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Maximality is Nothing But Continuity

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We show the connexions between the maximality of a monotone set-valued mapping and some continuity conditions on the domain of the mapping and on the mapping itself. We provide an original and simple proof of the famous characterization of maximal monotone mapping of Minty. Some words are said on monotone variational and generalized equilibrium problems.

Keywords: Monotonicity, maximal monotonicity, variational inequality problems, generalized equilibrium problems

1. Introduction and notation

An optimization problem is of the form

$$\text{Find } a \in C \text{ such that } \gamma(a,x) = g(x) - g(a) \geq 0 \quad \forall\, x \in C \tag{Ep}$$

where $g : \Omega \to \mathbb{R}$ and $C \subseteq \Omega \subseteq \mathbb{R}^n$. The problem is said to be convex if the constraint set C is convex and the objective function g is convex on Ω. In this case, the problem can be written as

$$\text{Find } a \in C \text{ s.t. } g'(a, x-a) = \sup[\langle a^*, x-a\rangle : a^* \in \partial g(a)] \geq 0 \quad \forall\, x \in C \tag{Cp}$$

where $\partial g(a)$ denotes the subdifferential of g at a and $g'(a,h)$ the directional derivative of g at a along the direction h. Convex optimization problems are particular cases of monotone variational inequality problems. Among the various formulations of these problems, the closest to (Cp) is as follows

$$\text{Find } a \in C \text{ s.t. } \gamma(a,x) = \sup[\langle a^*, x-a\rangle : a^* \in \Gamma(a)] \geq 0 \quad \forall\, x \in C \tag{Vip}$$

where $C \subseteq \Omega$ is convex and $\Gamma : \Omega \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is a monotone set-valued mapping.

If monotone variational inequality problems encompass convex optimization problems, there are real world monotone variational inequality problems which cannot be expressed as convex optimization problems. Maximality of Γ holds in these problems the

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role played by lower semicontinuity of the objective function g in convex optimization problems. It is defined in terms of inclusion: Γ is said to be maximal monotone on Ω if it is not contained in another monotone mapping on Ω. Maximality implies some forms of continuity of the domains (Sections 2 and 3) and of the mappings itself (Section 4 and 5). Conversely, we shall show that these continuities imply maximality. As a corollary, we provide a simple and original proof of the celebrated Minty's characterization of maximality.

In the past years a number of papers have appeared devoted to monotone equilibrium problems which are presumed to encompass monotone variational inequality problems. These problems are of the form

$$\text{Find } a \in C \text{ such that } \gamma(a,x) \geq 0 \quad \forall x \in C \tag{Eqp}$$

where the bifunction $\gamma : \Omega \times \Omega \to \mathbb{R}$ is such that, for all $x, y \in \Omega$, $\gamma(x,x) = 0$, $\gamma(x,.)$ is convex and $\gamma(x,y)+\gamma(y,x) \leq 0$. Usually an upper semicontinuity condition is required on the functions $\gamma(.,y)$. We shall show in Section 6 that the problem is nothing else than a classical monotone variational inequality problem and the equilibrium problem is nothing but a complicated formulation.

Now, let us precise the notation used throughout the paper. Given a subset $S \neq \emptyset$ of a topological vector space X, we denote by $\mathrm{co}\,(\mathrm{S})$ and $\overline{\mathrm{co}}\,(\mathrm{S})$ the convex hull and the closed convex hull of S respectively. Given $a \in S$, $\mathrm{lin}\,(\mathrm{S})$ denotes the linear space generated by $S - a$, this space does not depend on the point a chosen in S, and $\dim(S)$ denotes the dimension of $\mathrm{lin}\,(\mathrm{S})$. The affine space generated by S is $\mathrm{aff}\,(\mathrm{S}) = \mathrm{lin}\,(\mathrm{S}) + \mathrm{a}$.

Denote by X^* the topological dual space of X. With $G \subset X \times X^*$, we associate the set-valued mappings $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ and $\Gamma^{-1} : X^* \overset{\longrightarrow}{\longrightarrow} X$ such that

$$G = \{(x,x^*) : \, x^* \in \Gamma(x)\} = \{(x,x^*) : \, x \in \Gamma^{-1}(x^*)\}.$$

Thus G can be considered, up to a permutation of the two variables, as the graph of both mappings Γ and Γ^{-1}. The *domains* of Γ and Γ^{-1} are the projections of G on X and X^*, namely,

$$\mathrm{dom}\,(\Gamma) = \{x : \Gamma(x) \neq \emptyset\} = \mathrm{Pr}_{\mathrm{X}}(\mathrm{G}),$$
$$\mathrm{dom}\,(\Gamma^{-1}) = \{x^* : \Gamma^{-1}(x^*) \neq \emptyset\} = \mathrm{Pr}_{\mathrm{X}^*}(\mathrm{G}).$$

The *normal cone* at $x \in D$ to a convex set $D \subset X$ is

$$N_D(x) = \{x^* \in X^* : \, \langle x^*, y - x\rangle \leq 0 \; \forall y \in D\}.$$

In particular $N_D(x) = \{0\}$ when $x \in \mathrm{int}\,(\mathrm{D})$.

If for any sequence $\{(x_k, x_k^*)\}_k \subset G$ converging to (x,x^*) one has $(x,x^*) \in G$ then the mapping Γ is said to be *closed at* x and the mapping Γ^{-1} *closed at* x^*. If G is closed, $V \subset \mathrm{dom}\,(\Gamma)$ is open and there exists a compact set $K \supseteq \Gamma(V)$, then Γ is *usc on* V *in the sense of multivalued mappings*, i.e., for all $a \in V$ and all open $\Omega \supseteq \Gamma(a)$, there is a neighbourhood W of a such that $\Gamma(W) \subseteq \Omega$.

The *relative interior* $\mathrm{ri}\,(\mathrm{S})$ of S is the interior of S when considered as a subset of $\mathrm{aff}\,(\mathrm{S})$ and the *relative boundary* is the set of points in $\mathrm{cl}\,(\mathrm{S})$ which are not in $\mathrm{ri}\,(\mathrm{S})$. When X is finite dimensional and $S \subset X$ is convex and nonempty, then $\operatorname{aff}(S) = \operatorname{aff}(\operatorname{cl}(S))$, $\operatorname{ri}(S)$ is convex and nonempty, $\operatorname{cl}(S) = \operatorname{cl}(\operatorname{ri}(S))$ and $\operatorname{ri}(S) = \operatorname{ri}(\operatorname{cl}(S))$. Because some of the main results in this paper are based on this very important property, it is assumed throughout all the paper that $X = X^* = \mathbb{R}^n$.

By convention, $A + \emptyset = \emptyset$ for all $A \subseteq \mathbb{R}^n$.

2. Monotonicity: definitions

In this section, we briefly recall some definitions and basic facts on monotone mappings.

A set-valued mapping $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is said to be *monotone* if

$$\langle x^* - y^*, x - y\rangle \geq 0 \ \text{ for all } (x, x^*),\ (y, y^*) \in G, \tag{1}$$

where $G = \operatorname{graph}(\Gamma)$. Then, Γ^{-1} is monotone as well. Although the definition of monotonicity is often given for mappings, the true definition is on sets: a subset G of $\mathbb{R}^n \times \mathbb{R}^n$ is said to *monotone* if (1) holds.

G is said to be *maximal monotone* if G is monotone and if $G \subseteq F$ with F monotone implies $G = F$. Then the mappings Γ and Γ^{-1} associated with G are called maximal monotone. It is usual to associate with G the set

$$\widetilde{G} := \bigcap_{(y,y^*)\in G} \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \langle x^* - y^*, x - y\rangle \geq 0\}. \tag{2}$$

G is monotone if and only if $G \subseteq \widetilde{G}$. In this case,

$$\widetilde{G} = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n :\ G \cup \{(x, x^*)\} \text{ is monotone}\}. \tag{3}$$

From that one deduces that G is maximal monotone if and only if $G = \widetilde{G}$. In the line of this property, we say that a monotone mapping Γ is *maximal monotone at a* if $\Gamma(a) = \widetilde{\Gamma}(a)$ where $\widetilde{\Gamma} : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is the associated mapping with $\widetilde{G}$.

The mapping Γ is said to be *cyclically monotone* if for any finite ordered family $\{(x_i, x_i^*)\}_{i=0,1,\cdots,p} \subset G = \operatorname{graph}(\Gamma)$ with $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$, one has

$$\sum_{i=0}^{p} \langle x_i^*, x_{i+1}\rangle \leq \sum_{i=0}^{p} \langle x_i^*, x_i\rangle. \tag{4}$$

A cyclically monotone mapping is monotone. Setting $j = p + 1 - i$ in (4) one obtains

$$\sum_{j=0}^{p} \langle x_j, x_{j+1}^*\rangle \leq \sum_{j=0}^{p} \langle x_j, x_j^*\rangle.$$

Therefore, as for monotonicity, Γ is cyclically monotone if and only if Γ^{-1} is so. As for monotonicity, cyclic monotonicity is defined in terms of the set G as well as in terms of the associated mappings Γ and Γ^{-1}.

G is said to be *maximal cyclically monotone* if it is cyclically monotone and if $F \supseteq G$ with F cyclically monotone implies $G = F$. Then Γ and Γ^{-1} are said to be maximal cyclically monotone. In line with (3), we associate with G the set

$$\widehat{G} := \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : G \cup \{(x, x^*)\} \text{ is cyclically monotone}\}.$$

When G is cyclically monotone we also have

$$\widehat{G} = \bigcap_{J \in \mathcal{J}} \left\{ (x, x^*) : \langle x^*, a_1 \rangle + \langle a_p^*, x \rangle + \sum_{i=1}^{p-1} \langle a_i^*, a_{i+1} \rangle \leq \langle x^*, x \rangle + \sum_{i=1}^{p} \langle a_i^*, a_i \rangle \right\} \tag{5}$$

where $\mathcal{J}$ denotes the family of all ordered finite subsets $J = \{(a_i, a_i^*)\}_{i=1,\cdots,p}$ of G.

Hence one deduces that G is maximal cyclically monotone if and only if $G = \widehat{G}$. A cyclically monotone mapping Γ is said to be *maximal cyclically monotone at* a if $\Gamma(a) = \widehat{\Gamma}(a)$ where $\widehat{\Gamma} : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is associated with $\widehat{G}$.

Let us list some immediate additional properties of $\widetilde{G}$, $\widehat{G}$ and their associated mappings $\widetilde{\Gamma}$, $\widetilde{\Gamma}^{-1}$, $\widehat{\Gamma}$ and $\widehat{\Gamma}^{-1}$.

- $\widetilde{G}$ and $\widehat{G}$ are closed subsets of $\mathbb{R}^n \times \mathbb{R}^n$.
- $\widehat{G} \subseteq \widetilde{G}$.
- If $G \subset F$, then $\widetilde{F} \subseteq \widetilde{G}$ and $\widehat{F} \subseteq \widehat{G}$.
- For all $x \in \mathbb{R}^n$, $\widetilde{\Gamma}(x)$ and $\widehat{\Gamma}(x)$ are closed convex subsets of $\mathbb{R}^n$.
- For all $x^* \in \mathbb{R}^n$, $\widetilde{\Gamma}^{-1}(x^*)$ and $\widehat{\Gamma}^{-1}(x^*)$ are closed convex subsets of $\mathbb{R}^n$.

We observe that if G is cyclically monotone and maximal monotone, then $G \subseteq \widehat{G} \subseteq \widetilde{G} = G$ and therefore, G is also maximal cyclically monotone. One question: is a maximal cyclically monotone subset maximal monotone? Let us quote Rockafellar and Wets ([5], p. 546): "Every cyclically monotone mapping is monotone. It doesn't immediately follow that every maximal cyclically monotone mapping is maximal monotone". Their proof is the conjunction of two theorems, the first one says that a maximal cyclically monotone mapping is the subdifferential of a lower semicontinuous convex function and the second one says that the subdifferential of such a function is maximal monotone. The proof that we propose in Section 5 is a direct consequence of continuity properties of maximal (cyclically) monotone mappings.

3. Almost convex sets

Almost convex sets have been introduced by Himmelberg [2]. They are defined as follows

Definition 3.1. A subset C of a locally convex topological vector space E is said to be almost convex if for any neighbourhood U of $0 \in E$ and for any finite subset $\{x_1, \cdots, x_k\}$ of C there exist $z_1, \cdots, z_k \in C$ such that $z_i - x_i \in U$ for all i, and $\mathrm{co}(\{z_1, \cdots, z_k\}) \subset C$.

It is easily seen that convex sets are almost convex and closed almost convex sets are convex. It is a simple exercise to prove that the closure of an almost convex set is convex. But C is not necessarily almost convex when its closure is convex.

We shall give more speaking characterizations of almost convex sets in the case of finite dimensional spaces.

Proposition 3.2. *Let $\emptyset \neq C \subset \mathbb{R}^n$. Then, the three following conditions are equivalent*

1. *C is almost convex.*
2. $\mathrm{cl}\,(\mathrm{C})$ *is convex and* $\mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C})) = \mathrm{ri}\,(\mathrm{C})$.
3. $\mathrm{ri}\,(\mathrm{C})$ *is convex and* $\mathrm{cl}\,(\mathrm{C}) = \mathrm{cl}\,(\mathrm{ri}\,(\mathrm{C}))$.

Proof. i) Assume that *1.* holds. Then $\mathrm{cl}\,(\mathrm{C})$ is convex and therefore $\mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C}))$ is nonempty. Let some $a \in \mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C}))$ and $p = \dim(\mathrm{aff}\,(\mathrm{C}))$. There are $x_0, x_1, \cdots, x_p$ in $\mathrm{cl}\,(\mathrm{C})$ such that $a \in \mathrm{ri}\,(\mathrm{co}\,(\{x_0, x_1, \cdots, x_p\}))$. Next, there are $y_0, y_1, \cdots, y_p \in C$ such that $a \in \mathrm{ri}\,(\mathrm{co}\,(\{y_0, y_1, \cdots, y_p\}))$. Finally, since C is almost convex, there are $z_0, \cdots, z_p \in C$ such that $a \in \mathrm{ri}\,(\mathrm{co}\,(\{\mathrm{z}_0, \mathrm{z}_1, \cdots, \mathrm{z}_\mathrm{p}\}))$ and $\mathrm{co}\,(\{\mathrm{z}_0, \cdots, \mathrm{z}_\mathrm{p}\}) \subset \mathrm{C}$. It follows that $a \in \mathrm{ri}\,(\mathrm{C})$. We have proved that *1.* implies *2.*

ii) Assume that *2.* holds. Then $\mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C}))$ is convex and therefore $\mathrm{ri}\,(\mathrm{C})$ is convex. Also, $\mathrm{cl}\,(\mathrm{C}) = \mathrm{cl}\,(\mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C}))) = \mathrm{cl}\,(\mathrm{ri}\,(\mathrm{C}))$. We have proved that *2.* implies *3.*

iii) Finally, assume that *3.* holds. Fix some $a \in \mathrm{ri}\,(\mathrm{C})$. Let $x_1, \cdots, x_p \in C$ and $t \in (0,1)$. Take $z_i = x_i + t(a - x_i)$, $i = 1, \cdots, p$. Then $z_i \in \mathrm{ri}\,(\mathrm{C})$ and therefore the convex hull of points z_i is contained in $\mathrm{ri}\,(\mathrm{C})$ and therefore in C. We have proved that *3.* implies *1.* $\square$

Such characterizations do not exist in the infinite dimensional setting. Consider for instance the space E of real continuous functions on $[0,1]$. Next, consider the norm $\|f\| = \sup\,[|f(x)| : 0 \leq x \leq 1]$ and $C = C_1 \cup C_2$ where C_1 is the open unit ball and C_2 is the set of polynomial functions on $[0,1]$. Then C is almost convex, $\mathrm{cl}\,(\mathrm{C}) = \mathrm{E}$ and $\mathrm{int}\,(\mathrm{C}) = \mathrm{C}_1$. Clearly, *2.* and *3.* do not hold.

In $\mathbb{R}^n$ almost convexity can be considered as a kind of continuity on sets. Indeed, if C is almost convex then $\mathrm{ri}\,(\mathrm{co}\,(\mathrm{C})) = \mathrm{ri}\,(\mathrm{C})$ and therefore is convex. Moreover, if $x \in \mathrm{ri}\,(\mathrm{co}\,(\mathrm{C}))$ and $y \in C$, then $x + t(y - x) \in \mathrm{ri}\,(\mathrm{C}) \subseteq \mathrm{C}$ for all $t \in (0,1)$. There is no discontinuity in the portion of line connecting the two points x and y.

4. Continuity of the domains of monotone mappings

In this section, we show that the domains of maximal monotone and maximal cyclically monotone mappings are almost convex. Theorem 4.3 extend to cyclic monotone mappings results already known for monotone mappings (Theorem 4.2), the proof is more complex but is in the same spirit than the one given in [1] for monotone mappings. For the sake of comprehension and comparisons we give the two proofs. We begin with a lemma on monotone sets of finite cardinality.

Lemma 4.1. *Let $S = \{(x_i, x_i^*) : i = 1, 2, \cdots, p\}$ be a finite monotone subset of $\mathbb{R}^n \times \mathbb{R}^n$ and Σ be the mapping with graph S. Denote by C the convex hull of the points x_i. Let $\bar{x} \in \mathrm{int}\,(\mathrm{C})$. Then a neighbourhood $V \subset \mathrm{int}\,(\mathrm{C})$ of $\bar{x}$ and a compact K exist such that*

$$\emptyset \neq \widetilde{\Sigma}(x) = \bigcap_{i=0}^{p} \{x^* : \langle x^*, x - x_i \rangle \geq \langle x_i^*, x - x_i \rangle\} \subset K \quad \text{for all } x \in V.$$

It follows that $\mathrm{int}\,(\mathrm{C}) \subset \mathrm{dom}\,(\widetilde{\Sigma})$.

Proof. By assumption $C = \mathrm{co}\,(\mathrm{dom}\,(\Sigma))$. It results from (3) that

$$\widetilde{\Sigma}(x) = \bigcap_{i=0}^{p} \{x^* : \langle x^*, x - x_i\rangle \geq \langle x_i^*, x - x_i\rangle\} \text{ for all } x \in \mathbb{R}^n.$$

i) We first prove that $\widetilde{\Sigma}$ is bounded around $\bar{x}$. Otherwise, a sequence $\{(z_k, z_k^*)\} \subset \mathrm{graph}\,(\widetilde{\Sigma})$ exists such that $z_k \to \bar{x}$ and $\|z_k^*\| \to \infty$, when $k \to \infty$. Without loss of generality, we can assume that there is w^* such that the sequence $w_k^* = z_k^*/\|z_k^*\|$ converges to w^*. Then $\|w^*\| = 1$. Since $\langle z_k^* - x_i^*, z_k - x_i\rangle \geq 0$ it holds $\langle w^*, \bar{x} - x_i\rangle \geq 0$ for any i. This is not possible since $\bar{x} \in \mathrm{int}\,(\mathrm{C})$ and $w^* \neq 0$. It follows that there exist a neighbourhood $V \subset \mathrm{int}\,(\mathrm{C})$ of $\bar{x}$ and a compact K such that $\widetilde{\Sigma}(x) \subset K$ for all $x \in V$.

ii) Assume for contradiction that there is some $x \in V$ such that $\widetilde{\Sigma}(x) = \emptyset$. Let A be the $n \times p$ matrix whose columns are the vectors $(x - x_i)$ and a be the vector of $\mathbb{R}^p$ whose component a_i is $\langle x_i^*, x - x_i\rangle$. Then, there is no x^* such that $A^t x^* \geq a$. In view of the alternative theorems, a vector $u \geq 0$ exists such that $Au = 0$ and $\langle u, a\rangle > 0$. One can assume $\sum u_i = 1$. The equality $Au = 0$ implies $x = \sum u_i x_i$. On the other hand,

$$\langle u, a\rangle = -\sum_{i,j=0}^{p} u_i u_j \langle x_i^* - x_j^*, x_i - x_j\rangle.$$

Since Σ is monotone and $u \geq 0$, one obtains $\langle u, a\rangle \leq 0$. A contradiction with $\langle u, a\rangle > 0$. □

This result is now extended to the general case.

Theorem 4.2. *Assume that* $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ *is monotone and* $\bar{x} \in \mathrm{int}\,(\mathrm{co}\,(\mathrm{dom}\,(\Gamma)))$. *Then a neighbourhood* V *of* $\bar{x}$ *and a compact* K *exist such that*

$$\emptyset \neq \widetilde{\Gamma}(x) \subset K \quad \textit{for all } x \in V.$$

It follows that $\mathrm{dom}\,(\widetilde{\Gamma})$ *contains the interior of* $\mathrm{co}\,(\mathrm{dom}\,(\Gamma))$.

Proof. By assumption there exist a set $S = \{(x_i, x_i^*)\}_{\in I} \subset G = \mathrm{graph}\,(\Gamma)$ such that $\mathrm{card}\,(\mathrm{S}) < \infty$ and a point $\bar{x}$ in the interior of the convex hull of points x_i. Denote by Σ the mapping with graph S. Then Σ is monotone and $\widetilde{\Gamma}(x) \subset \widetilde{\Sigma}(x)$ for all x. Apply the lemma to Σ. Then there exist V and K such that

$$\widetilde{\Gamma}(x) \subset \widetilde{\Sigma}(x) = \bigcap_{i=0}^{p} \{x^* : \langle x^*, x - x_i\rangle \geq \langle x_i^*, x - x_i\rangle\} \subset K \quad \forall x \in V.$$

Next, assume for contradiction that there is some $x \in V$ with $\widetilde{\Gamma}(x) = \emptyset$. Then

$$\emptyset = \bigcap_{(y,y^*)\in G} \{x^* : \langle x^*, x - y\rangle \geq \langle y^*, x - y\rangle\},$$

$$\emptyset = \bigcap_{(y,y^*)\in G} \left[\{x^* : \langle x^*, x - y\rangle \geq \langle y^*, x - y\rangle\} \cap \widetilde{\Sigma}(x)\right].$$

The sets $\{x^* : \langle x^*, x-y\rangle \geq \langle y^*, x-y\rangle\} \cap \widetilde{\Sigma}(x)$ are compact. Thence a finite family $\{(x_i, x_i^*)\}_{i\in J} \subset G$ exists such that

$$\emptyset = \bigcap_{i\in J} \left[\{x^* : \langle x^*, x-x_i\rangle \geq \langle x_i^*, x-x_i\rangle\} \cap \widetilde{\Sigma}(x)\right].$$

Then,

$$\emptyset = \bigcap_{i\in I\cup J} \{x^* : \langle x^*, x-x_i\rangle \geq \langle x_i^*, x-x_i\rangle\}. \tag{6}$$

Denote by $T = \{(x_i, x_i^*)\}_{\in I\cup J}$ and by Θ the mapping with graph T. Then x is in the interior of the convex hull of the domain of Θ and, in view of the lemma, (6) cannot occur. The other claim follows. □

We shall prove that the same result holds for cyclic monotonicity. The proof is more complicated.

Theorem 4.3. *Assume that $\Gamma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a cyclically monotone mapping with graph G. Let $\bar{x}$ be in the interior of the convex hull of* dom (Γ)*. Then a neighbourhood V of $\bar{x}$ and a compact K exist such that*

$$\emptyset \neq \widehat{\Gamma}(x) \subset K \quad \textit{for all } x \in V.$$

It follows that dom $(\widetilde{\Gamma})$ *contains the interior of* co (dom (Γ)).

Proof. Let S and Σ be defined as in the proof of Theorem 4.2. Since $S \subset G$, S is cyclically monotone and $\widehat{\Gamma}(x) \subseteq \widehat{\Sigma}(x) \subseteq \widetilde{\Sigma}(x)$ for all $x \in \mathbb{R}^n$. We know by Lemma 4.1 that a neighbourhood $V \subset \mathrm{dom}\,(\widetilde{\Sigma})$ of $\bar{x}$ and a compact K exist such that

$$\widehat{\Gamma}(x) \subseteq \widehat{\Sigma}(x) \subseteq \widetilde{\Sigma}(x) = \bigcap_{i=0}^{p} B_i \subset K \quad \forall\, x \in V$$

where, for $i = 1, \cdots, p$,

$$B_i = \{x^* : \langle x^*, x-x_i\rangle \geq \langle x_i^*, x-x_i\rangle\}.$$

Next, assume for contradiction that there is some $x \in V$ such that $\widehat{\Sigma}(x) = \emptyset$. Then, in view of (5),

$$\emptyset = \cap_{A\in\mathcal{A}(x)} A,$$

where $A \in \mathcal{A}(x)$ if there exist (a_j, a_j^*), $j = 1, \cdots q$, such that

$$A = \left\{ x^* : \langle x^*, x-a_1\rangle \geq \sum_{j=1}^{q-1} \langle a_j^*, a_{j+1} - a_j\rangle + \langle a_q^*, x-a_q\rangle \right\}.$$

The sets $B_i = \{x^* : \langle x^*, x-x_i\rangle \geq \langle x_i^*, x-x_i\rangle\}$ belong to $\mathcal{A}(x)$ and their intersection is compact. Therefore, there exist a finite number of elements belonging to $\mathcal{A}(x)$ with empty intersection. Namely, there are positive integers r, q_j and $(a_{ij}, a_{ij}^*) \in G$ such that

$$\emptyset = \bigcap_{j=1}^{r} \left\{ x^* : \langle x^*, x - a_{1,j}\rangle \geq \sum_{i=1}^{q_j-1} \langle a_{i,j}^*, a_{i+1,j} - a_{i,j}\rangle + \langle a_{q_j,j}^*, x - a_{q_j,j}\rangle \right\}. \tag{7}$$

This means that there is no feasible solution for the linear system $B^t x^* \geq b$ where B is the $n \times r$ matrix whose column j is the vector $(x - a_{1,j})$ and b is the vector of $\mathbb{R}^r$ whose component j is $\sum_{i=1}^{q_j-1} \langle a_{i,j}^*, a_{i+1,j} - a_{i,j}\rangle + \langle a_{q_j,j}^*, x - a_{q_j,j}\rangle$.

In view of alternative theorems, a vector $u \geq 0$ exists such that

$$Bu = 0 \quad \text{and} \quad \langle u, b\rangle > 0. \tag{8}$$

Without loss of generality, we assume that $\sum u_j = 1$. It is clear that $Bu = 0$ implies $x = \sum_{k=1}^{r} u_k a_{1,k}$. Let us compute $\langle b, u\rangle$,

$$\begin{aligned} \langle b, u\rangle &= \sum_{j=1}^{r} u_j \left[\sum_{i=1}^{q_j-1} \langle a_{i,j}^*, a_{i+1,j} - a_{i,j}\rangle + \langle a_{q_j,j}^*, x - a_{q_j,j}\rangle \right], \\ \langle b, u\rangle &= \sum_{j,k=1}^{r} u_j u_k \alpha_{j,k} \end{aligned}$$

where

$$\alpha_{j,k} = \langle a_{q_j,j}^*, a_{1,k} - a_{q_j,j}\rangle + \sum_{i=1}^{q_j-1} \langle a_{i,j}^*, a_{i+1,j} - a_{i,j}\rangle.$$

Then,

$$\begin{aligned} 2\alpha_{j,k} = &\sum_{i=1}^{q_j-1} \langle a_{i,j}^*, a_{i+1,j} - a_{i,j}\rangle + \langle a_{q_j,j}^*, a_{1,k} - a_{q_j,j}\rangle + \cdots \\ &+ \sum_{l=1}^{q_k-1} \langle a_{l,k}^*, a_{l+1,k} - a_{l,k}\rangle + \langle a_{q_k,k}^*, a_{1,j} - a_{q_k,k}\rangle. \end{aligned}$$

One has $\alpha_{j,k} \leq 0$ because G is cyclically monotone and therefore $\langle b, u\rangle \leq 0$ in contradiction with (8). The result follows. $\square$

Next, we shall consider the case where $\operatorname{aff}(\operatorname{dom}(\Gamma)) \neq \mathbb{R}^n$. Let us denote by L the linear space $\operatorname{aff}(\operatorname{dom}(\Gamma)) - a$ where a is an arbitrary point of $\operatorname{dom}(\Gamma)$ and by $L^\perp$ the orthogonal space to L. Without loss of generality, we assume that the first n_1 components of $\mathbb{R}^n$ correspond to L and the last $n_2 = n - n_1$ to $L^\perp$. Let us define $\Sigma : \mathbb{R}^{n_1} \rightrightarrows \mathbb{R}^{n_1}$ by

$$x_1^* \in \Sigma(x_1) \Longleftrightarrow \exists\, x_2^* \text{ such that } (x_1^*, x_2^*) \in \Gamma(x_1 + a_1, a_2).$$

Γ is monotone if and only if Σ is so. It is easily seen that for all $x = (x_1, x_2) \in \mathbb{R}^n$

$$\widetilde{\Gamma}(x) = \widetilde{\Sigma}(x_1) \times \mathbb{R}^{n_2} \quad \text{and} \quad \widehat{\Gamma}(x) = \widehat{\Sigma}(x_1) \times \mathbb{R}^{n_2}.$$

Hence, one obtains generalizations of Theorems 4.2 and 4.3 to the case where aff (dom (Γ)) $\neq \mathbb{R}^n$. In particular, we have the following result on the domains of maximal monotone mappings and maximal cyclically monotone mappings.

Theorem 4.4 (Continuity of the domain). *If $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is maximal monotone or maximal cyclically monotone then its domain is almost convex. Furthermore for all $a \in \mathrm{ri}\,(\mathrm{dom}\,(\Gamma))$ there are a neighbourhood V of a and a compact $K \subset L$ such that one has*

$$\emptyset \neq \Gamma(x) = \Gamma(x) + L^{\perp} \subset K + L^{\perp} \quad \forall\, x \in V \cap (a + L)$$

where $L = \mathrm{lin}\,(\mathrm{dom}\,(\Gamma))$ and $L^{\perp}$ is the orthogonal space to L.

One sees that it is enough to study mappings with domains having the interior of their convex hull nonempty.

Let us relate the results of this section with some already known results in the literature.

i) First consider the case where Γ is maximal monotone, cyclically monotone and $\dim(\mathrm{dom}\,(\Gamma)) = \mathrm{n}$. Then the result of Theorem 4.4 is well known. The proof consists to says that there exists a convex function f such that Γ coincides with the subdifferential of f. Hence the domain of $\Gamma = \partial f$ is almost convex and Γ is locally bounded on the interior of its domain. Our proof is direct, it does not needs to make appeal to the function f.

ii) Theorem 26.2 in [6] says that if Γ is monotone and a belongs to interior of its domain, then Γ is locally bounded in a neighbourhood of a. Our Theorem 4.2 deals with $\widetilde{\Gamma}$ instead of Γ and adds a nonvacuity result. Also the proof in [6] makes appeal to the conjugate of a function obtained by fitzpatrication. Our proof is more direct and simple.

5. Continuity of monotone mappings

If Γ is maximal monotone (maximal cyclically monotone), then its graph G coincides with $\widetilde{G}$ ($\widehat{G}$) and therefore is closed. It follows that Γ is closed at any x, $\Gamma(x)$ is closed and convex for any x. If, in addition, $x \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$, then Γ is locally bounded in a neighbourhood of x and therefore upper semi-continuous at this point. The transposed results hold for Γ^{-1}.

Upper semi-continuity is a weak form of continuity for set-valued mappings as shown by the following example: $\Gamma : \mathbb{R} \overset{\longrightarrow}{\longrightarrow} \mathbb{R}$ is defined by $\Gamma(x) = \{0\}$ if $x \neq 0$ and $\Gamma(0) = [-1, 1]$. The graph of Γ is closed, Γ is usc at any x. The point $1 \in \Gamma(0)$ cannot be recovered from the knowledge of the values of $\Gamma(x)$ at points $x \neq 0$ in a neighbourhood of 0.

Maximal monotone (maximal cyclically monotone) mappings enjoy a very strong continuity property that we shall describe.

With $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, let us associate the set-valued mappings $\Gamma_l(\cdot, S)$ and $\Gamma_c(\cdot, S)$ defined as follows: for $x \in \mathbb{R}^n$

$$x^* \in \Gamma_l(x, S) \Longleftrightarrow \left\{ \begin{array}{l} \exists \text{ a sequence } \{(x_k, x_k^*)\}_k \subset [\mathrm{graph}\,(\Gamma)] \cap [S \times \mathbb{R}^n], \\ x_k \neq x \;\forall\, k, \quad \text{converging to } (x, x^*) \end{array} \right. \tag{9}$$

and

$$\Gamma_c(x,S) := \overline{\mathrm{co}}\,(\Gamma_l(x,S)). \tag{10}$$

One easily sees that

- If Γ is monotone, then $\Gamma_c(\cdot,S)$ is monotone and $\Gamma_c(x,S) \subset \widetilde{\Gamma}(x)$ for all x.
- If Γ is cyclically monotone, then $\Gamma_c(\cdot,S)$ is cyclically monotone and $\Gamma_c(x,S) \subset \widehat{\Gamma}(x)$ for all x.

The following theorem is proved in [1].

Theorem 5.1 (Continuity of the mapping). *Let* $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ *be monotone and* $S \subset \mathrm{dom}\,(\Gamma)$. *Let* $a \in D = \overline{\mathrm{co}}\,(\mathrm{dom}\,(\Gamma)$. *Assume that* $\mathrm{int}\,(\mathrm{D}) \neq \emptyset$ *and there exists an open neighbourhood* V *of* a *such that* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{D}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{S})$. *Then* $\widetilde{\Gamma}(a) = \Gamma_c(a,S) + N_D(a)$. *It follows that* Γ *is maximal monotone at* $a \in V \cap D$ *if and only if* $\Gamma(a)$ *coincides with* $\Gamma_c(a,S) + N_D(a)$.

Recall that $N_D(a) = \{0\}$ if $a \in \mathrm{int}\,(\mathrm{D})$. In this case, the condition reduces to $\widetilde{\Gamma}(a) = \Gamma_c(a,S)$.

We shall prove that the same result holds for cyclically monotone mappings

Theorem 5.2 (Continuity of the mapping). *Let* $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ *be cyclically monotone and* $S \subset \mathrm{dom}\,(\Gamma)$. *Let* $a \in D = \overline{\mathrm{co}}\,(\mathrm{dom}\,(\Gamma)$. *Assume that* $\mathrm{int}\,(\mathrm{D}) \neq \emptyset$ *and there exists an open neighbourhood* V *of* a *such that* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{D}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{S})$. *Then* $\widehat{\Gamma}(a) = \Gamma_c(a,S) + N_D(a)$. *It follows that* Γ *is maximal cyclically monotone at* $a \in V \cap D$ *if and only if* $\Gamma(a)$ *coincides with* $\Gamma_c(a,S) + N_D(a)$.

Proof. For any $x \in D$ one has $\widehat{\Gamma}(x) \subseteq \widetilde{\Gamma}(x) = \Gamma_c(x,S) + N_D(x)$. Let us prove the reverse inclusion. Let $x^* = y^* + z^*$ with $y^* \in \Gamma_c(x,S)$ and $z^* \in N_D(x)$. In view of (5) we must prove that for any finite ordered family $\{(a_i, a_i^*)\}_{i=1,\cdots,p}$ contained in graph (Γ) one has

$$\langle x^*, a_1\rangle + \langle a_p^*, x\rangle + \sum_{i=1}^{p-1}\langle a_i^*, a_{i+1}\rangle \leq \langle x^*, x\rangle + \sum_{i=1}^{p}\langle a_i^*, a_i\rangle.$$

On one hand, $\langle z^*, a_1\rangle \leq \langle z^*, x\rangle$ because z^* belongs to the normal cone at x to D and on the other hand

$$\langle y^*, a_1\rangle + \langle a_p^*, x\rangle + \sum_{i=1}^{p-1}\langle a_i^*, a_{i+1}\rangle \leq \langle y^*, x\rangle + \sum_{i=1}^{p}\langle a_i^*, a_i\rangle$$

because $\Gamma_c(.,S)$ is cyclically monotone. The result follows. □

Theorems 5.1 and 5.2 are easily transposed to the case where $\mathrm{aff}\,(\mathrm{dom}\,(\Gamma)) \neq \mathbb{R}^{\mathrm{n}}$. We are now ready to give a first characterization of maximal monotone and maximal cyclically monotone mappings in terms of continuity of the domain and of the mapping. Of course the same result holds for Γ^{-1}.

Theorem 5.3. *Let* $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ *be monotone (cyclically monotone),* $D \neq \emptyset$ *be the closed convex hull of* $\mathrm{dom}\,(\Gamma)$ *and* $S \subset D$ *be such that* $D = \mathrm{cl}\,(\mathrm{D} \cap \mathrm{S})$. *Then* Γ *is* *maximal monotone (maximal cyclically monotone) if and only if* $\mathrm{dom}(\Gamma)$ *is almost convex and* $\Gamma(x) = \Gamma_c(x, S) + N_D(x)$ *at any* $x \in D$.

As a direct consequence we get the equivalence of maximal monotonicity and maximal cyclic monotonicity. As announced at the end of Section 2, the proof does not make appeal to a convex function.

Corollary 5.4. *Assume that* $\Gamma : \mathbb{R}^n \overset{\rightarrow}{\longrightarrow} \mathbb{R}^n$ *is cyclically monotone. Then it is maximal cyclically monotone if and only if it is maximal monotone.*

Another consequence is the celebrated Minty's characterization of maximal monotone mappings [3]. Here again, we provide a proof which is simpler than the classical ones which are given.

Corollary 5.5 (Minty theorem). *Assume that* $\Gamma : \mathbb{R}^n \overset{\rightarrow}{\longrightarrow} \mathbb{R}^n$ *is monotone. Then* Γ *is maximal monotone if and only if for all* $x^* \in \mathbb{R}^n$ *there exists* $x \in \mathbb{R}^n$ *such that* $x^* \in x + \Gamma(x)$. *This* x *is unique.*

Proof. i) Clearly $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma + \mathrm{I})$. It is immediately shown that $\Gamma + I$ is monotone and $\Gamma_c(x, S) + x = (\Gamma + I)_c(x, S)$ for any $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$. Hence, from Theorem 5.3, Γ is maximal monotone if and only if $\Gamma + I$ is so.

Denote by Σ the inverse mapping of $\Gamma + I$ and let $(x^*, x), (y^*, y) \in \mathrm{graph}(\Sigma)$. Then,

$$\|x - y\|^2 \le \langle x^* - y^*, x - y\rangle \le \|x^* - y^*\| \|x - y\|.$$

It follows that, for all x^*, $\Sigma(x^*)$ is either empty or reduced to a singleton and

$$\|\Sigma(x^*) - \Sigma(y^*)\| \le \|x^* - y^*\| \quad \forall\, x^*, y^* \in \mathrm{dom}(\Sigma).$$

ii) Assume that Σ is maximal monotone. Assume for contradiction that $\mathrm{dom}(\Sigma) \neq \mathbb{R}^{\mathrm{n}}$. Since $\mathrm{dom}(\Sigma)$ is almost convex, there exists some a^* in its relative interior and b^* in its relative boundary. Let $a \in \Sigma(a^*)$. For all $t \in (0, 1)$, $a^* + t(b^* - a^*) \in \mathrm{ri}(\mathrm{dom}(\Sigma)) \subset \mathrm{dom}(\Sigma)$ and $\|\Sigma(a^* + t(b^* - a^*)) - \Sigma(a^*)\| \le \|b^* - a^*\|$. One deduces that $\Sigma(b^*) = \Sigma_c(b^*, \mathbb{R}^n)$ is nonempty. Next, in view of Theorem 5.3, $\Sigma(b^*)$ is unbounded. This cannot be since $\Sigma(b^*)$ is either empty or reduced to a singleton.

iii) Assume that $\mathrm{dom}(\Sigma) = \mathbb{R}^{\mathrm{n}}$. On one hand $\mathrm{dom}(\Sigma)$ is almost convex and on the other hand $\Sigma_c(x^*, \mathbb{R}^n) = \Sigma(x^*)$ for any x^* because Σ is single valued and continuous. Hence, again in view of Theorem 5.3, Σ is maximal monotone. □

Given $x \in \mathrm{dom}(\Gamma)$ and $h \in \mathbb{R}^n$, let us define

$$g_+(x, h) = \sup[\langle x^*, h\rangle : x^* \in \Gamma(x)], \tag{11}$$

$$g_-(x, h) = \inf[\langle x^*, h\rangle : x^* \in \Gamma(x)]. \tag{12}$$

By construction, $g_+(x, .)$ is convex, lower semi-continuous, positively homogeneous and one has $g_+(x, h) \ge g_-(x, h) = -g_+(x, -h)$. Moreover, in the case where Γ is monotone, it holds for all $x, y \in \mathrm{dom}(\Gamma)$,

$$g_-(x, y - x) \le g_+(x, y - x) \le g_-(y, y - x) \le g_+(y, y - x) \tag{13}$$

and

$$g_+(x, y-x) + g_+(y, x-y) \le 0. \tag{14}$$

Proposition 5.6. *Assume that Γ is maximal monotone and $a \in \mathrm{dom}\,(\Gamma)$. Then*

$$\Gamma(a) = \{x^* : \langle x^*, h\rangle \le g_+(a,h) \ \ \forall\, h \in \mathbb{R}^n\}, \tag{15}$$

Assume in addition that $a \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$. Then

$$-\infty < g_+(a,h) = \lim_{t\downarrow 0}[g_+(a+th,h)] = \lim_{t\downarrow 0}[g_-(a+th,h)] < \infty. \tag{16}$$

Moreover, for all $\varepsilon > 0$, there exists a neighbourhood V of a depending on ε such that $g_+(x,h) \le g_+(a,h) + \varepsilon$ for all $x \in V$.

Proof. (15) holds because $\Gamma(x)$ is a nonempty convex set. (13) implies

$$-\infty < g_+(a,h) \le \lim_{t\downarrow 0}[g_-(a+th,h)] = \lim_{t\downarrow 0}[g_+(a+th,h)] \le +\infty$$

In case where $a \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$, $\Gamma(a)$ is bounded and the inequality on the right is strict. Let us introduce

$$\Omega = \{x^* = a^* + b^* : \ a^* \in \Gamma(a), \ \|b^*\|\|h\| < \varepsilon\}.$$

Ω is an open set containing $\Gamma(a)$. Therefore there exists a neighbourhood V of a such that $\Gamma(V) \subseteq \Omega$. For all $x \in V$ one has $g_+(x,h) \le g_+(a,h) + \varepsilon$. Hence (16) follows. □

6. Monotone equilibrium problems

Recall that these problems are of the form:

$$\text{Find } a \in C \text{ such that } \gamma(a,x) \ge 0 \ \ \forall\, x \in C \tag{Eqp}$$

The problem is said to be monotone if:

1. C is convex;
2. $\gamma(x,x) = 0$ for all $x \in \Omega$;
3. $\gamma(x,.)$ is convex on Ω for all $x \in \Omega$;
4. $\gamma(x,y) + \gamma(y,x) \le 0$ for all $x, y \in \Omega$.

A monotone variational inequality problem can be formulated as a monotone generalized equilibrium problem: take $\gamma(x,y) = g_+(x, y-x)$. We shall show that a monotone equilibrium problem can also be formulated as a monotone variational inequality problem. Under some continuity conditions on the bifunction γ, the mapping in the variational inequality problem is maximal monotone.

Proposition 6.1. *Assume that we are given an open convex subset Ω of $\mathbb{R}^n$ and $g : \Omega \times \Omega \to \mathbb{R}$ fulfilling items 2, 3, 4 above. Let us define on Ω a mapping Γ by*

$$\Gamma(x) = \bigcap_{y\in\Omega}\{x^* \in \mathbb{R}^n : \langle x^*, y-x\rangle \le \gamma(x,y)\}. \tag{17}$$

Then Γ is monotone on Ω and $\emptyset \ne \Gamma(a) \subseteq \Gamma_c(a, \mathbb{R}^n)$ at any $a \in \Omega$. If, in addition, the function $\gamma(.,y)$ is upper semicontinuous for any y, then Γ is maximal monotone on Ω.

Proof. i) It follows from items 2 and 3 that $x^* \in \Gamma(x)$ if and only if for all $y \in \Omega$ and $t \in (0,1)$

$$t\langle x^*, y-x\rangle \leq \gamma(x, x+t(y-x)) - \gamma(x,x) \leq t[\gamma(x,y) - \gamma(x,x)].$$

Therefore,

$$\Gamma(x) = \bigcap_{h \in \mathbb{R}^n} \left\{ x^* \in \mathbb{R}^n : \langle x^*, h\rangle \leq \gamma'_x(h) = \lim_{t \downarrow 0} \frac{\gamma(x, x+th) - \gamma(x,x)}{t} \right\}. \tag{18}$$

The function γ'_x is finite, convex and positively homogeneous. Hence $\Gamma(x)$ is a nonempty compact convex set and thereby $\mathrm{dom}\,(\Gamma) = \Omega$ is almost convex

ii) Let $x, y \in \Omega$, $x^* \in \Gamma(x)$ and $y^* \in \Gamma(y)$. Then $\langle x^*, y-x\rangle \leq \gamma(x,y)$ and $\langle y^*, x-y\rangle \leq \gamma(y,x)$. Combining with item 4 we obtain $\langle x^* - y^*, x-y\rangle \geq 0$. Thus Γ is monotone on the open convex set Ω. Therefore it is locally bounded at any $a \in \Omega$ according to Theorem 4.2.

iii) Let $a \in \Omega$, we shall prove that $\Gamma(a) \subseteq \Gamma_c(a, \mathbb{R}^n)$. Since the sets $\Gamma(a)$ and $\Gamma_c(a, \mathbb{R}^n)$ are convex and compact, it is sufficient to prove that if a^* is an extremal point of $\Gamma(a)$ it belongs also to $\Gamma_c(a, \mathbb{R}^n)$. There is some $h \in \mathbb{R}^n$ such that $\langle x^*, h\rangle < \langle a^*, h\rangle$ for any $x^* \in \Gamma(a)$ with $x^* \neq a^*$. For k integer large enough so that $x_k = a + k^{-1}h \in \Omega$, let us take some $x_k^* \in \Gamma(x_k)$. The sequence $\{x_k^*\}$ is bounded and $\langle a^*, x_k - a\rangle \leq \langle x_k^*, x_k - a\rangle$ for any k. Let x^* be a cluster point of the sequence x_k^*, then $\langle a^*, h\rangle \leq \langle x^*, h\rangle$ which implies $a^* = x^* \in \Gamma_c(a, \mathbb{R}^n)$.

iv) Finally, assume that $\gamma(.,y)$ is upper semicontinuous for any y. In order to prove that Γ is maximal monotone on Ω, it is sufficient in view of Theorem 5.1 that $\Gamma(a) \supseteq \Gamma_c(a, \mathbb{R}^n)$ for any $a \in \Omega$. Because $\Gamma(a)$ is a compact convex set, it is enough to prove that if $\{x_k, x_k^*\} \subset \mathrm{graph}\,(\Gamma)$ is a sequence converging to (a, a^*), then $a^* \in \Gamma(a)$. Let any $y \in \Omega$. Because $\langle x_k^*, y - x_k\rangle \leq \gamma(x_k, y)$ and $\gamma(.,y)$ is upper semicontinuous, one has $\langle a^*, y-a\rangle \leq \gamma(a,y)$. Therefore $a^* \in \Gamma(a)$. □

Under the assumptions of Proposition 6.1, (Eqp) reduces to the simpler equivalent problem:

$$\text{Find } a \in C \text{ so that } \sup[\langle a^*, x-a\rangle : a^* \in \Gamma(a)] \geq 0 \quad \forall\, x \in C. \tag{Vip}$$

Indeed the generalized equilibrium formulation of (Vip) is

$$\text{Find } a \in C \text{ such that } \gamma'_a(x) \geq 0 \quad \forall\, x \in C.$$

The function γ'_a is simpler than the original function $\gamma(a,.)$: it is convex and positively homogeneous. Theoretical results on monotone generalized equilibrium problems are nothing else than consequences of results on variational equilibrium problems.

References

[1] J.-P. Crouzeix, E. Ocaña Anaya, W. Sosa Sandoval: A construction of a maximal monotone extension of a monotone map, ESAIM, Proc. 20 (2007) 93–104.

[2] C. J. Himmelberg: Fixed points of compact multifunctions, J. Math. Anal. Appl. 38 (1972) 205–207.

[3] G. J. Minty: Monotone (nonlinear) operators in Hilbert Space, Duke Math. J. 29 (1962) 341–346.

[4] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[5] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[6] S. Simons: From Hahn-Banach to Monotonicity, 2nd Ed., Lecture Notes in Mathematics 1693, Springer, Berlin (2008).