pitman, Boston (1983) 350–369.

[53] Y.-H. R. Tsai, Y. Giga, S. Osher: A level set approach for computing discontinuous solutions of Hamilton-Jacobi equations, Math. Comput. 72 (2003) 159–181.

[54] C. Villani: Topics in Optimal Transportation, American Mathematical Society, Providence (2003).

[55] L. C. Young: Lectures on the Calculus of Variations, Chelsea, New York (1980).