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On Maximal Domains for $\mathfrak{C}$-Convex Functions and Convex Extensions

Manfred Möller
The John Knopfmacher Centre of Applicable Analysis and Number Theory,
School of Mathematics, University of the Witwatersrand, WITS, 2050, South Africa
Manfred.Moller@wits.ac.za

Thabang M. J. Nthebe
School of Mathematics, University of the Witwatersrand,
WITS, 2050, South Africa
JohannesN@nedbank.co.za

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Let f be a real valued function with the domain $\mathrm{dom}(f)$ in some vector space X and let $\mathfrak{C}$ be the collection of convex subsets of X. The following two questions are investigated; 1. Do there exist maximal convex restrictions g of f with $\mathrm{dom}(g) \in \mathfrak{C}$? 2. If f is convex with $\mathrm{dom}(f) \in \mathfrak{C}$, do there exist maximal convex extension g of f with $\mathrm{dom}(g) \in \mathfrak{C}$? We will show that the answer to both questions is positive under a certain condition on $\mathfrak{C}$.

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1. Introduction

Convex functions play an important role in many fields of mathematics and its applications, e. g., convex and global optimization, differential inclusion theory, and mathematical economic and risk analysis. In this paper, we will address two questions associated with convex functions. Firstly, given an arbitrary real-valued function, find and characterize maximal convex restrictions. Secondly, what can be said about extensions and maximal extensions of convex functions?

The first question arises when one considers non-convex functions but wants to take advantage of properties of convex functions and therefore restricts the function to a domain on which it will be convex. One might also want to restrict the domains to certain classes of convex sets, such as closed or open convex sets. It turns out that an additional condition is needed to guarantee the existence of maximal convex restrictions under such general conditions.

Conditions for extensions of convex functions have been discussed by few other authors, and below we make mention of the few in different context and also identify the difference between previous publications and our results on such conditions. In [1], the authors consider set-valued functions $V : [t_0, \theta] \to \mathrm{comp}(\mathbb{R}^n)$, the collection of non-

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empty compact subsets in $\mathbb{R}^n$, whose graphs are convex. The authors give necessary and sufficent conditions for the existence of extensions of V to larger compact intervals in terms of upper and lower derivatives.

On the other hand, we will be concerned with convex functions in the usual sense, i. e., functions whose epigraphs are convex. This corresponds to convex set-valued functions mapping to semi-infinite intervals, and consequently, the results in [1] are disjoint to the results on convex extensions obtained in this paper.

In [4] an explicit necessary and sufficient condition is given such that a real-valued function from the boundary of a nonempty bounded open convex set $\Omega \subset \mathbb{R}^n$ has a Lipschitz continuous extension to a function on $\mathbb{R}^n$.

In [5] the authors consider a condition which is necessary and sufficient for the construction of a convex extension. This construction is related to global optimisation theory in that, the condition asserts and imposes upper bounds for the images of the constructible convex extension.

Another extension theorem was obtained in [3, Theorem 1], in which functions on non-convex domains in a vector space V are considered and the existence of convex extensions to all of V are discussed. Here the definition of convexity allows convex functions to take the values $\pm\infty$. The authors proclaim that it is worthwhile to define convexity of a function also on non-convex domains since these may occur in economics in a natural way, in particular in risk aversion problems.

Our main results in Section 3 prove that the existence of a maximal epigraph extension happens under certain conditions, namely, CUP and the pseudo-arbsorbing condition. For convex functions defined on intervals in $\mathbb{R}$, this maximal epigraph extension is unique.

Henceforth X denotes a real vector space, $f : \operatorname{dom} f \subseteq X \to \mathbb{R}$ denotes a real valued function with non-empty domain in X, and

$$\mathfrak{C} \subseteq \{C \subseteq A : \emptyset \neq C \text{ is convex}\}$$

denotes a non-empty class of convex non-empty subsets of A, for some fixed non-empty subset A in X.

Definition 1.1. A continuous real valued function $f : A \subseteq X \to \mathbb{R}$ is *convex* if A is convex and for any $x, y \in A$ and $\lambda \in [0, 1]$ we have $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$.

Definition 1.2. Let A be a non-empty subset of a vector space X.

1. A real function $f : A \subseteq X \to \mathbb{R}$ is $\mathfrak{C}$-*convex* if there is $C \in \mathfrak{C}$ such that $f|_C$ is a convex restriction of f.
2. Let $f : A \subseteq X \to \mathbb{R}$ be a $\mathfrak{C}$-convex function. Then a non-empty subset $M \in \mathfrak{C}$ satisfying
   (a) f is convex on M, and
   (b) there exists no convex set $P \in \mathfrak{C}$ such that $M \subset P \subseteq A$ and f is convex on P,

   is called *a $\mathfrak{C}$-maximal domain of convexity* ($\mathfrak{C}$-MDC) for f.

If $\mathfrak{C}$ contains a singleton, then every function $f : A \subseteq X \to \mathbb{R}$ is $\mathfrak{C}$-convex. Our objective is to discuss non-trivial $\mathfrak{C}$-convex functions, hence we make the following assumption:

Assumption 1.3. $\emptyset \neq \mathfrak{C} \subseteq \{C \subseteq A : \emptyset \neq C \text{ is convex and infinite}\} = \mathfrak{C}_0$.

There is one additional and vital condition we impose on $\mathfrak{C}$ and we define it as follows:

Definition 1.4. If for any chain $\mathfrak{B} \subset \mathfrak{C}$ we have $\bigcup_{B\in\mathfrak{B}} B \in \mathfrak{C}$, then we say $\mathfrak{C}$ satisfies the *Chain Union Property* (CUP).

Note that $\mathfrak{C}$ satisfying the CUP means that $\mathfrak{C}$ is chain-complete with every least upper bound of a chain being the union of the sets in the chain.

Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex. Then we denote by $\mathfrak{C}_f = \{F \in \mathfrak{C} : f|_F \text{ is convex}\}$ the collection of $\mathfrak{C}$-domains in X on which f is convex. Clearly $\mathfrak{C}_f$ is a subset of $\mathfrak{C}$, and since f is $\mathfrak{C}$-convex, $\mathfrak{C}_f \neq \emptyset$.

Since $\mathfrak{C}_0 \neq \emptyset$, A contains at least one non-trivial convex set C. It will become clear that the CUP for the collection $\mathfrak{C}_f$ is important in order to obtain the maximal elements in $\mathfrak{C}_f$.

2. Maximal Domains of Convexity

In this section we consider real not necessarily convex functions on subsets of a vector space X and discuss their maximal convex restrictions.

For any $x, y \in A$, denote by $[x, y]$ the line segment connecting x and y, that is, $[x, y] = \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$. Recall that a *chain* $\mathfrak{B}$ in $\mathfrak{C}$ is a family of sets in $\mathfrak{C}$ such that $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ whenever $B_1, B_2 \in \mathfrak{B}$. Furthermore, an element $F \in \mathfrak{C}$ is called *an upper bound* for $\mathfrak{B}$ if $B \subseteq F$ for each $B \in \mathfrak{B}$. Henceforth we shall denote by $\max\{B_1, B_2\}$ the larger of the two elements $B_1, B_2 \in \mathfrak{B}$.

Note that for every chain $\mathfrak{B}$ in $\mathfrak{C}$ it follows that $D = \bigcup_{B\in\mathfrak{B}} B$ is also a convex subset of A. The CUP therefore requires that the convex set D must belong to $\mathfrak{C}$. This condition is satisfied for a large class of collections $\mathfrak{C}$. As the most important example we have:

Example 2.1.

1) The collection $\mathfrak{C}_0 = \{C \subseteq A : C \text{ convex and infinite}\}$ satisfies the CUP.
2) Let X be a Banach space, and consider the collection $\mathfrak{C}' = \{C \in \mathfrak{C}_0 : C \text{ open}\}$. Clearly $\mathfrak{C}' \neq \emptyset$ if and only if $\operatorname{int}(A) \neq \emptyset$, and then $\mathfrak{C}'$ satisfies the CUP.

Remark 2.2. Henceforth we consider $\mathfrak{C}'$ only if $\operatorname{int}(A)$ is non-empty, and thus $\mathfrak{C}'$ satisfies Assumption 1.3.

Not all collections $\mathfrak{C}$ satisfy the CUP, hence we subsequently give examples of those collections $\mathfrak{C}$ with, and those lacking, this property:

Example 2.3. Let $A = \mathbb{R}$, $\mathfrak{C} = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$ be a collection of convex subsets in $\mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$.

(a) Let $B_n = (a_n, b_n) \in \mathfrak{C}$ with $a_n, b_n \in \mathbb{Q}$ $(n \in \mathbb{N})$ be such that $a_n \searrow \pi$ and $b_n \nearrow 2\pi$. Consequently $\mathfrak{B} = \{B_n : n \in \mathbb{N}\}$ is a chain in $\mathfrak{C}$. It follows that

$\bigcup_{n\in\mathbb{N}} B_n = (\pi, 2\pi)$, and that $(\pi, 2\pi) \notin \mathfrak{C}$ since $\pi \notin \mathbb{Q}$. Hence $\mathfrak{C}$ does not satisfy the CUP.

(b) Observe that $\mathfrak{C}_0$ is the collection of non-trivial intervals I in $\mathbb{R}$. It follows that $[(2k-1)\pi, 2k\pi], k \in \mathbb{Z}$ are maximal elements in $\mathfrak{C}_{0,f}$.

(c) $\mathfrak{C}_f$ contains no maximal element and fails the CUP.

(d) Let $\mathfrak{C}_{\text{int}}$ be the collection of open intervals. Then $A = \mathbb{R}$ satisfies Assumption 1.3, and $\mathfrak{C}_{\text{int}}$ has the CUP. Moreover $((2k-1)\pi, 2k\pi), k \in \mathbb{Z}$ are maximal elements in $\mathfrak{C}_{\text{int},f}$.

The above example shows that, in general, $\mathfrak{C}$-maximal domains of convexity may or may not exist. Below we will show that the CUP will guarantee the existence of $\mathfrak{C}$-MDC.

Proposition 2.4. *For any $\mathfrak{C}$-convex function $f : A \subseteq X \to \mathbb{R}$, if $\mathfrak{C}$ satisfies the* CUP*, so does $\mathfrak{C}_f$.*

Proof. Let $\mathfrak{B}$ be a chain in $\mathfrak{C}_f$. Then, since $\mathfrak{C}$ satisfies the CUP, $D = \bigcup_{B\in\mathfrak{B}} B \in \mathfrak{C}$. Since $f|_B$ is convex for each $B \in \mathfrak{B}$, also $f|_D$ is convex and therefore $D \in \mathfrak{C}_f$. It follows that $\mathfrak{C}_f$ satisfies the CUP. □

Proposition 2.5. *If $\mathfrak{C}$ satisfies the* CUP*, then $\mathfrak{C}$ contains a maximal element.*

Proof. Let $\mathfrak{B}$ be a chain in $\mathfrak{C}$. It follows that $\bigcup_{B\in\mathfrak{B}} B$ is an element of $\mathfrak{C}$, and thus each chain $\mathfrak{B}$ in $\mathfrak{C}$ has an upper bound $\bigcup_{B\in\mathfrak{B}} B$. Appealing to Zorn's lemma, it follows that $\mathfrak{C}$ contains a maximal element. □

Propositions 2.4 and 2.5 immediately lead to

Theorem 2.6. *Assume $\mathfrak{C}$ satisfies the* CUP *and let $f : A \subseteq X \to \mathbb{R}$ be a $\mathfrak{C}$-convex real function. Then $\mathfrak{C}_f$ contains a maximal element, that is, f has a $\mathfrak{C}$-MDC.*

If $\mathfrak{C}$ satisfies the CUP and $C \in \mathfrak{C}$, then also the set $\mathfrak{C}_C = \{B \in \mathfrak{C} : C \subseteq B\}$ satisfies the CUP. Clearly, any maximal element in $\mathfrak{C}_C$ is also a maximal element in $\mathfrak{C}$. Hemce the following two results:

Theorem 2.7. *Assume $\mathfrak{C}$ satisfies the* CUP*. Then the following hold.*

(*a*) *If C is an element of $\mathfrak{C}$, then there exists a maximal element, say M, in $\mathfrak{C}$ such that $C \subseteq M$.*

(*b*) *If C is an element of $\mathfrak{C}_f$ for some $\mathfrak{C}$-convex function $f : A \subseteq X \to \mathbb{R}$, then there exists a maximal element, say M, in $\mathfrak{C}_f$ such that $C \subseteq M$.*

Remark 2.8. Let $C \in \mathfrak{C}_f$ be fixed for some $\mathfrak{C}$-convex real function $f : A \subseteq X \to \mathbb{R}$. Then there may exist a maximal element M in $\mathfrak{C}_f$ such that C may not be contained in M.

For instance, if $\mathfrak{C} = \{I \subseteq \mathbb{R} : I \text{ an interval}\}$, $f(x) = \sin x$, and we choose $C \subseteq (\pi, 2\pi)$, then $f|_C$ is convex, $C \subset M = [\pi, 2\pi]$ and M is maximal in $\mathfrak{C}_f$. Furthermore, $M_0 = [-\pi, 0]$ is also a maximal element in $\mathfrak{C}_f$ with $C \nsubseteq M_0$.

On the other hand if $f : A \subseteq X \to \mathbb{R}$ is convex and $A \in \mathfrak{C}$, A is a unique maximal domain of f in $\mathfrak{C}_f$ and f is also $\mathfrak{C}$-convex. Observe that $\bigcup_{C\in\mathfrak{C}_f} C = A \in \mathfrak{C}_f$.

Suppose conversely that f is $\mathfrak{C}$-convex and has a unique maximal domain of convexity M. Then every $C \in \mathfrak{C}_f$ is contained in M, hence $M = \bigcup_{C\in\mathfrak{C}_f} C$ if $\mathfrak{C}$ satisfies the CUP.

We therefore have:

Proposition 2.9. *Suppose $\mathfrak{C}$ satisfies the* CUP.

(a) *$\mathfrak{C}$ has a unique maximal element if and only if $\bigcup_{C\in\mathfrak{C}} C \in \mathfrak{C}$.*
(b) *If $f : A \subseteq X \to \mathbb{R}$ is $\mathfrak{C}$-convex, then $\mathfrak{C}_f$ has a unique maximal element if and only if $\bigcup_{C\in\mathfrak{C}_f} C \in \mathfrak{C}_f$.*

3. Convex Extensions

In this section we consider convex functions on subsets of a vector space X and discuss their maximal convex extensions with respect to two different orderings on the set of all convex extensions.

Definition 3.1. Let g, h be real functions with $\mathrm{dom}(g), \mathrm{dom}(h) \subseteq X$. Then g is *an extension* of h, denoted by $g \succeq_{\text{ext}} h$, if $\mathrm{dom}(h) \subseteq \mathrm{dom}(g)$ and $g|_{\mathrm{dom}(h)} = h$. Moreover, if $f : Y \subseteq X \to \mathbb{R}$ is convex, then the set

$$\mathfrak{X}_f = \{g : \mathrm{dom}(g) \subseteq X \to \mathbb{R}, g \succeq_{\text{ext}} f, g \text{ convex}\}$$

is the collection of convex extensions of f.

Obviously, $\mathfrak{X}_f \neq \emptyset$ since $f \in \mathfrak{X}_f$. Our aim is to prove the existence of maximal extensions for any given convex f. If $\mathrm{dom}(f) \in \mathfrak{C}$, we also consider $\mathfrak{X}_{f,\mathfrak{C}} = \{g \in \mathfrak{X}_f : \mathrm{dom}(g) \in \mathfrak{C}\}$.

Theorem 3.2. *Let $\mathfrak{C}$ satisfy the* CUP. *For any convex function $f : Y \subseteq X \to \mathbb{R}$ with* $\mathrm{dom}(f) \in \mathfrak{C}$, *there exists a maximal convex extension in $\mathfrak{X}_{f,\mathfrak{C}}$ with respect to the ordering $\succeq_{ext}$.*

Proof. Let $\mathfrak{B}$ be a chain in $\mathfrak{X}_{f,\mathfrak{C}}$. Hence for each pair $g, h \in \mathfrak{B}$ such that $h \succeq_{\text{ext}} g$ we have $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \subseteq \mathrm{dom}(h)$. It follows that for each chain $\mathfrak{B}$ in $\mathfrak{X}_f$ there exists a chain $\mathrm{Dom}(\mathfrak{B})$ of domains of functions in $\mathfrak{B}$. By the definition of $\mathfrak{X}_{f,\mathfrak{C}}$ we have $B \in \mathfrak{C}$ for all $B \in \mathrm{Dom}(\mathfrak{B})$, and thus $\mathrm{Dom}(\mathfrak{B})$ is a chain in $\mathfrak{C}$. Moreover $\bigcup_{B\in\mathrm{Dom}(\mathfrak{X}_f)} B = D \in \mathfrak{C}$ by the CUP, and is also an upper bound for $\mathrm{Dom}(\mathfrak{B})$ in X.

We define $g^* : D \to \mathbb{R}$ as follows:
Let $x \in D$. Then there is $g \in \mathfrak{B}$ such that $x \in \mathrm{dom}(g)$ and we let $g^*(x) = g(x)$. Clearly, since $\mathfrak{B}$ is a chain, g^* is well defined and convex, and $g^* \succeq_{\text{ext}} g \succeq_{\text{ext}} f$ for all $g \in \mathfrak{B}$. Therefore $g^* \in \mathfrak{X}_{f,\mathfrak{C}}$ is an upper bound of $\mathfrak{B}$.

Since $\mathfrak{X}_{f,\mathfrak{C}}$ is partially ordered and each chain $\mathfrak{B}$ in $\mathfrak{X}_{f,\mathfrak{C}}$ has an upper bound g^* in $\mathfrak{X}_{f,\mathfrak{C}}$, it follows from Zorn's lemma that $\mathfrak{X}_{f,\mathfrak{C}}$ has a maximal element. □

Convex functions and convex epigraphs coincide and since we have been discussing extensions of convex function, we subsequently discuss their convex epigraphs and how to extend them and still preserve their convexity. Recall that for a function

$f : A \subseteq X \to \mathbb{R}$ its epigraph is defined and denoted by

$$\mathrm{epi}(f) = \{(x, \lambda) \in A \times \mathbb{R} : f(x) \leq \lambda\}.$$

It is well known and easily seen that the function $f : A \subseteq X \to \mathbb{R}$ is convex if and only if its epigraph is a convex subset of $X \times \mathbb{R}$.

Considering the collection of convex extensions of f as in Definition 3.1 above, we define the epigraph ordering $\sqsupseteq_{\mathrm{epi}}$ on $\mathfrak{X}_f$ as follows:

Definition 3.3. For any $g, h \in \mathfrak{X}_{f,\mathfrak{C}}$, $g \sqsupseteq_{\mathrm{epi}} h$ if and only if $\mathrm{epi}(h) \subseteq \mathrm{epi}(g)$. Equivalently, $\mathrm{dom}(h) \subseteq \mathrm{dom}(g)$ and $g(x) \leq h(x)$, $x \in \mathrm{dom}(h)$.

Clearly $\mathfrak{X}_{f,\mathfrak{C}}$ is partially ordered by $\sqsupseteq_{\mathrm{epi}}$, and $g \sqsupseteq_{\mathrm{epi}} f$ for all $g \in \mathfrak{X}_{f,\mathfrak{C}}$. We denote by $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}}) = \{\mathrm{epi}(g) : g \in \mathfrak{X}_{f,\mathfrak{C}}\}$ the collection of convex epigraphs $\mathrm{epi}(g)$ containing (or which are extensions of) $\mathrm{epi}(f)$.

There is one additional condition we deem important in our subsequent discussion, and that is stated as follows:

Definition 3.4. A subset A of X is said to be a *pseudo-absorbing* subset of X if for each $x \in X$ there exist $a, b \in A$ and $\alpha \in \mathbb{R}$ such that $x = a + \alpha(b - a)$.

Clearly, every absorbing subset of X is pseudo-absorbing.

Theorem 3.5. *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex and suppose that $\mathfrak{C}$ satisfy the* CUP *and that A is pseudo-absorbing in X. Then there exists a maximal epigraph extension of* $\mathrm{epi}(f)$ *in $X \times \mathbb{R}$, equivalently,* $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ *has a maximal element.*

Proof. Let $\mathfrak{B}$ be any chain in $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$. Then $\mathrm{epi}(f) \subseteq \widehat{B} = \bigcup_{B \in \mathfrak{B}} B \in X \times \mathbb{R}$. Moreover, for each $B \in \mathfrak{B}$ there exists $g \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $B = \mathrm{epi}(g)$, $g \sqsupseteq_{\mathrm{epi}} f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \in \mathfrak{C}$. Then $\mathfrak{K} = \{\mathrm{dom}(g) : \mathrm{epi}(g) \in \mathfrak{B}\}$ is a chain in $\mathfrak{C}$, and it follows that $D = \bigcup_{K \in \mathfrak{K}} K \in \mathfrak{C}$ since $\mathfrak{C}$ satisfies the CUP.

Obviously, $\widehat{B}$ is a convex subset of $X \times \mathbb{R}$. We are going to show that it is contained in the epigraph of some real convex function g^* in $\mathfrak{X}_{f,\mathfrak{C}}$. Indeed, we define the function $g^* : D \to \mathbb{R}$ as

$$g^*(x) = \inf\{g(x) \in \mathbb{R} : \mathrm{epi}(g) \in \mathfrak{B},\ x \in \mathrm{dom}(g)\} \quad (x \in D).$$

In order to show that $g^*(x) \in \mathbb{R}$ for all $x \in D$, we first observe that $g^*(x) = f(x)$ if $x \in \mathrm{dom}(f)$. Now fix $x \in D \setminus \mathrm{dom}(f)$. Appealing to the pseudo-absorbing property of A, it follows that there exist $a, b \in A = \mathrm{dom}(f)$ and $\alpha \in \mathbb{R}$ such that $x = a + \alpha(b - a)$. Since A is convex, $x \notin [a, b]$, and therefore we may assume without loss of generality that $a \in [x, b]$. Thus there exists $\lambda \in (0, 1)$ such that $a = \lambda x + (1 - \lambda)b$ and hence

$$g(a) = g(\lambda x + (1 - \lambda)b) \leq \lambda g(x) + (1 - \lambda)g(b)$$

for each $g \in \mathfrak{X}_{f,\mathfrak{C}}$ with $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$, which leads to

$$g(x) \geq \frac{1}{\lambda}(g(a) - (1 - \lambda)g(b)) = \frac{1}{\lambda}(f(a) - (1 - \lambda)f(b)).$$

It follows that $g^*(x) \geq \frac{1}{\lambda}(g(a) - (1-\lambda)g(b))$ for each $g \in \mathfrak{X}_{f,\mathfrak{C}}$ with $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$. Consequently, $g^*(x) > -\infty$ for each $x \in D$.

Now we are going to show that g^* is convex. Let $x, y \in D$ and $\lambda \in [0,1]$. By definition of g^*, for any $\varepsilon > 0$ there exist $g_1, g_2 \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $\mathrm{epi}(g_1), \mathrm{epi}(g_2) \in \mathfrak{B}$, $x \in \mathrm{dom}(g_1)$, $y \in \mathrm{dom}(g_2)$, $g_1(x) \leq g^*(x) + \varepsilon$ and $g_2(y) \leq g^*(y) + \varepsilon$. Since $\mathfrak{B}$ is a chain, $g_1 \sqsupseteq_{\mathrm{epi}} g_2$ or $g_2 \sqsupseteq_{\mathrm{epi}} g_1$, and we denote by g the maximum of g_1 and g_2. Then $g(x) \leq g_1(x)$ and $g(y) \leq g_2(y)$. Therefore,

$$\begin{aligned} g^*(\lambda x + (1-\lambda)y) &\leq g(\lambda x + (1-\lambda)y) \\ &\leq \lambda g(x) + (1-\lambda)g(y) \\ &\leq \lambda(g^*(x) + \varepsilon) + (1-\lambda)(g^*(y) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\begin{aligned} g^*(\lambda x + (1-\lambda)y) &\leq \lim_{\varepsilon \to 0} \lambda(g^*(x) + \varepsilon) + (1-\lambda)(g^*(y) + \varepsilon) \\ &= \lambda g^*(x) + (1-\lambda)g^*(y). \end{aligned}$$

Thus we have shown that g^* is convex. Together with $D \in \mathfrak{C}$ and $g^*|_{\mathrm{dom}(f)} = f$ this leads to $g^* \in \mathfrak{X}_{f,\mathfrak{C}}$ and hence $\mathrm{epi}(g^*) \in \mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$.

Finally, we show that $\mathrm{epi}(g^*)$ is an upper bound of $\mathfrak{B}$. To this end take $(x, \alpha) \in \widehat{B}$. Then there exists $g \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$. It follows that $\alpha \geq g(x) \geq g^*(x)$ and hence $(x, \alpha) \in \mathrm{epi}(g^*)$. Consequently $\widehat{B} \subseteq \mathrm{epi}(g^*) \in \mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$, and hence $\mathrm{epi}(g^*)$ is an upper bound of $\mathfrak{B}$ in $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$. By Zorn's lemma, $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ has a maximal element. □

To conclude this paper, we will briefly discuss the uniqueness of the maximal elements in $\mathfrak{X}_{f,\mathfrak{C}}$.

We consider convex functions on intervals on the real line and first recall

Theorem 3.6 ([2, Theorem 1.3.3, p. 21]). *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be convex. Then f is continuous on the interior* $\mathrm{int}(A)$ *of A and has finite left and right derivatives at each point of* $\mathrm{int}(A)$. *Moreover, $x < y$ in* $\mathrm{int}(A)$ *implies*

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y).$$

Consequently $f'(x) \leq f'(y)$ provided they exist.

Conversely, we have

Proposition 3.7. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function on an open interval A, and assume that f has finite left and right derivatives at each point of A satisfying*

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$$

for all $x < y$ in A. Then f is convex.

Although, in general, the existence proof for maximal elements is not constructive, for $\mathrm{dom}(f) \subset \mathbb{R}$ it is mathematical folklore, and relatively straightforward to show, that there is a unique maximal element in $\mathrm{Epi}(\mathfrak{X}_f)$, that is

Proposition 3.8. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be convex and $A = \mathrm{dom}(f)$ be non-trivial. Then $\mathrm{Epi}(\mathfrak{X}_f)$ has a unique maximal element.*

This maximal extension f^* can be constructed in two steps.

Step 1. Let a and b be the left and right endpoints of $\mathrm{dom}(f)$, respectively. Here $a = -\infty$ and $b = \infty$ are possible. If $\alpha \in \{a, b\} \cap \mathbb{R}$, $\lim_{x\to\alpha^\pm} f(x)$ denotes $\lim_{x\to a^+} f(x)$ if $\alpha = a$ and $\lim_{x\to b^-} f(x)$ if $\alpha = b$. Putting

$$\mathfrak{L} = \Big\{\alpha \in \{a, b\} : \alpha \in \mathbb{R} \setminus \mathrm{dom}(f),\ \lim_{x\to\alpha^\pm} f(x) \text{ exists}\Big\},$$

define the function $\hat{f}$ on $\mathrm{dom}(f) \cup \mathfrak{L}$ by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f), \\ \lim_{t\to x^\pm} f(t) & \text{if } x \in \mathfrak{L}. \end{cases}$$

Step 2. The function f^* is defined as follows:

$$f^*(x) = \begin{cases} \hat{f}(x) & \text{if } x \in \mathrm{dom}(\hat{f}) \\ T_{\hat{f},a}(x) & \text{if } x < a,\ a \in \mathrm{dom}(\hat{f}),\ (\hat{f})'_+(a) \text{ exists}, \\ T_{\hat{f},b}(x) & \text{if } x > b,\ b \in \mathrm{dom}(\hat{f}),\ (\hat{f})'_-(b) \text{ exists}, \end{cases}$$

where $T_{\hat{f},a}(x) = \hat{f}(a)+(x-a)(\hat{f})'_+(a)$ and $T_{\hat{f},b}(x) = \hat{f}(b)+(x-b)(\hat{f})'_-(b)$ are the tangent lines to $\hat{f}$ at the endpoints of $\mathrm{dom}(\hat{f})$. With the aid of Theorem 3.6 and Proposition 3.7 it can now be shown that $\mathrm{epi}(f^*)$ is the unique maximal element in $\mathrm{Epi}(\mathfrak{X}_f)$.

Remark 3.9. 1. Even for convex functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$ it depends on $\mathfrak{C}$ if $\mathfrak{X}_{f,\mathfrak{C}}$ has a unique maximal element. For example, if $\mathfrak{C}$ is the collection of all finite intervals (a, b) with $0 < b - a \le 10$ and $f(x) = x^2$ with $\mathrm{dom}(f) = (0, 1)$, then $\mathfrak{C}$ satisfies the CUP, and with f^* being the maximal extension of f in $\mathfrak{X}_f$, every function $f^*|_{(-c,10-c)}$ with $0 \le c \le 9$ is a maximal element in $\mathfrak{X}_{f,\mathfrak{C}}$ with respect to $\succeq$.

2. We are not aware of any uniqueness result if $f : A \subseteq X \to \mathbb{R}$ is convex with $\dim X \ge 2$, and it may be the case that there are f such that there is more than one maximal element in $\mathrm{Epi}(\mathfrak{X}_f)$.

References

[1] Kh. G. Guseinov, S. A. Duzce, O. Ozer: Convex extensions of the convex set valued maps, J. Math. Anal. Appl. 314 (2006) 672–688.

[2] C. Niculescu, L.-E. Persson: Convex Functions and Their Applications, Springer, New York (2006).

[3] H. J. M. Peters, P. P. Wakker: Convex functions on non-convex domains, Econ. Lett. 22 (1987) 251–254.

[4] W. Rzymowski: Convex extension preserving Lipschitz constant, J. Math. Anal. Appl. 212 (1997) 30–37.

[5] M. Tawarmalani, N. V. Sahinidis: Convex extensions and envelopes of lower semi-continuous functions, Math. Program., Ser. A 93 (2002) 247–263.