Journal of Convex Analysis
Volume 17 (2010), No. 2, 405–411

Explicit Constructive Approximation to Symmetrization via Iterated Polarizations

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Dedicated to Al Baernstein II on the occasion of his 70th birthday.

Received: March 6, 2009
Revised manuscript received: March 31, 2009

We construct an explicit approximation to a rearranged function u^* of u via iterated polarizations of u.

1. Introduction

Two-point rearrangement or polarization is a powerful tool to establish numerous functional inequalities. The breakthrough paper of Al Baernstein II, [1], opened the way to many interesting works.

Let us first recall that for any nonnegative real valued function $u : \mathbb{R}^N \to \mathbb{R}_+$ and any half space H of $\mathbb{R}^N$ **containing the origin** $O_{\mathbb{R}^N}$, we define the two-point rearrangement (or polarization) of u with respect to H by:

$$u^H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{elsewhere;} \end{cases} \tag{1}$$

here σ_H is the reflection with respect to H.

We say that a nonnegative function u is a **symmetrizable** function if $\mu\{x \in \mathbb{R}^N : u(x) > t\} < \infty$ for all $t > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^N$. $\mathbf{u}^*$ is the unique function equimeasurable with u (see Section 2) such that $u^*(x) = h(|x|)$ with h non-increasing and right-continuous.

Inspired by the work of Al Baernstein II, F. Brock and Y. Solynin proved in [4] that for a nonnegative function $u \in L^p(\mathbb{R}^N)$, its Schwarz symmetrization u^* is the limit of iterated polarizations. Namely

$$u_n = u^{H_1 \dots H_n} = \left[\left(u^{H_1}\right)^{H_2^{\cdot^{\cdot^{\cdot}}}} \right]^{H_n} \to u^* \in L^p\left(\mathbb{R}^N\right) \quad \text{provided that } u \in L^p_+\left(\mathbb{R}^N\right). \tag{2}$$

ISSN 0944-6532 / $ 2.50

It turned out that (2) is extremely useful to prove many rearrangement inequalities. Indeed, it reduces complicated symmetrization inequalities to easier combinatorial problems. More precisely, to prove the generalized Hardy-Littlewood inequalities [3, 5], it is sufficient to establish:

$$\int_{\mathbb{R}^N} F\left(u_1(x),\ldots,u_n(x)\right)\,dx \le \int_{\mathbb{R}^N} F\left(u_1^H(x),\ldots,u_n^H(x)\right)\,dx \tag{3}$$

for any H. Hence using (2) and suitable growth conditions, we get

$$\int_{\mathbb{R}^N} F\left(u_1(x),\ldots,u_n(x)\right)\,dx \le \int_{\mathbb{R}^N} F\left(u_1^*(x),\ldots,u_n^*(x)\right)\,dx. \tag{4}$$

The same approach applies to establish the generalized Riesz inequality. C. Draghici [3] and A. Burchard and the author [5] proved that:

$$\begin{aligned} &\int_{\mathbb{R}^N}\cdots\int_{\mathbb{R}^N} F\left(u_1(x_1),\ldots,u_n(x_n)\right)\prod_{i<j} K_{i,j}\left(d(x_i,x_j)\right)\,dx_1\ldots\,dx_n \qquad (5)\\ \le &\int_{\mathbb{R}^N}\cdots\int_{\mathbb{R}^N} F\left(u_1^H(x_1),\ldots,u_n^H(x_n)\right)\prod_{i<j} K_{i,j}\left(d(x_i,x_j)\right)\,dx_1\ldots\,dx_n \end{aligned}$$

for any $H, u_1, \ldots, u_2$ are symmetrizable functions and $K_{i,j}$ are non-increasing kernels. Hence using again the limiting procedure (2), we obtain:

$$\begin{aligned} &\int_{\mathbb{R}^N}\cdots\int_{\mathbb{R}^N} F\left(u_1(x_1),\ldots,u_n(x_n)\right)\prod_{i<j} K_{i,j}\left(d(x_i,x_j)\right)\,dx_1\ldots\,dx_n \\ \le &\int_{\mathbb{R}^N}\cdots\int_{\mathbb{R}^N} F\left(u_1^*(x_1),\ldots,u_n^*(x_n)\right)\prod_{i<j} K_{i,j}\left(d(x_i,x_j)\right)\,dx_1\ldots\,dx_n. \qquad (6) \end{aligned}$$

Moreover two-point rearrangement enables us to study equality cases in (4) and (6). Once again it reduces a very hard functional analysis problem to a much less difficult combinatorial one. Indeed, to study equality cases in the generalized Hardy-Littlewood and Riesz inequalities it is sufficient to determine equality cases in (3) and (5) (respectively). This was done by A. Burchard and the author in [3, Theorem 2]. The other key tool was the following result

$$v = v^* \Leftrightarrow v = v^H \quad \forall H \in \mathcal{H}. \tag{7}$$

Polarization is an efficient tool to establish Pólya-Szegö inequality. Indeed, if u is a nonnegative function in the Sobolev space $W^{1,p}(\mathbb{R}^N)$, then u^H is also in the same space, [8], and $|\nabla u|_p = |\nabla u^H|_p$, which implies that $|\nabla u|_p = |\nabla u^{H_1\ldots H_n}|_p$, the lower semi-continuity of $|\cdot|_p$ and (2) enables us to conclude that

$$|\nabla u|_p \ge |\nabla u^*|_p. \tag{8}$$

Recently, M. Squassina and the author, [8], have extended this inequality to integrands depending on u and its gradient. Our main ingredient was polarization. Generalized Pólya-Szegö inequality has numerous applications in quasilinear equations.

However the study of equality cases in the Generalized Pólya-Szegö inequality needs subtle informations about the iterated polarizations. The method of Brock and Solynin to establish (2) is based on maximization techniques in which one cannot have concrete informations about the maximum and consequently the sequence constructed in [4, Lemma 6.1] is very abstracted. In [9, 10], Van Schaftingen has slightly improved the construction of [4] but his proof is not direct either and the underlying ideas he used are also based on implicit maximization problem.

The goal of this paper is to give an explicit construction of a sequence (u_n) obtained by iterated polarizations of u with respect to some half spaces H. This construction is a key ingredient in establishing equality cases in the generalized Pólya-Szegö inequality.

After the submission of the paper, the author learned that Jean Van Schaftingen had obtained the main result of the present article independently in a recent preprint [11]. We would like to point out that the same version of this paper was sent to some colleagues on December 11, 2008.

2. Notation and Definitions

- All statements about measurability refer to the Lebesgue measure μ in $\mathbb{R}^N$ unless it is indicated ($N \in \mathbb{N}^*$).
- In an integral where no domain of integration (variable of integration) is indicated, it is to be understood that the integration extends over all $\mathbb{R}^N$ (respectively the variable of integration is $d\mu$)
- $M(\mathbb{R}^N)$ is the set of real valued measurable functions, for $p > 1$; $L^p(\mathbb{R}^N) = \left\{u \in M(\mathbb{R}^N) : |u|_p < \infty\right\}$ where $|u|_p = \left(\int |u|^p\right)^{\frac{1}{p}}$, $|\cdot|$ is the euclidean norm in $\mathbb{R}^N$. $L_p^+(\mathbb{R}^N)$ is the cone of non-negative functions of $L_p(\mathbb{R}^N)$.
- The set of symmetrizable functions $F_N = \left\{u \in M_+(\mathbb{R}^N) : \mu\left\{x \in \mathbb{R}^N : u(x) > t\right\} < \infty \ \forall t > 0\right\}$; $M_+(\mathbb{R}^N)$ is the set of nonnegative measurable functions.
- If u and v are in F_N, we say that u is **equimeasurable with** v if $\mu\{x \in \mathbb{R}^N : u(x) > t\} = \mu\{x \in \mathbb{R}^N : v(x) > t\}$ $\forall t > 0$; we write $u \sim v$.
- For $u \in F_N$, its **Schwarz symmetrization** $\mathbf{u}^*$ is the unique function such that $u \sim u^*$ with $u^*(x) = h(|x|); h : (0, \infty) \to \mathbb{R}_+$ is nonincreasing and right continuous. When $u = u^*$, we say that u is **Schwarz symmetric**. When h is strictly decreasing, we say that u^* is strictly decreasing too (for a more detailed account, see [7]).
- It is well-known that for any half space H containing $O_{\mathbb{R}^N}$, we have that: u, u^H and u^* are equimeasurable [1], therefore:

$$|u|_p = |u^H|_p = |u^*|_p \ \text{ for any } u \in L_p^+(\mathbb{R}^N). \tag{9}$$

Topology of Half spaces: $\mathcal{H}$ denotes the set of closed half spaces H of $\mathbb{R}^N$ containing $O_{\mathbb{R}^N}$.

We equip it with the endowed norm ensuring that $H_n \to H$ if and only if there is a sequence of isometries $i_n : \mathbb{R}^N \to \mathbb{R}^N$ such that $H_n = i_n(H)$ and i_n converges to identity when n goes to infinity.

3. Preliminaries and Main Result

Before we state our main Theorem, let us prove some intermediate results. We start by giving Lemma 3.1 which can be very useful for many other purposes.

3.1. Polarization Inequalities and strict Inequalities:

Lemma 3.1. *Let u, v be two elements of F_N, $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a Borel measurable function such that:*

i) $F(w_1, w_2)+F(z_1, z_2) \le F(\max(z_1, w_1), \max(z_2, w_2))+F(\min(z_1, w_1), \min(z_2, w_2))$ *for any $z_1 \neq w_1$ and $z_2 \neq w_2$.*
Then:

$$\int F(u, v) \le \int F(u^H, v^H) \quad \text{for any } H \in \mathcal{H}$$

provided that both integrals are finite.

In addition, if $v = v^*$ and is strictly radially decreasing, and if i) holds with strict inequality, then the following holds: If $\int F(u, v) = \int F(u^H, v^H)$ $\forall H \in \mathcal{H}$ then $u = u^H$ for any $H \in \mathcal{H}$ and $u = u^*$.

Proof. Using the integrability assumptions, we can write:

$$\int F(u(x), v(x))\, dx = \int_H F(u(x), v(x)) + \int_H F(u(\sigma_H(x)), v(\sigma_H(x)))$$

$$\int F\left(u^H(x), v^H(x)\right) dx = \int_H F\left(u^H(x), v^H(x)\right) + \int_H F\left(u^H(\sigma_H(x)), v^H(\sigma_H(x))\right)$$

Therefore $\int F\left(u^H(x), v^H(x)\right) - \int F(u(x), v(x)) = \int_H F\left(u^H(x), v^H(x)\right) - F(u(x), v(x))$
$- F(u(\sigma_H(x)), v(\sigma_H(x))) + F(u(\sigma_H(x)), v(\sigma_H(x)))$ (*)

For $x \in H$, set $z_1 = u(x), z_2 = v(x), w_1 = u(\sigma_H(x)), w_2 = v(\sigma_H(x))$. Then $\max(z_1, w_1) = u^H(x)$ and $\min(z_1, w_1) = u^H(\sigma_H(x))$. The result follows thanks to $i)$.

Now if $v = v^*$ is strictly decreasing and (*) equals zero then $(u(x) - u(\sigma_H(x)))$ and $(v(x) - v(\sigma_H(x)))$ have the same sign (up to a set of measure zero).

Since $v = v^H$ $\forall H \in \mathcal{H}$, we certainly have that $v(x) \ge v(\sigma_H(x))$ for almost every $x \in H$ (here we used that $|\sigma_H(x)| \ge |x|$ for $x \in H$).

Hence $u(x) \ge u(\sigma_H(x))$ for a.e. $x \in H$. Using (7), the result follows. □

Corollary 3.2 (Hardy-Littlewood inequality for polarization). *For any $u \in L^p_+(\mathbb{R}^N)$ and $v \in L^q_+(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$:*

1. $\int uv \leq \int u^H v^H \ \forall H \in \mathcal{H}$.
If additionally $v = v^*$ is strictly decreasing then:
2. *If* $\int uv = \int u^H v \ \forall H \in \mathcal{H}$, *then* $u = u^*$.

Proof. Set $F(r,s) = rs$ and apply Lemma 3.1. □

Remark 3.3. A similar result was proved in [7] with u^H replaced by u^* and v^H replaced by v^*.

Corollary 3.4 (non-expansivity of two-point rearrangement). *For any* $u, v \in L^p_+(\mathbb{R}^N)$, $|u^H - v^H|_p \leq |u - v|_p \ \forall H \in \mathcal{H}$.

Proof. Set $F(r,s) = -|r-s|^p$ and apply Lemma 3.1. □

Remark 3.5. Corollary 3.4 tells us that if u_n is a sequence in $L^p_+(\mathbb{R}^N)$ converging to u in $L^p(\mathbb{R}^N)$, then $u_n^H \to u^H$ in $L^p(\mathbb{R}^N)$ for any $H \in \mathcal{H}$.

Remark 3.6. A similar result [3, Therorem 1] is obtained for Schwarz symmetrization. It implies that if $u_n \to u$ in $L^p(\mathbb{R}^N)$ then $u_n^* \to u^*$.

3.2. Density Result

Lemma (Brock, Solynin [4, Lemma 6.1]). *Let* $u \in L^p_+(\mathbb{R}^N)$ *and* $(H_n)_{n\geq 1}$ *be a sequence of closed half spaces of* $\mathbb{R}^N$ *containing* $0_{\mathbb{R}^N}$. *Then:* $u_n = u^{H_1 \dots H_n} = \left[\left(u^{H_1}\right)^{H_2^{\cdot^{\cdot^{\cdot}}}} \right]^{H_n}$ *is relatively compact in* $L^p(\mathbb{R}^N)$.

Proof. We will use [2, Theorem IV.25 p72].

First using (9), it follows that $\int |u^{H_1 \dots H_n}| = \int |u^{H_1 \dots H_{n-1}}|^p = \dots \int |u|^p$.

Therefore (u_n) is bounded in $L^p(\mathbb{R}^N)$.

On the other hand, given $\varepsilon > 0$, we can find $R > 0$ such that $\int_{|x|>R} |u|^p < \varepsilon$.

Since $\int_{|x|>R} |u_{n+1}|^p \leq \int_{|x|>R} |u_n|^p \leq \dots \leq \int_{|x|>R} |u|^p$, we can deduce that for any $n \in \mathbb{N}$: $\int_{|x|>R} |u_n|^p \leq \varepsilon$.

Now for any τ_δ a family of strictly decreasing Schwarz symmetric functions such that $\tau_\delta(t) = \tau(t/\delta)\delta^{-N}$, where τ is radial, radially nonincreasing, has compact support, and $\int \tau = 1$, there is a positive δ satisfying:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^p \tau_\delta(x-y)\, dx\, dy < \varepsilon.$$

By [3, Theorem 2], we have:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^p \tau_\delta(x-y)\, dx\, dy < \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^p \tau_\delta(x-y)\, dx\, dy < \varepsilon.$$

Finally applying the Riesz-Fréchet-Kolmogorov theorem [2], the conclusion follows. □

Theorem. *Let $u \in L^p_+(\mathbb{R}^N)$, $(H_n)_{n\geq 1}$ be a dense sequence in the set of closed half spaces containing $0_{\mathbb{R}^N}$.*

Define $(u_n)_{n\geq 0}$ by $\begin{cases} u_0 = u \\ u_{n+1} = u_n^{H_1\dots H_{n+1}}. \end{cases}$

Then $u_n \to u^*$ in $L^p(\mathbb{R}^N)$.

Proof. Using Lemma 3.2, (up to a subsequence) u_n converges to v in $L^p(\mathbb{R}^N)$. Let f be a strictly decreasing Schwarz symmetric function in $L^q_+(\mathbb{R}^N)$ with $\frac{1}{p}+\frac{1}{q}=1$.

It follows, using m times [Part (1), Corollary 3.2], that:

$$\int u_n^{H_1\dots H_m} f \leq \int u_{n+1} f \quad \text{for any } m\in\mathbb{N} \text{ with } m\leq n. \tag{10}$$

Using Remark 3.5, we obtain by letting n go to infinity:

$$\int v^{H_1\dots H_m} f \leq \int vf. \tag{11}$$

On the other hand, using once again [Part (1), Corollary 3.2], we know that:

$$\int vf \leq \int v^{H_1} f \leq \int v^{H_1 H_2} f \leq \dots \int v^{H_1\dots H_m} f. \tag{12}$$

(11) together with (12) imply that:

$$\int vf = \int v^{H_1} f = \int v^{H_1 H_2} f = \dots \int v^{H_1\dots H_m} f. \tag{13}$$

Hence it follows by [Part (2), Corollary 3.2] that $v = v^{H_1}$, $v^{H_1} = v^{H_1 H_2}$, …, $v^{H_1\dots H_{m-1}} = v^{H_1\dots H_m}$.

Therefore $v = v^{H_1}$, $v^{H_1} = \left(v^{H_1}\right)^{H_2} = v^{H_2} = v$. It follows that $v = v^{H_k}$ for $1\leq k\leq m$. But this is true for any $m\leq n$, from which we deduce that:

$$v = v^{H_k} \quad \forall k\in\mathbb{N}. \tag{14}$$

Now since $(H_n)_{n\geq 1}$ is dense in $\mathcal{H}$, for any $H\in\mathcal{H}$ we can find a subsequence (we will also denote it (H_n)) such that there exists $(i_n)_{n\geq 1}$ a sequence of isometries such that i_n converges to the identity with $H_n = i_n(H)$. Hence:

$$v^{H_n} \to v^H. \tag{15}$$

(14) together with (15) imply that $v = v^H$ $\forall H\in\mathcal{H}$ and therefore using (7):

$$v = v^*. \tag{16}$$

To conclude, we need to prove that $u^* = v^*$. Remark 3.6 tells us that:

$$u_n^* \to v^* \text{ in } L^p(\mathbb{R}^N). \tag{17}$$

On the other hand, we know that:

$$u^* \sim u \sim u_n \sim u_n^*. \tag{18}$$

$u_n^* \to v^*$ and $u_n^* \sim u^* \Rightarrow v^* \sim u^*$, thus $u^* = v^*$. □

Acknowledgements. The author is very grateful to F. Brock who brought to his attention the problem and for many fruitful discussions with him during his visit to AUB in June 2008.

References

[1] A. Baernstein II: A unified approach to symmetrization, in: Partial Differential Equations of Elliptic Type (Cortona, 1992), A. Alvino et al. (ed.), Cambridge Univ. Press, Cambridge (1994) 47–91.

[2] H. Brezis: Analyse Fonctionnelle. Théorie et Applications, Masson, Paris (1984).

[3] A. Burchard, H. Hajaiej: Rearrangement inequalities for functionals with monotone integrands, J. Funct. Anal. 233 (2006) 561–582.

[4] F. Brock, Y. Solynin: An approach to symmetrization via polarization, Trans. Amer. Math. Soc. 352 (2000) 1759–1796.

[5] F. Brock: A general rearrangement inequality à la Hardy-Littlewood, J. Inequal. Appl. 5 (2000) 309–320.

[6] C. Draghici: Rearrangement inequalities with application to ratios of heat kernels, Potential Anal. 22 (2005) 351–374.

[7] H. Hajaiej: Cases of equality and strict inequality in the extended Hardy-Littlewood inequalities, Proc. R. Soc. Edinb., Sect. A, Math. 135 (2005) 643–661.

[8] H. Hajaiej, M. Squassina: Generalized Pólya-Szegö inequality and applications to some quasilinear problems, under review (2009).

[9] J. Van Schaftingen: Approximation of symmetrizations and symmetry of critical points, Topol. Methods Nonlinear Anal. 28 (2006) 61–85.

[10] J. Van Schaftingen: Universal approximation of symmetrizations by polarizations, Proc. Amer. Math. Soc. 134 (2006) 177–186.

[11] J. Van Schaftingen: Explicit approximation of the symmetric rearrangement by polarizations, Arch. Math. 93 (2009) 181–190.