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Some Explicit Examples of Minimizers for the Irrigation Problem

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We construct some examples of explicit solutions to the problem

$$\min_{\gamma} \int_{\Omega} d_{\gamma}(x)\, dx$$

where the minimum is over all connected compact sets $\gamma \subset \overline{\Omega} \subset \mathbb{R}^2$ of prescribed one-dimensional Hausdorff measure.

More precisely we show that, if γ is a $C^{1,1}$ curve of length l with curvature bounded by $1/R$, $l \leq \pi R$ and $\varepsilon \leq R$, then γ is a solution to the above problem with Ω being the ε-neighbourhood of γ. In particular, $C^{1,1}$ regularity is optimal for this problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and let Σ_l denote, for $l > 0$, the class of all compact connected sets $\gamma \subset \overline{\Omega}$ whose one-dimensional Hausdorff measure does not exceed l. The variational problem

$$\min_{\gamma \in \Sigma_l} \int_{\Omega} d_{\gamma}(x)\, dx, \tag{1}$$

where $d_{\gamma}(x) = \min_{y \in \gamma} |y - x|$ is the distance function to γ, is called the *optimal irrigation* problem, and was first studied in [1]. The reader is referred to [1, 2] for an introduction to the problem, for its connections with mass transportation theory, and for several results such as the existence of minimizers and their topological properties. In particular, we mention the fact that each minimizer is topologically equivalent to a binary tree, in the sense that it is a connected union of finitely many branches (Lipschitz curves) without closed loops, and no more than three branches can meet at one point. Moreover, the regularity of the minimizers has been investigated in [6], where local $C^{1,1}$ regularity of each branch was proved, away from possible triple junctions and corner points. Further qualitative properties of minimizers can be found in [5, 7].

Some interesting numerical experiments are also available (see [1]). As it was pointed out in [1, 2], however, no example of explicit solution was known, not even in particular cases. The difficulty was mainly due to the fact that, even though several necessary optimality consitions have been obtained (see [1, 2, 6]), sufficient conditions seem hard to find, and the fact that an optimal γ must be loop-free rules out several candidate minimizers which would otherwise seem quite natural: for instance, if Ω is a disk, one

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might guess that for small l the optimal γ is a circle, but this is not the case since it would violate the no-loop property. In fact, the numerical experiments in [1] suggest that in this case, at least for small l, the optimal γ is some horseshoe-shaped curve around the center of the disk. Larger values of the parameter l give rise to different shapes, and in particular minimizers can have triple junctions: indeed, it is not difficult to prove that certain configurations of Ω (e.g., the union of an equilateral triangle of side 1 and three thin rectangles, each of sides 1 and $\lambda >> 1$, each rectangle having one short side in common with one side of the triangle) can force any minimizer to have a triple junction, at least for suitable values of l (e.g., $l = O(\lambda)$ in the mentioned example). One expects that quite ramificated structures may arise for large values of l, but numerical simulations become harder and harder, due to the myriad of possible topologies of the admissible configurations.

In this paper, we show how one can construct some explicit examples of minimizers. In particular we prove that any curve of class $C^{1,1}$, with length l and curvature bounded by $1/R$, is a solution to (1), provided that $l \leq \pi R$ and Ω is a suitable tubular neighbourhood of γ. Throughout the paper, the term *curve* denotes as usual a map $\gamma : [0, l] \mapsto \mathbb{R}^2$ with a certain regularity. We use the same symbol γ, however, to denote also the *range* $\gamma([0, l])$ of the map: thus, in particular, if l denotes the length of the curve, our γ can be regarded as a compact, connected set which can compete in (1). We think that no confusion should arise with this convention: we explicitly point out, however, that the symbol γ always denotes the range of the curve when it appears in the distance function $d_\gamma(x) = \min_{y\in\gamma} |y - x|$.

Theorem 1.1 (construction of minimizers). *Let $\gamma : [0, l] \mapsto \mathbb{R}^2$ be a curve of class $C^{1,1}$, parameterized by arclentgh, of length l. Let $R > 0$ be a number such that the curvature bound*

$$|\gamma''(t)| \leq \frac{1}{R} \quad \text{for a.e. } t \in [0, l] \tag{2}$$

holds true, and suppose that $l \leq \pi R$. Then, for every $\lambda \in (0, R)$, the curve γ is a solution to (1)*, where Ω is defined as*

$$\Omega = \left\{ x \in \mathbb{R}^2 \mid d_\gamma(x) < \lambda \right\}.$$

As a consequence, it turns out that $C^{1,1}$ regularity (see [6]) is optimal for the minimizers of (1) (at least if no smoothness assumption is made concerning the boundary of Ω).

Note that, since γ is supposed to be of class $C^{1,1}$, the curvature bound (2) is satisfied by any number $R > 0$ such that $R^{-1} \geq \|\gamma''\|_{L^\infty}$: the real geometric assumption on γ is in fact the inequality $l \leq \pi R$ which, being scale invariant, can be considered as a shape factor. However it is clear that, given any curve of class $C^{1,1}$, any sufficiently short (compact, connected) portion of it is a minimizer, for a suitable Ω.

A more general – but perhaps less effective – result is the following.

Theorem 1.2. *Let γ be a connected compact subset of $\mathbb{R}^2$, having finite one-dimensional Hausdorff measure l. Suppose also that, for some $\lambda > 0$ and for all $s \in (0, \lambda]$, the sub-level set of the distance function*

$$A_s = \left\{ x \in \mathbb{R}^2 \mid d_\gamma(x) < s \right\}$$

has an area given by

$$\operatorname{meas}(A_s) = 2ls + \pi s^2. \tag{3}$$

Then, with the choice $\Omega = A_\lambda$, the set γ solves the irrigation problem (1).

Note that, when γ is a smooth *non closed* curve and $s > 0$ is sufficiently small, the set A_s – a tubular neighbourhood of γ – is the union of a thin curvilinear rectangle (which follows the bends of γ) and two half disks at the endpoints of γ: the area of the rectangle (under smoothness assumptions on γ and smallness assumptions on s) is expected to be $2ls$, whereas each half disk has area $\pi s^2/2$. If these three sets are mutually disjoint, one can sum their areas and get an intuitive explanation of (3): on the other hand, even under smoothness assumptions on γ, the distorted rectangle and the two half disks may overlap, the rectangle will self-overlap if γ is not injective, etc., hence the validity of (3) is by no means guaranteed for a generic compact connected set γ – not even for a smooth curve (note that, in the case of a closed curve, the two half disks are not even present).

It turns out that, when γ is a curve, the assumptions of Theorem 1.1 are sufficient to guarantee that γ as a set satisfies (3), even though proving this sufficiency has required us some work, based on certain sharp relationships between length, curvature and level sets of the distance function. These facts, some of which are essentially known in global geometry (see for instance [3], which was kindly brought to our attention by Frank Morgan), are discussed in detail in Section 2. To make this presentation self contained, however, we have chosen to include detailed proofs of all the lemmas, based on certain integral inequalities which might be of some interest in other frameworks as well (see Lemmas 2.2 and 2.3).

These facts from the global geometry of curves allow us to obtain Theorem 1.1 from Theorem 1.2. The proof of the latter theorem, however, is based on the following estimate from below of the minimum value in (1), explicitly in terms of l and the area of Ω, which we believe is of independent interest:

Theorem 1.3 (sharp estimate from below). *Let $\Omega \subset \mathbb{R}^2$ be a Borel set of finite measure, and let $\gamma \subset \mathbb{R}^2$ be a connected compact set, having finite one-dimensional Hausdorff measure l. Then*

$$\int_\Omega d_\gamma(x)\,dx \geq \frac{2\sigma(l^2 + \pi \operatorname{meas}(\Omega)) - l \operatorname{meas}(\Omega)}{3\pi}, \tag{4}$$

where

$$\sigma = \frac{-l + \sqrt{l^2 + \pi \operatorname{meas}(\Omega)}}{\pi},$$

is the non negative root of the equation $2l\sigma + \pi\sigma^2 = \operatorname{meas}(\Omega)$.

One can easily check, by elementary computations, that the right hand side of (4) is always positive. It turns out that this estimate is sharp in some cases, which enables us to explicitly construct minimizers, i.e. sets γ which achieve this lower bound and are hence optimal.

We think it would be very interesting to construct explicit examples of minimizers γ having at least one triple junction. So far, we have not been able to do so (even though such minimizers can be proved to exist, as it was mentioned above).

Finally, we observe that the techniques used in this paper also work for more general integrand functions than in (1), e.g. integrands of the kind $F(d_\gamma(x))$, with F positive and increasing.

We end this introduction by two elementary examples, to illustrate our results.

Example 1.4 (segment). Let γ be a segment with endpoints P, Q and length $l = |Q-P|$. Of course the curvature is zero, hence (2) (with the natural parameterization $\gamma(t) = P + t(Q-P)/l$, $0 \le t \le l$) is satisfied for any value of $R > 0$. Hence, for every $\lambda > 0$, we can apply Theorem 1.1 by choosing R large enough. Then, clearly, the resulting sublevel set Ω is the convex envelope of the two disks of radius λ, centered at P and Q, and the segment γ solves the irrigation problem with this choice of Ω. Moreover, an elementary computation shows that, in this case, the estimate (4) is optimal.

We point out that a segment γ is the only example we know, of a compact commected set of length l such that (3) holds true for *every* s. Indeed, we believe that a segment is the only compact set with this property.

Example 1.5 (arc of circle). Let γ be an arc of circle of radius R and angular width $\theta \le \pi$. Then clearly $l = \theta R \le \pi R$, and (2) is satisfied with the natural parameterizations of γ. Hence we can apply Theorem 1.1 for every $\lambda \in [0, R]$, and γ results to be optimal for the corresponding choices of Ω.

2. Some auxiliary results

In this section we establish some auxiliary results which will be useful in the sequel. We recall that, for every non empty set $\gamma \subset \mathbb{R}^2$, we denote by d_γ the distance function to γ, i.e.

$$d_\gamma(x) = \inf\{|y-x| \mid y \in \gamma\}, \quad x \in \mathbb{R}^2$$

(the infimum is in fact a minimum, if γ is a closed set).

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a measurable set and let $\gamma \subset \mathbb{R}^2$ be a compact connected set. Then*

$$\operatorname{meas}(\{x \in \Omega \mid d_\gamma(x) \le s\}) \le \min\left\{\operatorname{meas}(\Omega), 2ls + \pi s^2\right\}, \quad s > 0, \tag{5}$$

where l denotes the one-dimensional Hausdorff measure of γ.

Proof. It is clear that the left hand side cannot exceed meas(Ω). On the other hand, the inequality

$$\operatorname{meas}\left(\{x \in \mathbb{R}^2 \mid d_\gamma(x) \le s\}\right) \le 2ls + \pi s^2, \quad s > 0,$$

has been proved in [4] as a lemma. □

Lemma 2.2. *Assume $0 < l \le \pi$, and $\theta : [0,l] \mapsto \mathbb{R}$ is a Lipschitz function such that $\theta(0) = 0$ and $|\theta'(t)| \le 1$ almost everywhere. Then*

$$\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt\right)^2 \ge 2(1-\cos l). \tag{6}$$

Moreover, equality holds if and only if either $\theta(t) = t$ or $\theta(t) = -t$.

Proof. For any two numbers $s, t \in [0,l]$ we have

$$0 \le |\theta(t) - \theta(s)| \le |t-s| \le l \le \pi.$$

Therefore, since the cosine function is decreasing on the interval $[0,\pi]$, we have

$$\cos(\theta(t)-\theta(s)) = \cos|\theta(t)-\theta(s)| \ge \cos|t-s| = \cos(t-s) \quad \forall s,t \in [0,l]. \tag{7}$$

By Fubini Theorem, we have

$$\begin{aligned}
&\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt\right)^2 \\
&= \int_0^l \int_0^l (\cos\theta(t)\cos\theta(s) + \sin\theta(t)\sin\theta(s))\;dtds = \int_0^l \int_0^l \cos(\theta(t)-\theta(s))\,dtds.
\end{aligned}$$

Therefore, (7) yields

$$\begin{aligned}
&\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt\right)^2 \ge \int_0^l \int_0^l \cos(t-s)\,dtds \\
&= \int_0^l \int_0^l (\cos t\cos s + \sin t \sin s)\,dtds = \left(\int_0^l \cos t\,dt\right)^2 + \left(\int_0^l \sin t\,dt\right)^2 \\
&= (\sin l)^2 + (1-\cos l)^2 = 2 - 2\cos l
\end{aligned}$$

and the claim follows. □

Lemma 2.3. *Assume $0 < l \le \pi$, and $\theta : [0,l] \mapsto \mathbb{R}$ is a Lipschitz function such that $\theta(0) = 0$ and $|\theta'(t)| \le 1$ almost everywhere. Then*

$$\begin{aligned}
\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt + 1\right)^2 \ge 1, \\
\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt - 1\right)^2 \ge 1,
\end{aligned} \tag{8}$$

and, equality holds in the first inequality if and only if $\theta(t) = -t$, whereas it occurs in the second if and only if $\theta(t) = t$. Moreover,

$$\int_0^l \cos\theta(t)\,dt \ge \sin l. \tag{9}$$

Proof. Since $\theta(0)=0$ and θ is 1-Lipschitz, then $\theta(t)$ has a linear growth, i.e.

$$|\theta(t)| \le t \quad \forall t \in [0,l]. \tag{10}$$

Therefore, since $\cos(z)$ is decreasing on the interval $[0,\pi]$, we have

$$\cos\theta(t) \ge \cos t \quad \forall t \in [0,l]. \tag{11}$$

Integrating this inequality over $[0,l]$, we obtain (9). Since $\sin l \ge 0$, squaring (9) we find

$$\left(\int_0^l \cos\theta(t)\,dt\right)^2 \ge (\sin l)^2. \tag{12}$$

Note that, expanding the second square, the two inequalities in (8) are equivalent to the two inequalities

$$\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt\right)^2 \ge -2\int_0^l \sin(\theta(t))\,dt,$$

$$\left(\int_0^l \cos(\theta(t))\,dt\right)^2 + \left(\int_0^l \sin(\theta(t))\,dt\right)^2 \ge 2\int_0^l \sin(\theta(t))\,dt.$$

Therefore, if

$$\left|\int_0^l \sin\theta(t)\,dt\right| \le 1-\cos l, \tag{13}$$

then (8) follows from Lemma 2.2. If equality occurs, still from Lemma 2.2 we conclude that θ is a linear function.

It remains to consider the case where

$$\left|\int_0^l \sin\theta(t)\,dt\right| > 1-\cos l. \tag{14}$$

Note that this condition implies that $l>\pi/2$. Indeed, if $l\le\pi/2$, then the sine function is increasing on $[0,l]$, and we would have from (10)

$$|\sin\theta(t)| = \sin|\theta(t)| \le \sin t \quad \forall t\in[0,l].$$

Therefore, it would follow

$$\left|\int_0^l \sin\theta(t)\,dt\right| \le \int_0^l \sin t\,dt = 1-\cos l,$$

which would violate (14).

But $\pi/2 < l \le \pi$ implies that $\cos l < 0$, hence from (14) we have

$$\int_0^l \sin\theta(t)\,dt - 1 > -\cos l > 0,$$

and hence

$$\left(\int_0^l \sin\theta(t)\,dt - 1\right)^2 > (\cos l)^2.$$

Adding this inequality to (12) reveals that strict inequality occurs in (8). □

The following Proposition is a particular form of the so callled Schur's Lemma in global geometry (see [3]). We present a self contained proof based on the last two lemmas.

Proposition 2.4. *Let $\gamma : [0,l] \mapsto \mathbb{R}^2$ be a curve of class $C^{1,1}$, parameterized by arclength, with curvature satisfying $|\gamma''(t)| \le 1/R$ for some $R > 0$ and a.e. $t \in [0,l]$. Suppose, moreover, that $l \le \pi R$.*

Then γ is injective. If, moreover, D is a disk of radius R tangent to $\gamma'(t_0)$ at the point $\gamma(t_0)$, $t_0 \in [0,l]$, then $\gamma(t)$ cannot enter the interior of D.

Proof. We begin with the last claim. By separately considering the two curves $t \mapsto \gamma(t+t_0)$ for $0 \le t \le l - t_0$, and $t \mapsto \gamma(t_0 - t)$ for $0 \le t \le t_0$, we can assume that $t_0 = 0$. Moreover by scaling, i.e. by considering the curve $t \mapsto R^{-1}\gamma(Rt)$ for $0 \le t \le l/R$, we can assume that $R = 1$ (and $l \le \pi$). Finally, by a rigid motion and possibly a reflection, we can also assume that $\gamma(0) = (0,0)$, that $\gamma'(0) = (1,0)$ and that D is the disk of radius one, centered at the point $(0,1)$. Writing $\gamma'(t)$ as $(\cos(\theta(t)), \sin(\theta(t)))$ for a suitable function $\theta : [0,l] \mapsto \mathbb{R}$, one easily checks that $\theta(t)$ satisfies the assumptions of Lemma 2.3.

Then, since we have

$$\gamma(T) = \int_0^T \gamma'(t)\,dt = \left(\int_0^T \cos(\theta(t))\,dt, \int_0^T \sin(\theta(t))\,dt\right), \quad 0 < T \le l \le 1,$$

from the second inequality in (8) it follows that $\gamma(T)$ cannot belong to the interior of D.

Finally, (6) reveals that $|\gamma(l) - \gamma(0)|^2 = |\gamma(l)|^2 \ge 2(1-\cos l)$, hence $\gamma(l) \ne \gamma(0)$. But the same argument can be applied to the restriction of γ to any subinterval $[t_1, t_2] \subseteq [0,l]$, hence γ is injective. □

Remark 2.5. The assumption that $l \le \pi R$ is optimal. Indeed, for fixed $\varepsilon > 0$ and $R > 0$, let γ be the path in Figure 2.1, obtained by glueing together (with a C^1 link) a segment of length εR from a point P_1 to a point P_2, and a circular arc of radius R and angular width $\pi + \varepsilon$ from P_2 to P_3.

Note that γ has total length $(\pi + 2\varepsilon)R$, and it admits a natural $C^{1,1}$ parameterization by arclength, with curvature bounded by $1/R$.

However, one can easily check that the endpoint P_3 is interior to a disk of radius R, which is tangent to γ at P_1 (the tangent disk is represented as a dashed circle in Figure 2.1).

3. Proof of the main results

Remark 3.1 (level sets of the distance function). If γ is a non empty set in $\mathbb{R}^2$, then the level sets of the distance function d_γ satisfy

$$\text{meas}\left(\{x \in \mathbb{R}^2 \mid d_\gamma(x) = s\}\right) = 0 \ \text{ for } \textit{every } s \in \mathbb{R} \tag{15}$$

(note that if we require this to hold only for *almost every* s, then the claim would be trivial since it holds true for any measurable function). Indeed, it is well known that

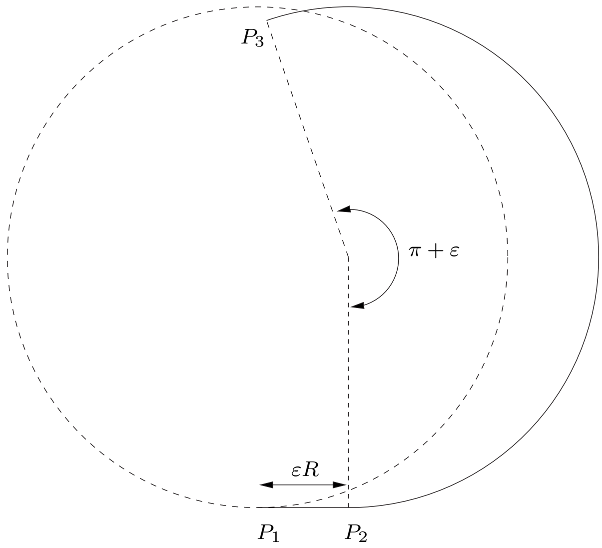

Figure 2.1: A curve with length $(\pi + 2\varepsilon)R$ and curvature bounded by $1/R$, which gets inside a tangent disk of radius R.

d_γ is 1-Lipschitzian and moreover $|\nabla d_\gamma(x)| = 1$ for almost every x. On the other hand, we also have that $\nabla d_\gamma = 0$ almost everywhere on *any* given level set of d_γ (this is a well known property of functions in $W^{1,1}_{\mathrm{loc}}$). Combining these two facts, we obtain that on any given level set of d_γ, both $|\nabla d_\gamma(x)| = 1$ and $|\nabla d_\gamma(x)| = 0$ are satisfied almost everywhere, and hence every level set of d_γ has zero Lebesgue measure.

As a consequence of (15), we have that

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^2 \mid d_\gamma(x) < s\right\}\right) = \operatorname{meas}\left(\left\{x \in \mathbb{R}^2 \mid d_\gamma(x) \le s\right\}\right) \quad \text{for every } s \in \mathbb{R}.$$

We will implicitly use this fact in the sequel, without further reference. Note that this holds true, more generally, in $\mathbb{R}^d$.

Proof of Theorem 1.3. We have by the slicing formula

$$\int_\Omega d_\gamma(x)\,dx = \int_0^\infty \operatorname{meas}\left(\{x \in \Omega \mid d_\gamma(x) > s\}\right)\,ds. \tag{16}$$

On the other hand, as Ω has finite measure, one has for $s > 0$

$$\operatorname{meas}\left(\{x \in \Omega \mid d_\gamma(x) > s\}\right) = \operatorname{meas}(\Omega) - \operatorname{meas}\left(\{x \in \Omega \mid d_\gamma(x) \le s\}\right),$$

and using (5) we obtain

$$\begin{aligned} \operatorname{meas}\left(\{x \in \Omega \mid d_\gamma(x) > s\}\right) &\ge \operatorname{meas}(\Omega) - \min\left\{\operatorname{meas}(\Omega), 2ls + \pi s^2\right\} \qquad (17) \\ &= \left(\operatorname{meas}(\Omega) - 2ls - \pi s^2\right)^+ = p(s), \quad s > 0 \end{aligned}$$

where the last equality defines the function $p(s)$. It is easy to check that $p(s) = 0$ for $s > \lambda$, whereas $p(s) = \operatorname{meas}(\Omega) - 2ls - \pi s^2$ for $s \in [0, \lambda]$. Therefore, plugging (17) into

(16), we finally obtain

$$\begin{aligned}\int_\Omega d_\gamma(x)\,dx \geq \int_0^\infty p(s)\,ds = \int_0^\lambda p(s)\,ds = \int_0^\lambda \left(\operatorname{meas}(\Omega) - 2ls - \pi s^2\right)\,ds \\ = \lambda \operatorname{meas}(\Omega) - l\lambda^2 - \frac{\pi}{3}\lambda^3 = \lambda\left(\operatorname{meas}(\Omega) - l\lambda - \frac{\pi}{3}\lambda^2\right).\end{aligned}$$

On the other hand, λ satisfies $\pi\lambda^2 = \operatorname{meas}(\Omega) - 2l\lambda$, which can be used to eliminate $\pi\lambda^2$ within the last brackets, thus obtaining

$$\int_\Omega d_\gamma(x)\,dx \geq \lambda\left(\frac{2}{3}\operatorname{meas}(\Omega) - \frac{l\lambda}{3}\right) = \frac{2}{3}\lambda \operatorname{meas}(\Omega) - \frac{l\lambda^2}{3}.$$

Finally, on eliminating λ^2 once more, one obtains (4). □

Proof of Theorem 1.2. The optimality of γ follows if we show that equality occurs in (4). Indeed, by a direct inspection of the previous proof, it is clear that the inequality sign in (4) is only due to the inequality sign in (17), which in turn is a consequence of (5).

But the assumptions of Theorem 1.2 (and the choice $\Omega = A_\lambda$) guarantee that equality occurs in (5) for all $s \in [0, \lambda]$, which reflects into equality in (17) and hence in (4) as well. □

Proof of Theorem 1.1. From Proposition 3.2 below, it follows immediately that the curve γ satisfies the assumptions of Theorem 1.2, for every $\lambda \in (0, R)$. □

Proposition 3.2. *Let $\gamma : [0, l] \mapsto \mathbb{R}^2$ be a curve of class $C^{1,1}$, parameterized by arc length, satisfying for some constant $R > 0$ the curvature bound*

$$|\gamma''(t)| \leq 1/R \quad \text{for a.e. } t \in [0, l] \tag{18}$$

and the length bound

$$l \leq \pi R. \tag{19}$$

Then the sublevel sets of the distance function d_γ are such that

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^2 \mid d_\gamma(x) < s\right\}\right) = 2ls + \pi s^2 \quad \forall s \in (0, R). \tag{20}$$

Proof. If $l = 0$ the claim is trivial, hence suppose that $l > 0$. Take $s \in (0, R)$, consider the sublevel set

$$A_s = \left\{x \in \mathbb{R}^2 \mid d_\gamma(x) < s\right\} \tag{21}$$

and define the map

$$F : (0, l) \times (-s, s) \mapsto \mathbb{R}^2, \qquad F(t, z) = \gamma(t) + z\nu(t),$$

where $\nu(t)$ is the unit normal vector to γ at t, obtained by a $\pi/2$ counterclockwise rotation of the velocity vector $\gamma'(t)$. Clearly, we can regard t as a coordinate along γ and z as a coordinate perpendicular to γ.

We wish to show that F is injective. For, suppose there are two pairs $(t_1, z_1) \neq (t_2, z_2)$ such that

$$0 < t_1 \leq t_2 < l, \qquad |z_1| < s, \qquad |z_2| < s$$

and

$$F(t_1, z_1) = F(t_2, z_2).$$

Seeking a contradiction, suppose first that $t_1 = t_2$. Then $z_1 \nu(t_1) = z_2 \nu(t_1)$, hence $z_1 = z_2$, thus violating the condition that $(t_1, z_1) \neq (t_2, z_2)$.

Now suppose that $t_1 < t_2$, and let $r := \max\{|z_1|, |z_2|\}$. If, say, $r = |z_1|$, let S_r be the circle of radius r centered at $F(t_1, z_1)$ (which is tangent to the curve, at the point $\gamma(t_1)$). By Lemma 2.4, the curve cannot get into either of the two disks of radius R which are tangent to the curve at $\gamma(t_1)$. Therefore, since $R > r$ and S_r is internally tangent to one of these two disks, the curve cannot get inside the smaller circle S_r, and can touch it only at the point $\gamma(t_1)$. On the other hand, since $t_2 \neq t_1$ and γ is injective by Proposition 2.4, we deduce that $\gamma(t_2)$ is outside the circle S_r, that is,

$$|F(t_1, z_1) - \gamma(t_2)| > r \geq |z_2|,$$

but this is a contradiction since the assumption $F(t_1, z_1) = F(t_2, z_2)$ implies in particular that

$$|F(t_1, z_1) - \gamma(t_2)| = |z_2|.$$

Finally, the case where $r = |z_2|$ can be treated in a similar way.

Now let $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the rotation matrix such that $\nu(t) = U\gamma'(t)$. Then $F(t, z) = \gamma(t) + zU\gamma'(t)$, and the Jacobian of F is given by

$$\begin{aligned} \left|\det\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial z}\right)\right| &= \left|\det\left(\gamma'(t) + zU\gamma''(t),\, U\gamma'(t)\right)\right| \\ &= \left|\det\left(\gamma'(t),\, U\gamma'(t)\right) + z\det(U)\det\left(\gamma''(t),\, \gamma'(t)\right)\right| \\ &= \left|1 + z\det\left(\gamma''(t),\, \gamma'(t)\right)\right| \end{aligned}$$

since $|\gamma'(t)| = 1$ and $\det(U) = 1$. Note, however, that since $|z| < R$, and $|\gamma''(t)| \leq 1/R$, we have

$$\left|z\det\left(\gamma''(t),\, \gamma'(t)\right)\right| < 1$$

and hence

$$\left|\det\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial z}\right)\right| = 1 + z\det\left(\gamma''(t),\, \gamma'(t)\right).$$

Therefore, letting $\mathcal{R} \subset \mathbb{R}^2$ denote the rectangle $(0, l) \times (-s, s)$, we have by the area formula

$$\begin{aligned} \operatorname{meas}(F(\mathcal{R})) &= \iint_{\mathcal{R}} \left|\det\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial z}\right)\right| dtdz \\ &= \iint_{\mathcal{R}} \left(1 + z\det\left(\gamma''(t),\, \gamma'(t)\right)\right) dtdz = \operatorname{meas}(\mathcal{R}) = 2ls \end{aligned}$$

(note that the function $z \det\big(\gamma''(t),\, \gamma'(t)\big)$ is odd in the variable z, hence its integral over $\mathcal{R}$ is zero). Recalling (21), from $|F(t,z) - \gamma(t)| = |z| < s$ it follows that

$$F(\mathcal{R}) \subseteq A_s.$$

Now consider the two half disks of radius s

$$D_1 = \left\{\gamma(0) + v \mid v \in \mathbb{R}^2,\ |v| < s,\ v \cdot \gamma'(0) < 0\right\},$$
$$D_2 = \left\{\gamma(l) + v \mid v \in \mathbb{R}^2,\ |v| < s,\ v \cdot \gamma'(l) > 0\right\}.$$

Since also $D_1 \subset A_s$ and $D_2 \subset A_s$, the proof will be completed if we show that $F(\mathcal{R})$, D_1 and D_2 are mutually disjoint. Indeed, in this case we would have

$$\begin{aligned}\operatorname{meas}(A_s) &\geq \operatorname{meas}(F(\mathcal{R}) \cup D_1 \cup D_2)\\ &= \operatorname{meas}(F(\mathcal{R})) + \operatorname{meas}(D_1) + \operatorname{meas}(D_2) = 2ls + \pi s^2,\end{aligned}$$

whereas the opposite inequality follows from (5) with $\Omega = \mathbb{R}^2$.

Note that by time inversion (i.e. by considering the reparameterization $t \mapsto \gamma(l-t)$) the two half disks play a symmetric role, hence it is enough to show that $D_1 \cap D_2 = \emptyset$, and that $D_1 \cap F(\mathcal{R}) = \emptyset$. We end this proof by separately considering these two claims. We preliminarly observe that, by scaling (i.e. by considering the curve $t \mapsto R^{-1}\gamma(Rt)$ for $0 \leq t \leq l/R$) we can suppose that $R = 1$ and $l \leq \pi$. Moreover, by a rigid motion we can also assume that $\gamma(0) = (0,0)$ and $\gamma'(0) = (1,0)$, and we may write $\gamma'(t)$ as $(\cos\theta(t), \sin\theta(t))$ for a suitable 1-Lipschitz function θ such that $\theta(0) = 0$, as in Lemma 2.3. We will tacitly do this in the rest of the proof.

Claim i): $D_1 \cap \overline{D_2} = \emptyset$. Suppose, on the contrary, that this is not the case. By our assumptions, this means that there is a vector $w \in \mathbb{R}^2$ such that

$$|w| \leq 1, \quad w \cdot \gamma'(l) \geq 0, \quad \text{and} \quad \gamma(l) + w \in D_1 \tag{22}$$

(the first two conditions are equivalent to $\gamma(l) + w \in \overline{D_2}$). From (9) we obtain that

$$\gamma(l) \cdot (1,0) = \int_0^l \cos\theta(t)\, dt \geq \sin l \geq 0,$$

hence $\gamma(l)$ is in the closed right half plane (whereas D_1 is in the open left half plane). Since $\gamma(l) + w \in D_1$ and $|w| \leq 1$, it is clear that $\gamma(l)$ must also belong to the tubular neighbourhood of widht 1 of $\overline{D_1}$. Moreover, since $\gamma'(0) = (1,0)$, by Proposition 2.4 γ cannot enter the two open disks of radius one cantered at $(0, \pm 1)$. Putting these conditions together, we obtain in particular that $\gamma(l) \in T$ where

$$T = \left\{(x,y) \mid x^2 + (y-1)^2 \geq 1,\ x^2 + (y+1)^2 \geq 1,\ 0 \leq x \leq 1,\ |y| \leq 1\right\}.$$

In particular, we must have $|\gamma(l)|^2 \leq 2$. This condition, recalling that $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$, can be written as

$$\left(\int_0^l \cos\theta(t)\, dt\right)^2 + \left(\int_0^l \sin\theta(t)\, dt\right)^2 \leq 2.$$

Combining with (6), we obtain $2(1-\cos l) \leq 2$, that is $\cos l \geq 0$, which reflects into the stronger condition on l

$$l \leq \frac{\pi}{2}. \tag{23}$$

Therefore, as $\theta(t)$ is 1-Lipschitzian and $\theta(0) = 0$, we must have $|\theta(t)| \leq \pi/2$ and in particular

$$\gamma'(l) \cdot (1,0) = \cos\theta(l) \geq 0. \tag{24}$$

Since $\gamma'(l) = (\cos\theta(l), \sin\theta(l))$, the first two conditions in (22) imply that we can write

$$w = (\rho\cos(\theta(l)+\varphi), \rho\sin(\theta(l)+\varphi)), \quad 0 \leq \rho \leq 1, \ |\varphi| \leq \frac{\pi}{2},$$

for suitable ρ and φ. The we can compute

$$\begin{aligned}(\gamma(l)+w)\cdot(1,0) &= \int_0^l \cos\theta(t)\,dt + \rho\cos(\theta(l)+\varphi) \\ &= \int_0^l \cos\theta(t)\,dt + \rho\cos\varphi\cos\theta(l) - \rho\sin\varphi\sin\theta(l).\end{aligned}$$

From (24) and $|\varphi| \leq \pi/2$ it follows that $\cos\varphi\cos\theta(l) \geq 0$, hence using also (9) we find

$$\begin{aligned}(\gamma(l)+w)\cdot(1,0) &\geq \int_0^l \cos\theta(t)\,dt - \rho\sin\varphi\sin\theta(l) \geq \int_0^l \cos\theta(t)\,dt - |\sin\theta(l)| \\ &\geq \sin l - |\sin\theta(l)| = \sin l - \sin|\theta(l)|.\end{aligned}$$

On the other hand, since $\theta(0) = 0$ and θ is 1-Lipschitzian, $|\theta(l)| \leq l$: since $\sin t$ is increasing in $[0,\pi/2]$, from (23) it follows that the last expression is non negative. Hence

$$(\gamma(l)+w)\cdot(1,0) \geq 0$$

which is a contradiction since it is not compatible with the last condition in (22) (recall that D_1 is contained in the open left half plane). Therefore, the claim is proved.

Claim ii): $D_1 \cap F(R) = \emptyset$. In fact, this is a consequence of Claim *i)*. For, suppose that for some $t_0 \in (0,l)$ and for some $z \in (-s,s)$ we have

$$F(t_0,z) = \gamma(t_0) + z\nu(t_0) \in D_1.$$

We can consider the curve $\widehat{\gamma}(t)$ defined as the restriction of $\gamma(t)$ to the interval $[0,t_0]$. Clearly $\widehat{\gamma}$ has length $\widehat{l} = t_0 < l$, and satisfies the same assumptions as the original curve γ. If we denote $\widehat{D}_1$ and $\widehat{D}_2$ the two half disks defined exactly as D_1 and D_2, but relative to $\widehat{\gamma}$, it is clear that $\widehat{D}_1 = D_1$ and $F(t_0,z) \in \overline{\widehat{D}_2}$. Hence, our claim *i)* applied to $\widehat{\gamma}$ would yield a contradiction. $\square$

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