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Measurability and Selections of Multi-Functions in Banach Spaces

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We study the existence of measurable selectors for multi-functions whose values are weakly compact subsets of a Banach space. On one hand, we characterize multi-functions having strongly measurable selectors. On the other hand, we prove that every scalarly measurable multi-function admits scalarly measurable selectors. No separability assumptions are required for the range spaces.

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1. Introduction

Let p be the projection from $Y \times X$ onto Y, $B \subset Y \times X$ a set and define $\Omega = p(B)$. A uniformization of B is a function $f : \Omega \to X$ such that $(t, f(t)) \in B$ for every $t \in \Omega$. Notice that with the aid of the axiom of choice such a uniformization f always exists. The problem is how *nice* can f be chosen when B is *nice*? For instance if B is Borel (Y and X are topological spaces) can f be chosen being Borel (or analytic) measurable? The study of the existence of nice uniformizations for Borel sets when $Y = X = \Omega = [0, 1]$ attracted the attention of leading mathematicians from the very beginning of the XX century such as Baire, Borel, Hadamard, Lebesgue, von Neumann, Novikov, Kondo, Yankov, Luzin, Sierpinski, etc. and precipitated the birth and flourishment of the *descriptive set theory.* More recent authors contributing to this topic are, amongst others, Kuratowski, Ryll-Nardzewski, Sion, Larman, Mauldin,

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Pol, Saint-Raymond, etc. Notice that for our given B we naturally can define the multi-function $F : \Omega \to 2^X$ that at each $t \in \Omega$ is given by

$$F(t) := \{x \in X : \ (t, x) \in B\}.$$

With this language properties of B are just properties of the graph of F defined as $\mathrm{Graph}(F) := \{(t, x) : t \in \Omega, x \in F(t)\}$ and a uniformization of B is just a selector of F, *i.e.*, a single valued function $f : \Omega \to X$ such that $f(t) \in F(t)$ for every $t \in \Omega$.

When dealing with general multi-functions $F : \Omega \to 2^X$ the domain Ω is usually a measurable or a topological space and the range X is usually a topological space. In this setting analysts, topologists and applied mathematicians soon realized that many times when one needs to find a *nice* selector f for F, the starting point is not an hypothesis about $\mathrm{Graph}(F)$: one usually just know that F is lower or upper semicontinuous, measurable, Effros measurable, scalarly measurable, etc. A striking and pretty useful selection theorem relevant for this paper is the following:

Theorem A (Kuratowski-Ryll Nardzewski, [10]). *Let (Ω, Σ) be a measurable space and X a separable metric space. Let $F : \Omega \to 2^X$ be a multi-function with complete non-empty values satisfying that*

$$\{t \in \Omega : \ F(t) \cap O \neq \emptyset\} \in \Sigma \ \text{ for each open set } O \subset X. \tag{E}$$

Then F admits a $\Sigma-$Borel(X) measurable selector f.

The development in the theory of the existence of measurable selections has been many times linked to its applications to control theory, differential inclusions, mathematical models in economy and integration of multi-functions. People doing integration of multi-functions yearned over the years for a selection theorem where the range space need not be separable, but as far as we know they could only get hold of Kuratowski-Ryll Nardzewski's theorem that just works in the separable case. In this paper we get rid of the separability constraints for the range space and we find two strong selection results for multi-functions with values weakly compact sets of a Banach space, see Theorem 2.5 and Theorem 3.8 presented below.

Throughout this paper (Ω, Σ, μ) is a complete finite measure space and X is a real Banach space. By 2^X we denote the family of all non-empty subsets of X and by $cl(X)$, $k(X)$, $wk(X)$ and $cwk(X)$ we denote, respectively, the subfamilies of 2^X made up of norm closed, norm compact, weakly compact and convex weakly compact subsets of X.

Recall that a multi-function $F : \Omega \to 2^X$ that satisfies property (E) in Theorem 1 is said to be Effros measurable. A single valued function $f : \Omega \to X$ is strongly measurable if it is the μ-almost everywhere (shortly, μ-a.e.) limit of a sequence of Σ-simple X-valued functions defined in Ω. Strongly measurable selectors have been classically found along this line: assume that X is separable and take a multi-function $F : \Omega \to cl(X)$ Effros measurable, then apply Kuratowski-Ryll Nardzewski's theorem to produce a $\Sigma-$Borel(X,norm) measurable selector f of F and then with the help of Pettis' measurability theorem, *cf.* [6, Theorem 2, p. 42], conclude that f is strongly measurable.

Our first main result for non necessarily separable Banach spaces reads as follows:

Theorem 2.5. *For a multi-function $F : \Omega \to wk(X)$ the following statements are equivalent:*

(*i*) *F admits a strongly measurable selector.*

(*ii*) *For every $\varepsilon > 0$ and each $A \in \Sigma$ with $\mu(A) > 0$ there exist $B \subset A$, $B \in \Sigma$ with $\mu(B) > 0$, and $D \subset X$ with $\operatorname{diam}(D) \leq \varepsilon$ such that $F(t) \cap D \neq \emptyset$ for every $t \in B$.*

We write $\delta^*(x^*, C) := \sup\{x^*(x) : x \in C\}$ for any set $C \subset X$ and any x^* in the dual Banach space X^*. A multi-function $F : \Omega \to 2^X$ is said to be scalarly measurable if for each $x^* \in X^*$ the function $t \mapsto \delta^*(x^*, F(t))$ is measurable. In particular a single valued function $f : \Omega \to X$ is scalarly measurable if the composition $x^* \circ f$ is measurable for every $x^* \in X^*$.

Here is our second main result in this paper:

Theorem 3.8. *Every scalarly measurable multi-function $F : \Omega \to wk(X)$ admits a scalarly measurable selector.*

As far as we know, one of the standard ways of looking for selectors of scalarly measurable multi-functions $F : \Omega \to cwk(X)$ has been to assume that X is separable: in this case scalar measurability of F is equivalent to its Effros measurability, *cf.* [4, Theorem III.37] or [1, Corollary 4.10 (a)], and then F has indeed a strongly measurable selector as commented above. Outside the separable setting only two particular cases of Theorem 3.8 are known to be true with the extra hypothesis of the multi-function taking moreover convex values, namely, when X^* is weak*-separable, *cf.* [12, Proposition 6], and when X is reflexive, *cf.* [3, Theorem 5.1].

Whereas Theorem 2.5 is proved using techniques and reduction arguments which are not alien to other previous selection results in the literature, the proof of Theorem 3.8 relies on the convergence of martingales provided by the Radon-Nidodým property of convex weakly compact sets in Banach spaces.

Notation and terminology

Our standard references are [4] and [9] for multi-functions and convex analysis, [6] for measurability in Banach spaces and [5] for general concepts in functional analysis.

For the real Banach space $(X, \|\cdot\|)$ we denote by B_X the closed unit ball and S_X the unit sphere. For a set $D \subset X$ we define

$$\operatorname{diam}(D) := \sup_{x,y \in D} \|x - y\|$$

and we denote by $\operatorname{co}(D)$ the convex hull of D. Given a multi-function $F : \Omega \to 2^X$ and $C \subset X$, we write

$$F^-(C) := \{t \in \Omega : \ F(t) \cap C \neq \emptyset\}.$$

2. Multi-functions admitting strongly measurable selectors

This section is devoted to prove Theorem 2.5 as presented in the introduction.

Let Σ^+ be the family of all $A \in \Sigma$ with $\mu(A) > 0$. Given $A \in \Sigma^+$, the collection of all subsets of A belonging to Σ^+ is denoted by Σ_A^+.

We isolate first the following definition.

Definition 2.1. We say that a multi-function $F : \Omega \to 2^X$ satisfies property (P) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ and $D \subset X$ with $\operatorname{diam}(D) \leq \varepsilon$ such that $F(t) \cap D \neq \emptyset$ for every $t \in B$ (*i.e.* $B \subset F^-(D)$).

The notion of fragmented (σ-fragmented) multi-function introduced in [8] is a topological counterpart to our property (P) above.

The next proposition provides the first examples of functions with property (P) in the single valued case: the experienced reader will recognize easily that in this situation property (P) offers a characterization of strongly measurable functions. A very short proof is sketched for the sake of a wider audience:

Proposition 2.2. *For a function $f : \Omega \to X$ the following statements are equivalent:*

(i) *f satisfies property* (P)*.*
(ii) *For each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exist $B \in \Sigma_A^+$ with* $\operatorname{diam}\big(f(B)\big) \leq \varepsilon$.
(iii) *f is strongly measurable.*

Proof. The statement in (ii) is just a different way of writing (i). The implication (iii) $\Rightarrow$ (ii) follows from Egoroff's theorem, [7, Theorem 1, p. 94], the definition of strongly measurable function as limit μ-a.e. of a sequence of simple functions and the fact that simple functions satisfy (ii). The proof of (ii) $\Rightarrow$ (iii) can be done using an *exhaustion argument* to prove that for every $\varepsilon > 0$ there is a pairwise disjoint sequence (A_n) in Σ and a sequence (x_n) in X such that the function $g : \Omega \to X$ defined by $g = \sum_{n=1}^{\infty} x_n \mathbb{1}_{A_n}$ satisfies $\|f - g\| \leq \varepsilon$ μ-a.e. Now an appeal to [6, Corollary 3, p. 42] gives us that f is strongly measurable. $\square$

Proposition 2.3. *Let $F : \Omega \to 2^X$ be a multi-function.*

(i) *If there exists a multi-function $G : \Omega \to 2^X$ satisfying property* (P) *such that $G(t) \subset F(t)$ for μ-a.e. $t \in \Omega$, then F satisfies property* (P) *as well.*
(ii) *If F admits strongly measurable selectors, then F satisfies property* (P)*.*

Proof. The proof of (i) follows straightforwardly from the definition of (P) and (ii) follows from (i). $\square$

Note that if X is separable and $F : \Omega \to cl(X)$ is Effros measurable then F has property (P) because F admits a strongly measurable selector as commented in the introduction. There are however multi-functions with values in $k(\mathbb{R})$ with property (P) which are not Effros measurable. A simple example follows. Let $h : [0,1] \to \mathbb{R}^+$ be a non-measurable function and define $F : [0,1] \to k(\mathbb{R})$ by $F(t) := [0, h(t)]$. Then F has measurable selectors and so it satisfies property (P). However, F is not Effros measurable, because $F^-((a,+\infty)) = h^{-1}((a,+\infty))$ for all $a \in \mathbb{R}^+$.

It is known that, when X is separable, a $cwk(X)$-valued function is Effros measurable if and only if it is scalarly measurable, *cf.* [4, Theorem III.37] or [1, Corollary 4.10 (a)]. Without the convexity assumption we still have the following:

Lemma 2.4. *Suppose that X is separable. Let $F : \Omega \to wk(X)$ be a multi-function. The following statements are equivalent:*

(i) *F is Effros measurable.*

(ii) *$F^-(W) \in \Sigma$ for every set $W \subset X$ which can be written as a finite intersection of closed half-spaces.*

Proof. It is well known (*cf.* [4, Theorem III.30]) that (i) is equivalent to saying that $F^-(C) \in \Sigma$ for every norm closed set $C \subset X$. In particular, (i) implies (ii).

We divide the proof of (ii) $\Rightarrow$ (i) into several steps.

Step 1. $F^-(rB_X) \in \Sigma$ for all $r > 0$. Let (x_k^*) be a sequence in B_{X^*} such that

$$\|x\| = \sup_{k\in\mathbb{N}} x_k^*(x) \quad \text{for all } x \in X.$$

Then $rB_X = \bigcap_{n\in\mathbb{N}} W_n$, where $W_n := \bigcap_{k=1}^n \{x \in X : x_k^*(x) \leq r\}$ for all $n \in \mathbb{N}$. Since F has weakly compact values and (W_n) is a decreasing sequence of weakly closed sets, it is not difficult to check that $F^-(rB_X) = \bigcap_{n\in\mathbb{N}} F^-(W_n) \in \Sigma$.

Step 2. $F^-(C) \in \Sigma$ for every closed ball $C \subset X$. Write $C = x_0 + rB_X$ with $r > 0$ and $x_0 \in X$. The multi-function $G : \Omega \to wk(X)$ given by $G(t) := -x_0 + F(t)$ also has the property that $G^-(W) \in \Sigma$ for every finite intersection of closed half-spaces $W \subset X$, hence *Step 1* applied to G tells us that $F^-(C) = G^-(rB_X) \in \Sigma$.

Step 3. $F^-(U) \in \Sigma$ for every norm open set $U \subset X$. Indeed, this follows from *Step 2* and the fact that U can be written as a countable union of closed balls, because of the separability of X. The proof is over. □

We are now ready to prove the main result of this section.

Theorem 2.5. *For a multi-function $F : \Omega \to wk(X)$ the following statements are equivalent:*

(i) *F admits a strongly measurable selector.*

(ii) *F satisfies property* (P).

(iii) *There exist a set of measure zero $\Omega_0 \in \Sigma$, a separable subspace $Y \subset X$ and a multi-function $G : \Omega \setminus \Omega_0 \to wk(Y)$ that is Effros measurable and such that $G(t) \subset F(t)$ for every $t \in \Omega \setminus \Omega_0$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 2.3.

Let us prove (ii) $\Rightarrow$ (iii). We fix a decreasing sequence of positive real numbers (ε_m) converging to 0. Now our arguments are divided in several steps.

Step 1. Combining property (P) and a standard exhaustion argument, we can find a countable partition (up to a μ-null set) $\Gamma_1 = (A_{n,1})$ of Ω in Σ^+ and a sequence $(D_{n,1})$ of subsets of X with $\operatorname{diam}(D_{n,1}) \leq \varepsilon_1$ such that $F(t) \cap D_{n,1} \neq \emptyset$ for every $t \in A_{n,1}$ and every $n \in \mathbb{N}$. Observe that the set $V_{n,1,1} := \bigcap_{t \in A_{n,1}} (F(t) + \varepsilon_1 B_X)$ contains $D_{n,1}$ and so is non-empty for every $n \in \mathbb{N}$. The set $E_1 := \Omega \setminus \bigcup_{n \in \mathbb{N}} A_{n,1} \in \Sigma$ has measure zero.

The same argument, but now with ε_2 instead of ε_1, allows us to find a countable partition (up to a μ-null set) $\Gamma_2 = (A_{n,2})$ of Ω in Σ^+ such that the set $V_{n,2,2} := \bigcap_{t \in A_{n,2}} (F(t) + \varepsilon_2 B_X)$ is non-empty for every $n \in \mathbb{N}$. Since $\varepsilon_1 \geq \varepsilon_2$, we also have $V_{n,2,1} := \bigcap_{t \in A_{n,2}} (F(t) + \varepsilon_1 B_X) \neq \emptyset$ for every $n \in \mathbb{N}$. Again, the set $E_2 := \Omega \setminus \bigcup_{n \in \mathbb{N}} A_{n,2} \in \Sigma$ has measure zero.

In this way, we can find a sequence $\Gamma_m = (A_{n,m})$ of countable partitions (up to a μ-null set E_m) of Ω in Σ^+ such that the sets $V_{n,m,k} := \bigcap_{t \in A_{n,m}} (F(t) + \varepsilon_k B_X)$ are non-empty for every $k \leq m$ and every $n \in \mathbb{N}$. Clearly, the set $\Omega_0 := \bigcup_{m \in \mathbb{N}} E_m$ has measure zero.

Take $v_{n,m,k} \in V_{n,m,k}$ for every $k \leq m$ and every $n \in \mathbb{N}$, and let Y be the closed linear span of all the $v_{n,m,k}$'s, so that Y is *separable*. Since F has weakly compact values, it is clear that each $W_{n,m,k} := V_{n,m,k} \cap Y$ is weakly closed (and non-empty). Given $k \leq m$, set $F_{m,k} : \Omega \setminus \Omega_0 \to 2^Y$ by $F_{m,k} := \sum_{n \in \mathbb{N}} W_{n,m,k} \mathbb{1}_{A_{n,m}}$. Observe that for each set $C \subset Y$ we have $F^-_{m,k}(C) \in \Sigma$.

Given $k \in \mathbb{N}$, we define $F_k : \Omega \setminus \Omega_0 \to 2^Y$ by $F_k(t) := \mathrm{cl_w}\big(\bigcup_{m \geq k} F_{m,k}(t)\big)$. It is easy to see that for each weakly open set $U \subset Y$ we have $F^-_k(U) \in \Sigma$.

Step 2. Fix $t \in \Omega \setminus \Omega_0$. For each $m \in \mathbb{N}$, let $n_m(t) \in \mathbb{N}$ be such that $t \in A_{n_m(t),m}$. Observe that for each $k \in \mathbb{N}$ we have $F_k(t) \supset F_{k+1}(t)$ because the inequality $\varepsilon_{k+1} \leq \varepsilon_k$ allows us to write

$$\begin{aligned}\bigcup_{m \geq k+1} F_{m,k+1}(t) &= \bigcup_{m \geq k+1} \bigcap_{s \in A_{n_m(t),m}} \big(F(s) + \varepsilon_{k+1} B_X\big) \cap Y \\ &\subset \bigcup_{m \geq k} \bigcap_{s \in A_{n_m(t),m}} \big(F(s) + \varepsilon_k B_X\big) \cap Y = \bigcup_{m \geq k} F_{m,k}(t).\end{aligned}$$

Set $G(t) := \bigcap_{k \in \mathbb{N}} F_k(t)$. We will prove that the weakly closed set $G(t)$ is non-empty and contained in $F(t)$. To this end, observe first that for every $k \in \mathbb{N}$ we have

$$G(t) \subset F_k(t) = \mathrm{cl_w}\Big(\bigcup_{m \geq k} \bigcap_{s \in A_{n_m(t),m}} \big(F(s) + \varepsilon_k B_X\big) \cap Y\Big) \subset F(t) + \varepsilon_k B_X. \tag{1}$$

For each $k \in \mathbb{N}$ we take $x_k \in F_k(t)$ and write $x_k = y_k + z_k$ with $y_k \in F(t)$ and $z_k \in \varepsilon_k B_X$ (bear in mind (1)). Since the sequence (y_k) is contained in the weakly compact set $F(t)$, it has a weak cluster point $y \in F(t)$. Since $z_k \to 0$ in norm as $k \to \infty$, we conclude that y is also a weak cluster point of (x_k). Taking into account that $F_{k+1}(t) \subset F_k(t)$ for all $k \in \mathbb{N}$, it follows that $y \in \bigcap_{k \in \mathbb{N}} F_k(t) = G(t)$ and so $G(t) \neq \emptyset$. A similar argument shows that $G(t) \subset F(t)$.

Step 3. It follows that G is a multi-function on $\Omega \setminus \Omega_0$ taking values in $wk(Y)$. We next prove that G is Effros measurable. To this end we will apply Lemma 2.4. Fix $W \subset Y$ of the form $W = \bigcap_{i=1}^p \{y \in Y : y_i^*(y) \leq a_i\}$, where $y_i^* \in Y^*$ and $a_i \in \mathbb{R}$ for all $1 \leq i \leq p$. For each $k \in \mathbb{N}$, we define

$$O_k := \bigcap_{i=1}^p \{y \in Y : \ y_i^*(y) < a_i + 1/k\}.$$

Each O_k is weakly open in Y and so $F_k^-(O_k) \in \Sigma$. Observe that $O_{k+1} \subset \overline{O_{k+1}} \subset O_k$ for all $k \in \mathbb{N}$ and that $W = \bigcap_{k\in\mathbb{N}} O_k$. We claim that

$$G^-(W) = \bigcap_{k\in\mathbb{N}} F_k^-(O_k) \in \Sigma.$$

The inclusion "$\subset$" is clear. Conversely, take $t \in \bigcap_{k\in\mathbb{N}} F_k^-(O_k)$. Select a point $x_k \in F_k(t) \cap O_k$ for all $k \in \mathbb{N}$. Since $F(t)$ is weakly compact, the sequence (x_k) has a weak cluster point $x \in G(t)$ (imitate the proof that $G(t) \neq \emptyset$). Moreover, $x \in \bigcap_{k\in\mathbb{N}} \overline{O_k} = \bigcap_{k\in\mathbb{N}} O_k = W$. It follows that $t \in G^-(W)$. This proves the claim and shows that G is Effros measurable. The proof of *(ii)* $\Rightarrow$ *(iii)* is finished.

The arguments to prove *(iii)* $\Rightarrow$ *(i)* have been discussed previously in this paper. We can use Kuratowski-Ryll Nardzewski's theorem to find a Σ-Borel(Y, norm) measurable selector $g : \Omega \setminus \Omega_0 \to Y$ of G. Now we define $f : \Omega \to X$ as $f(t) := g(t)$ for every $t \in \Omega \setminus \Omega_0$ and $f(t)$ as any point in $F(t)$ for $t \in \Omega_0$. Then f is a selector of F that is strongly measurable according to Pettis' measurability theorem, *cf.* [6, Theorem 2, p. 42]. □

3. Scalarly measurable selectors

In order to prove the existence of scalarly measurable selectors for any scalarly measurable $wk(X)$-valued function, Theorem 3.8, we need some previous work. Given $C \in wk(X)$ and $x^* \in X^*$, we write

$$C|^{x^*} := \{x \in C : \; x^*(x) = \max x^*(C)\}$$

and

$$C|_{x^*} := \{x \in C : \; x^*(x) = \min x^*(C)\}.$$

Observe that $C|_{x^*} = C|^{-x^*}$ and that both $C|^{x^*}$ and $C|_{x^*}$ belong to $wk(X)$.

Lemma 3.1. *Let $C \in wk(X)$, consider $L := \overline{\text{co}}(C) \in cwk(X)$ and fix $x^* \in X^*$. Then $L|^{x^*} = \overline{\text{co}}(C|^{x^*})$.*

Proof. Since $\max x^*(C) = \max x^*(L)$, we have $C|^{x^*} \subset L|^{x^*}$ and (since $L|^{x^*}$ is closed and convex) we conclude that $\overline{\text{co}}(C|^{x^*}) \subset L|^{x^*}$.

Let us prove now the other inclusion. Since $L|^{x^*}$ is weakly compact and convex, the Krein-Milman theorem, [5, Theorem V.7.4], ensures that $L|^{x^*} = \overline{\text{co}}(\text{Ext}(L|^{x^*}))$. So it suffices to check that $\text{Ext}(L|^{x^*}) \subset C|^{x^*}$. Observe first that $\text{Ext}(L|^{x^*}) \subset \text{Ext}(L)$. On the other hand, $L = \overline{\text{co}}(C)$ and so, by the so-called "converse" of the Krein-Milman theorem, [5, Theorem V.7.8], $\text{Ext}(L) \subset C$. It follows that $\text{Ext}(L|^{x^*}) \subset C \cap L|^{x^*} = C|^{x^*}$. □

Lemma 3.2. *Let $F : \Omega \to wk(X)$ be a multi-function and consider the multi-function $G : \Omega \to cwk(X)$ defined by $G(t) := \overline{\text{co}}(F(t))$ for all $t \in \Omega$. Then F is scalarly measurable if and only if G is scalarly measurable.*

Proof. Bear in mind that $\delta^*(x^*, F) = \delta^*(x^*, G)$ for all $x^* \in X^*$. □

Given a multi-function $F : \Omega \to wk(X)$ and $x^* \in X^*$, we define the multi-functions $F|_{x^*}, F|^{x^*} : \Omega \to wk(X)$ by $F|_{x^*}(t) := F(t)|_{x^*}$ and $F|^{x^*}(t) := F(t)|^{x^*}$.

Lemma 3.3. *Let $F : \Omega \to wk(X)$ be a scalarly measurable multi-function and $x^* \in X^*$. Then $F|^{x^*}$ and $F|_{x^*}$ are scalarly measurable.*

Proof. Since $F|_{x^*} = F|^{-x^*}$, it suffices to prove that $F|^{x^*}$ is scalarly measurable. Let $G : \Omega \to cwk(X)$ be the scalarly measurable multi-function given by $G(t) := \overline{\text{co}}(F(t))$ (use Lemma 3.2). According to [12, Lemme 3], the multi-function $G|^{x^*}$ is scalarly measurable. In view of Lemma 3.1, we have

$$G|^{x^*}(t) = \overline{\text{co}}(F|^{x^*}(t)) \quad \text{for all } t \in \Omega,$$

and another appeal to Lemma 3.2 ensures that $F|^{x^*}$ is scalarly measurable. □

Lemma 3.4. *Let $F_n : \Omega \to wk(X)$ be a sequence of scalarly measurable multi-functions such that $F_n(t) \supset F_{n+1}(t)$ for every $n \in \mathbb{N}$ and every $t \in \Omega$. Then the multi-function $G : \Omega \to wk(X)$ given by $G(t) := \bigcap_{n\in\mathbb{N}} F_n(t)$ is scalarly measurable.*

Proof. Note that for every $t \in \Omega$ the set $G(t)$ is not empty and weakly compact. For each $x^* \in X^*$ we have $\delta^*(x^*, G) = \inf_{n\in\mathbb{N}} \delta^*(x^*, F_n)$ and therefore $\delta^*(x^*, G)$ is measurable. Indeed, fix $t \in \Omega$. Clearly $(\delta^*(x^*, F_n)(t))$ is a decreasing sequence bounded from below by $\delta^*(x^*, G)(t)$. For each $n \in \mathbb{N}$ we select $x_n \in F_n(t)$ such that $x^*(x_n) = \delta^*(x^*, F_n)(t) = \max x^*(F_n(t))$. Let $x \in G(t)$ be a weak cluster point of (x_n). Then $\delta^*(x^*, G)(t) \geq x^*(x)$ and $x^*(x)$ is a cluster point of $x^*(x_n)$, hence $x^*(x) = \inf_{n\in\mathbb{N}} \delta^*(x^*, F_n)(t) = \delta^*(x^*, G)(t)$. □

For a multi-function $F : \Omega \to wk(X)$ we will also use the notation $\delta_*(x^*, F)$ to denote the function given, for each $x^* \in X^*$, by $t \mapsto \delta_*(x^*, F)(t) := \inf x^*(F(t))$.

Remark 3.5. Let $F : \Omega \to wk(X)$ be a scalarly measurable multi-function. Then $\delta_*(x^*, F)$ and $\delta^*(x^*, F)$ are measurable for all $x^* \in X^*$ and we can consider

$$\Delta F := \sup_{x^* \in S_{X^*}} \int_\Omega \big(\delta^*(x^*, F) - \delta_*(x^*, F)\big)\, d\mu \in [0, \infty].$$

Evidently, if $\Delta F = 0$ then every selector f of F is scalarly measurable, because for every $x^* \in S_{X^*}$ one has $\delta^*(x^*, F) = x^* \circ f = \delta_*(x^*, F)$ μ-a.e.

The following theorem is a particular case of the main result of this section, Theorem 3.8 below. We include a detailed proof which advances some of the ideas used in the general case and provides a "constructive" method for finding selectors which might be of interest for applications.

Theorem 3.6. *Let $F : \Omega \to k(X)$ be a scalary measurable multi-function. Then F admits a scalary measurable selector.*

Proof. We divide the proof into two cases.

Particular case: Assume there is $M > 0$ such that, for each $x^* \in S_{X^*}$, we have $|\delta^*(x^*, F)| \leq M$ μ-a.e.

Clearly, the assumption ensures that for each $x^* \in S_{X^*}$ we have $|\delta_*(x^*, F)| \leq M$ μ-a.e. and that $\Delta F \leq 2M < \infty$. Let us define a sequence of scalarly measurable multi-functions $F_n : \Omega \to k(X)$ with $F_n(t) \supset F_{n+1}(t)$ for every $n \in \mathbb{N}$ and every $t \in \Omega$, as follows. Set $F_1 := F$ and, if F_n is already defined, then set $F_{n+1} := F_n|^{x_n^*}$, where $x_n^* \in S_{X^*}$ is selected in such a way that

$$\int_\Omega \big(\delta^*(x_n^*, F_n) - \delta_*(x_n^*, F_n)\big)\, d\mu \geq \frac{\Delta F_n}{2}. \tag{2}$$

By Lemma 3.3, each F_n is scalarly measurable. The multi-function $G : \Omega \to k(X)$ given by $G(t) := \bigcap_{n\in\mathbb{N}} F_n(t)$ is scalarly measurable after Lemma 3.4 and we have that $G(t) \subset F(t)$ for all $t \in \Omega$. To prove the theorem in the particular case we are dealing with it is sufficient to show that $\Delta G = 0$. Our proof is by contradiction. Suppose that $\Delta G > 0$. Then for each $n \in \mathbb{N}$ we have $\Delta F_n \geq \Delta G > 0$ and (2) yields

$$\int_\Omega \big(\delta^*(x_n^*, F_n) - \delta_*(x_n^*, F_n)\big)\, d\mu \geq \frac{\Delta G}{2} > 0.$$

By Lebesgue's dominated convergence theorem, there is a point $t_0 \in \Omega$ at which the function $\delta^*(x_n^*, F_n) - \delta_*(x_n^*, F_n)$ does not tend to 0 as $n \to \infty$. Set

$$\varepsilon_n := \delta^*(x_n^*, F_n)(t_0) - \delta_*(x_n^*, F_n)(t_0) \ \text{ for every } n \in \mathbb{N}.$$

By passing to a subsequence we may assume that $\inf_{n\in\mathbb{N}} \varepsilon_n = \varepsilon > 0$. For each $n \in \mathbb{N}$ we pick $x_n \in F_n(t_0)$ with $x_n^*(x_n) = \delta_*(x_n^*, F_n)(t_0)$. Then, given $m > n$, we have $x_m \in F_m(t_0) \subset F_{n+1}(t_0) = F_n|^{x_n^*}(t_0)$ and so $x_n^*(x_m) = \delta^*(x_n^*, F_n)(t_0)$, hence

$$\|x_m - x_n\| \geq x_n^*(x_m - x_n) = \varepsilon_n \geq \varepsilon.$$

Since all x_n's belong to the norm compact set $F(t_0)$, we reach a contradiction that finishes the proof of this case.

General case. Since $\{\delta^*(x^*, F) : x^* \in S_{X^*}\} \subset \mathbb{R}^\Omega$ is a pointwise bounded family of measurable functions, we can find a countable partition $E_1, E_2, \dots$ of Ω in Σ and a sequence (M_n) of positive real numbers such that, for each $n \in \mathbb{N}$ and each $x^* \in S_{X^*}$, we have $|\delta^*(x^*, F)| \leq M_n$ μ-a.e. on E_n (cf. [11, Proposition 3.1]). The particular case already proved can be applied to the restriction of F to each E_n ensuring that F admits a scalarly measurable selector. The proof is over. $\square$

In order to deal with Lemma 3.7 below we need to introduce some terminology. As usual, $\{0,1\}^{<\mathbb{N}} = \bigcup_{k\in\mathbb{N}} \{0,1\}^k$. Given $\tau = (\tau_n) \in T := \{0,1\}^{\mathbb{N}}$ and $m \in \mathbb{N}$, we write $\tau|m = (\tau_1, \dots, \tau_m) \in \{0,1\}^m$. Given $k \in \mathbb{N}$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in \{0,1\}^k$, we denote $\sigma \frown 0 = (\sigma_1, \dots, \sigma_k, 0)$ and $\sigma \frown 1 = (\sigma_1, \dots, \sigma_k, 1)$.

Let λ be the usual Borel product probability measure on T. For every $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the σ-algebra on T generated by the first n coordinate projections, *i.e.* the (finite) σ-algebra generated by the sets

$$H_\sigma := \{\tau \in T : \ \tau|n = \sigma\}, \quad \sigma \in \{0,1\}^n.$$

Lemma 3.7. *Let $A \in wk(X)$, $x^* \in S_{X^*}$ and $x^*_\sigma \in S_{X^*}$ for every $\sigma \in \{0,1\}^{<\mathbb{N}}$. Consider the family of sets $A_\sigma \in wk(X)$, $\sigma \in \{0,1\}^{<\mathbb{N}}$, defined recurrently as follows:*

(a) $A_{(0)} := A|_{x^*}$ *and* $A_{(1)} := A|^{x^*}$;
(b) $A_{\sigma\frown 0} := A_\sigma|_{x^*_\sigma}$ *and* $A_{\sigma\frown 1} := A_\sigma|^{x^*_\sigma}$ *for every* $\sigma \in \{0,1\}^{<\mathbb{N}}$.

Then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\sigma\in\{0,1\}^n} \big(\max x^*_\sigma(A_\sigma) - \min x^*_\sigma(A_\sigma)\big) = 0.$$

Proof. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and for each $\sigma \in \{0,1\}^{<\mathbb{N}}$ also fix $x_\sigma \in A_\sigma$. For $\sigma \in \{0,1\}^{<\mathbb{N}}$ we denote by z_σ the weak limit along $\mathcal{U}$ of the sequence

$$\underbrace{x_\sigma, \dots, x_\sigma,}_{\text{length of } \sigma}\ \frac{x_{\sigma\frown 0} + x_{\sigma\frown 1}}{2},\ \frac{x_{\sigma\frown 0\frown 0} + x_{\sigma\frown 0\frown 1} + x_{\sigma\frown 1\frown 0} + x_{\sigma\frown 1\frown 1}}{4},\ \cdots \tag{3}$$

Note that the sequence in (3) lies in the set $\overline{\mathrm{co}}(A)$ that is weakly compact by the Krein-Smulyan theorem, [5, Theorem V.7.14], and therefore the existence of z_σ is ensured and moreover $z_\sigma \in \overline{\mathrm{co}}(A)$.

The following assertions are easily checked to be true for every $\sigma \in \{0,1\}^{<\mathbb{N}}$:

(α) $z_\sigma = \frac{1}{2}(z_{\sigma\frown 0} + z_{\sigma\frown 1})$;
(β) $x^*_\sigma(z_{\sigma\frown 0}) = \min x^*_\sigma(A_\sigma)$ and $x^*_\sigma(z_{\sigma\frown 1}) = \max x^*_\sigma(A_\sigma)$;
(γ) $x^*_\sigma(z_\sigma) = \frac{1}{2}\big(\max x^*_\sigma(A_\sigma) + \min x^*_\sigma(A_\sigma)\big)$.

Given $n \in \mathbb{N}$ we define $g_n : T \to X$ by the formula

$$g_n := \sum_{\sigma\in\{0,1\}^n} z_\sigma \mathbb{1}_{H_\sigma}.$$

Property (α) above says that $\{z_\sigma\}_{\sigma\in\{0,1\}^{<\mathbb{N}}}$ is a tree and therefore the sequence $(g_n, \mathcal{T}_n)_{n\in\mathbb{N}}$ is a *martingale*. Since $(g_n, \mathcal{T}_n)_{n\in\mathbb{N}}$ is a martingale such that $\bigcup_{n\in\mathbb{N}} \mathcal{T}_n$ generates $\mathrm{Borel}(T)$ and each g_n takes values in the weakly compact convex set $\overline{\mathrm{co}}(A)$, it follows that there exists the limit of (g_n) in the norm topology of $L^1(\lambda, X)$, *cf.* [2, Theorems 3.6.1 and 2.3.6]. In particular:

$$\lim_{n\to\infty} \int_T \|g_{n+1}(\tau) - g_n(\tau)\|\, d\lambda(\tau) = 0. \tag{4}$$

Fix $n \in \mathbb{N}$. Note that property (β) implies that for any $\tau \in T$ we have

$$x^*_{\tau|n}\big(g_{n+1}(\tau)\big) = \begin{cases} \min x^*_{\tau|n}(A_{\tau|n}) & \text{if } \tau_{n+1} = 0, \\ \max x^*_{\tau|n}(A_{\tau|n}) & \text{if } \tau_{n+1} = 1. \end{cases}$$

For any $\sigma \in \{0,1\}^n$ and every $\tau \in H_\sigma$ we use property (γ) to deduce that

$$x^*_\sigma\big(g_n(\tau)\big) = \frac{1}{2}\big(\max x^*_\sigma(A_\sigma) + \min x^*_\sigma(A_\sigma)\big)$$

and so

$$\|g_{n+1}(\tau) - g_n(\tau)\| \geq \big|x_\sigma^*\big(g_{n+1}(\tau) - g_n(\tau)\big)\big| = \frac{1}{2}\big(\max x_\sigma^*(A_\sigma) - \min x_\sigma^*(A_\sigma)\big).$$

Consequently, we have

$$\begin{aligned} & 2\int_T \|g_{n+1}(\tau) - g_n(\tau)\|\, d\lambda(\tau) \\ &= 2\sum_{\sigma\in\{0,1\}^n} \int_{H_\sigma} \|g_{n+1}(\tau) - g_n(\tau)\|\, d\lambda(\tau) \\ &\geq \sum_{\sigma\in\{0,1\}^n} \big(\max x_\sigma^*(A_\sigma) - \min x_\sigma^*(A_\sigma)\big)\lambda(H_\sigma) \\ &= \frac{1}{2^n}\sum_{\sigma\in\{0,1\}^n} \big(\max x_\sigma^*(A_\sigma) - \min x_\sigma^*(A_\sigma)\big), \end{aligned}$$

which combined with (4) finishes the proof. □

Theorem 3.8. *Every scalarly measurable multi-function $F : \Omega \to wk(X)$ admits a scalary measurable selector.*

Proof. As in the proof of Theorem 3.6, we can assume without loss of generality that there is $M > 0$ such that for each $x^* \in S_{X^*}$ we have $|\delta^*(x^*, F)| \leq M$ μ-a.e. and $|\delta_*(x^*, F)| \leq M$ μ-a.e. We divide the proof into two steps.

Step 1. For every $\varepsilon > 0$ there exists a scalarly measurable multi-function $G : \Omega \to wk(X)$ such that $G(t) \subset F(t)$ for all $t \in \Omega$ and $\Delta G \leq \varepsilon$.

Our proof is by contradiction. Suppose there is $\varepsilon > 0$ such that $\Delta G > \varepsilon$ for every scalarly measurable multi-function $G : \Omega \to wk(X)$ such that $G(t) \subset F(t)$ for all $t \in \Omega$. We define recurrently, for each $\sigma \in \{0,1\}^{<\mathbb{N}}$, a functional $x_\sigma^* \in S_{X^*}$ and a scalarly measurable multi-function $F_\sigma : \Omega \to wk(X)$ with $F_\sigma(t) \subset F(t)$ for all $t \in \Omega$, as follows. Since $\Delta F > \varepsilon$, we can find $x^* \in S_{X^*}$ such that

$$\int_\Omega \big(\delta^*(x^*, F) - \delta_*(x^*, F)\big)\, d\mu > \varepsilon.$$

Set $F_{(0)} := F|_{x^*}$ and $F_{(1)} := F|^{x^*}$, so that both $F_{(0)}$ and $F_{(1)}$ are scalarly measurable (by Lemma 3.3). Assume now that for some $\sigma \in \{0,1\}^{<\mathbb{N}}$ the multi-function F_σ is already constructed. Then $\Delta F_\sigma > \varepsilon$ and we can select $x_\sigma^* \in S_{X^*}$ such that

$$\int_\Omega \big(\delta^*(x_\sigma^*, F_\sigma) - \delta_*(x_\sigma^*, F_\sigma)\big)\, d\mu > \varepsilon. \tag{5}$$

Then we set $F_{\sigma\frown 0} := F_\sigma|_{x_\sigma^*}$ and $F_{\sigma\frown 1} := F_\sigma|^{x_\sigma^*}$, which are scalarly measurable (again by Lemma 3.3).

Fix $n \in \mathbb{N}$ and define the measurable function $a_n : \Omega \to \mathbb{R}$ by

$$a_n(t) := \frac{1}{2^n}\sum_{\sigma\in\{0,1\}^n} \big(\delta^*(x_\sigma^*, F_\sigma)(t) - \delta_*(x_\sigma^*, F_\sigma)(t)\big).$$

Clearly, for each $n \in \mathbb{N}$ we have $|a_n| \leq 2M$ μ-a.e. Moreover, given any $t \in \Omega$, Lemma 3.7 applied to the weakly compact set $F(t)$ ensures that $\lim_{n\to\infty} a_n(t) = 0$. By Lebesgue's dominated convergence theorem we have $\lim_{n\to\infty} \int_\Omega a_n \, d\mu = 0$. However for each $n \in \mathbb{N}$ inequality (5) implies that

$$\int_\Omega a_n \, d\mu = \frac{1}{2^n} \sum_{\sigma\in\{0,1\}^n} \int_\Omega \big(\delta^*(x^*_\sigma, F_\sigma) - \delta_*(x^*_\sigma, F_\sigma)\big) \, d\mu > \varepsilon.$$

This contradiction finishes the proof of the first step.

Step 2. By the first step, we can find a scalarly measurable multi-function $F_1 : \Omega \to wk(X)$ such that $F_1(t) \subset F(t)$ for all $t \in \Omega$ and $\Delta F_1 \leq 1$. Again, the first step applied to F_1 ensures the existence of a scalarly measurable multi-function $F_2 : \Omega \to wk(X)$ such that $F_2(t) \subset F_1(t)$ for all $t \in \Omega$ and $\Delta F_2 \leq 1/2$. In this way, we can find a sequence of scalarly measurable multi-functions $F_n : \Omega \to wk(X)$ with $\Delta F_n \leq 1/n$ such that $F_{n+1}(t) \subset F_n(t)$ for every $t \in \Omega$. Then the multi-function $G : \Omega \to wk(X)$ given by $G(t) := \bigcap_{n\in\mathbb{N}} F_n(t)$ is scalarly measurable (by Lemma 3.4) and $\Delta G = 0$ because $0 \leq \Delta G \leq \Delta F_n$ for every $n \in \mathbb{N}$. Consequently, every selector of G (which in turn is a selector of F) is scalary measurable. The proof is over. □

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