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Topologies Associated with Kuratowski-Painlevé Convergence of Closed Sets

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The main purpose of this paper is to identify topologies on the closed subsets $\mathscr{C}(X)$ of a Hausdorff space X that are sequentially equivalent to classical Kuratowski-Painlevé convergence K. This reduces to a study of upper topologies sequentially equivalent to upper Kuratowski-Painlevé convergence K^+, where we are of course led to consider the sequential modification of upper Kuratowski-Painlevé convergence. We characterize those miss topologies induced by a cobase of closed sets that are sequentially equivalent to K^+, with special attention given to X first countable. Separately in the final section, we revisit Mrowka's theorem on the compactness of Kuratowski-Painlevé convergence.

Keywords: Fell topology, Kuratowski-Painlevé convergence, hyperspace, hit-and-miss topology, modification, sequential modification, subsequential selector

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1. Introduction

Given a Hausdorff topological space $\langle X, \mathscr{T}\rangle$ we denote by $\mathscr{C}(X)$ the family of all closed subsets of X. We recall that given a net $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ in $\mathscr{C}(X)$, the *upper closed limit* and the *lower closed limit* of the net are defined as

$$\mathrm{Ls}\ A_\lambda := \{x \in X : U_x \cap A_\lambda \neq \varnothing \text{ cofinally for every neighborhood } U_x \text{ of } x\};$$
$$\mathrm{Li}\ A_\lambda := \{x \in X : U_x \cap A_\lambda \neq \varnothing \text{ residually for every neighborhood } U_x \text{ of } x\}.$$

Befitting their names, both are closed subsets of X (see, e.g., [6, Proposition 5.2.2]). The net $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ is said to be:

- K^+-convergent or *upper Kuratowski-Painlevé convergent* to A if $\mathrm{Ls}\ A_\lambda \subseteq A$;

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- K^--convergent or *lower Kuratowski-Painlevé convergent* to A if $A \subseteq \mathrm{Li}\ A_\lambda$;
- K-convergent or *Kuratowski-Painlevé convergent* to A if it is K^+-convergent and K^--convergent to A, i.e. $\mathrm{Ls}\ A_\lambda = \mathrm{Li}\ A_\lambda = A$.

This kind of convergence for sequences of sets was introduced by Painlevé and Hausdorff, later receiving broad dissemination in this context through the monograph of Kuratowski [36]. It resurfaced again for filtered families of closed sets in the seminal article of Choquet [15], followed by the monograph of Berge [10], and it is from this perspective that it is often studied (see, e.g., [25, 38]). One important fact about K-convergence is its compactness without restriction [6, 37]. It has been applied to lower semicontinuous extended real-valued functions as associated with their (closed) epigraphs, especially to convex functions (see, e.g., [5, 6, 26, 42, 43]).

In general, Kuratowski-Painlevé convergence is not topological, i.e. we cannot find a topology on $\mathscr{C}(X)$ such that the convergence of nets in this topology is equivalent to their Kuratowski-Painlevé convergence [25, 35]. Nevertheless, there is a finest topology whose convergence is weaker than the Kuratowski-Painlevé convergence. This topology has been historically called the *convergence topology* and will be denoted by $\tau\mathrm{K}$. In a similar way, the *upper Kuratowski topology* $\tau\mathrm{K}^+$ is the finest topology coarser than the upper Kuratowski-Painlevé convergence.

It can be proved that the finest hit-and-miss topology coarser than $\tau\mathrm{K}$ is the Fell topology (see Remark 4.3) and this topology coincides with the Kuratowski-Painlevé convergence in a Hausdorff space if and only if it is locally compact ([35]).

It is well-known that convergence in the lower Vietoris topology is equivalent to lower Kuratowski-Painlevé convergence. Dolecki, Greco and Lechicki [25] were perhaps the first to investigate when the co-compact topology (the upper Fell topology) is equal to the upper Kuratowski topology $\tau\mathrm{K}^+$. The topological spaces for which this equality is verified are called *consonant*. Examples of consonant topological spaces are: Čech-complete spaces, Hausdorff k_ω-spaces, etc. (see [1, 2, 11, 12, 13, 14, 19, 31, 38, 40]).

In this paper, we obtain some new results on the sequential modification of the upper Kuratowski-Painlevé convergence, that is, the strongest sequential topology that is sequentially coarser than the convergence. We are also interested in those miss topologies that have the same convergent sequences as the upper Kuratowski-Painlevé convergence. The spaces where this coincidence is true for the co-compact topology are called *sequentially consonant* [16], but it seems to us more natural to focus attention on the miss topologies generated by cobases of closed countably compact sets, as in an arbitrary Hausdorff space, the miss topology generated by the cobase of all closed countably compact sets is the finest miss topology sequentially coarser than the upper Kuratowski convergence. We completely characterize those miss topologies generated by a cobase which have the same convergent sequences as the upper Kuratowski-Painlevé convergence (see Theorem 4.4). We look more carefully at such miss topologies when the underlying topology of the space is first countable, where we introduce the notion of *subsequential selector* for the convergent sequences on the space. We also obtain some two-sided results, e.g., we characterize in the context of a metric space the sequentiality of the Fell topology $\mathscr{T}_F$ and the convergence topology $\tau\mathrm{K}$, and we revisit Mrowka's Theorem on the compactness of Kuratowski-Painlevé convergence for nets of closed sets.

2. Preliminaries

All topological spaces will be assumed to be Hausdorff and to consist of at least two points. We denote the closure, set of limit points and interior of a subset A of a Hausdorff space X by $\mathrm{cl}(A)$, A' and $\mathrm{int}(A)$, respectively. Let $\mathscr{C}_0(X)$ denote the nonempty closed subsets of X. We define idempotent operators Σ, $\Downarrow$ and $\Uparrow$ on subfamilies of $\mathscr{C}(X)$ as follows:

(1) $\Sigma(\mathscr{A}) := \{E : E$ is a finite union of elements of $\mathscr{A}\}$;
(2) $\Downarrow \mathscr{A} := \{E : E$ is a closed subset of some element of $\mathscr{A}\}$;
(3) $\Uparrow \mathscr{A} := \{E : E$ is a closed superset of some element of $\mathscr{A}\}$.

Let $\langle X, d\rangle$ be a metric space. For $x \in X$ and $\varepsilon > 0$, we write $S_\varepsilon(x)$ for the open ball of radius ε with center x. If A is a nonempty subset of X and $x \in X$, we put $d(x, A) := \inf\{d(x, a) : a \in A\}$. We agree to the convention $d(x, \varnothing) = \infty$. With this in mind, the *Wijsman topology* $\mathscr{T}_{W_d}$ determined by the metric d is the weak topology on $\mathscr{C}(X)$ induced by the family of distance functions $\{d(x, \cdot) : x \in X\}$ [6, 17].

By a *cobase* for a topological space $\langle X, \mathscr{T}\rangle$, we mean a family of nonempty closed sets Δ which contains the singletons and such that $\Sigma(\Delta) = \Delta$ [39]. Evidently, the largest cobase is $\mathscr{C}_0(X)$ and the smallest is the set of nonempty finite subsets $\mathscr{F}_0(X)$. Given a family $\mathscr{B}$ of closed subsets the smallest cobase containing them is $\Sigma(\mathscr{B} \cup \mathscr{F}_0(X))$. We call this the *cobase generated by* $\mathscr{B}$. We call a cobase *compact* (resp. *countably compact*) if its members are all compact subsets (resp. countably compact subsets) of X.

We list some important cobases in a Hausdorff space; the last two are particular to a metric space $\langle X, d\rangle$:

(1) $\mathscr{K}_0(X) := \{F \in \mathscr{C}_0(X) : F$ is compact$\}$;
(2) $\{F \in \mathscr{C}_0(X) : g(F)$ is bounded$\}$ where g is a real-valued function on X;
(3) $\Delta^{\mathrm{seq}} := \Sigma(\{\widehat{\alpha} : \alpha \in \mathrm{seq}(X)\})$, where $\mathrm{seq}(X)$ is the set of all convergent sequences in X and $\widehat{\alpha}$ is the range of $\alpha \in \mathrm{seq}(X)$ along with its unique limit point;
(4) $\{F \in \mathscr{C}_0(X) : F$ is countably compact$\}$;
(5) $\{F \in \mathscr{C}_0(X) : F$ is bounded$\}$;
(6) $\{F \in \mathscr{C}_0(X) : \langle F, d\rangle$ is complete$\}$.

We call a cobase Δ *Urysohn* [6, 20] if whenever V is open and $D \in \Delta$ with $D \subseteq V$, there exists $D_1 \in \Delta$ such that $D \subseteq \mathrm{int}(D_1) \subseteq D_1 \subseteq V$. A weaker condition is that the cobase be *local*: $\forall x \in X$, whenever V is a neighborhood of x, there exists $D_1 \in \Delta$ with $x \in \mathrm{int}(D_1) \subseteq D_1 \subseteq V$. In a Hausdorff space, $\mathscr{C}_0(X)$ is local if and only if the topology is regular, and in a regular Hausdorff space, $\mathscr{C}_0(X)$ is Urysohn if and only if the topology is normal.

Given a cobase Δ for $\langle X, \mathscr{T}\rangle$, the *hit-and-miss topology* $\mathscr{T}_\Delta$ on $\mathscr{C}(X)$ [6, 39] is the supremum $\mathscr{T}_{LV} \vee \mathscr{T}_\Delta^+$ where $\mathscr{T}_{LV}$ is the *lower Vietoris topology* having as a subbase $\mathscr{C}(X)$ plus all sets of the form

$$\{F \in \mathscr{C}(X) : F \cap V \neq \varnothing\} \quad (V \text{ is open}),$$

and the miss topology $\mathscr{T}_\Delta^+$ has as a base all sets of the form

$$\{F \in \mathscr{C}(X) : F \subseteq X \backslash D\} \quad (D \in \Delta \cup \{\varnothing\}).$$

When $\Delta = \mathscr{K}_0(X)$, $\mathscr{T}_\Delta^+$ is called the *co-compact topology* and $\mathscr{T}_\Delta$ is called the *Fell topology* which we denote by $\mathscr{T}_C$ and $\mathscr{T}_F$ respectively. We define the *co-countably compact topology* $\mathscr{T}_{CC}$ as the topology $\mathscr{T}_\Delta^+$ when Δ is the family of all closed and countably compact sets. When $\Delta = \mathscr{C}_0(X)$, we obtain the classical *Vietoris topology* $\mathscr{T}_V$. The following important fact is left as an exercise for the reader.

Theorem 2.1. *Let Δ_1 and Δ_2 be cobases for a Hausdorff space $\langle X, \mathscr{T}\rangle$. Then $\mathscr{T}_{\Delta_1}^+ \subseteq \mathscr{T}_{\Delta_2}^+$ if and only if whenever $A \in \mathscr{C}(X)$ and $B_1 \in \Delta_1$ with $A \cap B_1 = \varnothing$, there exists $B_2 \in \Delta_2$ with $B_1 \subseteq B_2$ and $A \cap B_2 = \varnothing$.*

Example 2.2. Let Δ_1 be the cobase in $\mathbb{R}$ generated by all closed and bounded intervals with rational endpoints, and let Δ_2 be the cobase in $\mathbb{R}$ generated by all closed and bounded intervals with irrational endpoints. While the cobases are different, by virtue of the last result, they generate the same miss topologies.

Letting $A = \varnothing$ in the last theorem, we see that a necessary condition for $\mathscr{T}_{\Delta_1}^+ \subseteq \mathscr{T}_{\Delta_2}^+$ is that $\Delta_1 \subseteq \Downarrow \Delta_2$. This immediately implies that when Δ_2 is a compact (resp. countably compact) cobase and $\mathscr{T}_{\Delta_1} \subseteq \mathscr{T}_{\Delta_2}$, then also Δ_1 is a compact (resp. countably compact) cobase. The condition is not sufficient: again in $\mathbb{R}$, let $\Delta_1 = \mathscr{K}_0(X)$ and let Δ_2 be the cobase generated by $\{[r, \infty) : r \in \mathbb{R}\}$.

Seemingly more generally, we could define the miss topology determined by a closed cover $\mathscr{B}$ of X containing the singletons, but since no stronger topology results by replacing $\mathscr{B}$ by $\Sigma(\mathscr{B})$, we have elected to make this part of our definition.

The point of view of [25] leads to a useful expression of the closed subsets of an arbitrary miss topology on $\mathscr{C}(X)$. Let us denote by $\mathscr{F}^\sharp$ the *grill* of a family of subsets $\mathscr{F}$, i.e. $\mathscr{F}^\sharp = \{A \subseteq X : A \cap F \neq \varnothing \text{ for all } F \in \mathscr{F}\}$ [10, 25]. Let Δ be a cobase of closed sets. Evidently, a base for the closed sets with respect to $\mathscr{T}_\Delta^+$ is the family

$$\{C \in \mathscr{C}(X) : C \cap D \neq \varnothing\} \quad (D \in \Delta \cup \{\varnothing\}).$$

Consequently, a nonempty family of closed subsets $\mathscr{F}$ is $\mathscr{T}_\Delta^+$-closed if there exists $\Delta_1 \subseteq \Delta$ such that $\mathscr{F} = \Delta_1^\sharp \cap \mathscr{C}(X)$. From this, we can prove the following result.

Proposition 2.3. *Let Δ be a cobase in a topological space $\langle X, \mathscr{T}\rangle$, and let $\mathscr{F}$ be a nonempty family of closed subsets. Then $\mathscr{F}$ is closed with respect to $\mathscr{T}_\Delta^+$ if and only if $\mathscr{F} = \Uparrow \mathscr{F}$ and whenever G is an open set such that $G \in \mathscr{F}^\sharp$ then there exists $D_G \in \Delta$ such that $D_G \subseteq G$ and $D_G \in \mathscr{F}^\sharp$.*

Proof. Of course, $\mathscr{F}$ is closed if and only if there exists $\Delta_1 \subseteq \Delta$ such that $\mathscr{F} = \Delta_1^\sharp \cap \mathscr{C}(X)$. If this occurs, then it obvious that $\mathscr{F} = \Uparrow \mathscr{F}$. Now suppose G is open with $G \in \mathscr{F}^\sharp$, yet $\forall D \in \Delta_1$, D fails to be a subset of G. Clearly, $X \backslash G \in \Delta_1^\sharp \cap \mathscr{C}(X) = \mathscr{F}$, which contradicts $G \cap F \neq \varnothing$ for each $F \in \mathscr{F}$. Thus there exists some $D \in \Delta_1$ with $D \subseteq G$, and we can put $D_G = D$.

For sufficiency, for each open set G in the grill of $\mathscr{F}$, pick D_G with the required property. We claim that $\mathscr{F} = \{D_G : G \text{ open and } G \in \mathscr{F}^\sharp\}^\sharp \cap \mathscr{C}(X)$. As one inclusion is trivial, we just verify $\mathscr{F} \supseteq \{D_G : G \text{ open and } G \in \mathscr{F}^\sharp\}^\sharp \cap \mathscr{C}(X)$. To this end, suppose $E \in \{D_G : G \text{ open and } G \in \mathscr{F}^\sharp\}^\sharp \cap \mathscr{C}(X)$ and $E \notin \mathscr{F}$. Since no closed subset of E

can belong to $\mathscr{F}$, for each $F \in \mathscr{F}$, we have $F \cap (X \backslash E) \neq \varnothing$. As a result, $X \backslash E \in \mathscr{F}^\sharp$, but $E \cap D_{X \backslash E} = \varnothing$. $\square$

By a *convergence* Q on a set X (see [23, 24]), we mean a function that assigns to each net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ in X a possibly empty subset of X, called the Q-*limits* of the net. When $a \in X$ is a Q-limit of $\langle x_\lambda \rangle_{\lambda \in \Lambda}$, we will write

$$\langle x_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathrm{Q}}{\to} a.$$

We will assume here that all convergences are *isotone*, that is the set of limits for a subnet of a net includes those of the original net, and that constant nets are convergent to the repeated value.

We say that a convergence Q is *stronger* or *finer* than another convergence P and write $\mathrm{Q} \geq \mathrm{P}$ if

$$\langle x_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathrm{Q}}{\to} a \Rightarrow \langle x_\lambda \rangle_{\lambda \in \Lambda} \overset{\mathrm{P}}{\to} a.$$

With respect to this partial order, the set of convergences on X becomes a complete lattice. A topology $\mathscr{T}$ on X induces an isotone convergence in an unambiguous way, and with this in mind, the symbols $\mathrm{Q} \leq \mathscr{T}$, $\mathscr{T} \leq \mathrm{Q}$ and $\mathscr{T} = \mathrm{Q}$ make sense where Q is a convergence. If a convergence is induced by a topology, it will be called a *topological convergence*.

It is important to note that while the topologies on X also form a complete lattice with respect to inclusion, they do not form a complete sublattice of the lattice of convergences. More precisely, while the lattice of topologies is a complete join semilattice of the lattice of convergences (whence we can write $\bigvee_{i \in I} \mathscr{T}_i$ unambiguously), the intersection of a family of topologies can produce a convergence strictly coarser than the meet of the convergences determined by the topologies. While a convergence Q may not be induced by a topology, there is always a finest topology $\tau\mathrm{Q}$ whose convergence is coarser than Q, which we call here the *modification* of Q. A subset A of X is closed in this modification topology if and only if A is stable under Q-limits of nets in A [17, Lemma 2.1]. When $\tau\mathrm{Q} = \mathrm{Q}$ or equivalently $\tau\mathrm{Q} \geq \mathrm{Q}$, then the convergence is topological. We note that the word *topologization* seems to be used interchangeably with modification in the literature. We have chosen the usage we have as it is a lot easier to say. In the special case of Kuratowski-Painlevé convergence, the modification is nothing but the convergence topology alluded to above, while in the case of upper Kuratowski-Painlevé convergence, it coincides with the upper Kuratowski topology.

The following fact, which allows us under certain conditions to obtain the modification of a convergence from above, is crucial in a fundamental paper on hyperspace topologies of Costantini, Levi and Pelant [17].

Proposition 2.4 (cf. [17, Proposition 2.2]). *Let* Q *be a convergence on a set* X*. Suppose in the lattice of convergences we have* $\mathrm{Q} = \bigwedge\{\mathscr{T} : \mathscr{T}$ *is a topology on* X *and* $\mathrm{Q} \leq \mathscr{T}\}$*. Then* $\tau\mathrm{Q} = \bigcap\{\mathscr{T} : \mathrm{Q} \leq \mathscr{T}\}$.

Proof. First since the discrete topology is the supremum of all convergences, the set $\{\mathscr{T} : \mathrm{Q} \leq \mathscr{T}\}$ is nonempty. Let $\mathscr{T}_0$ be a topology coarser than Q; if $\mathrm{Q} \leq \mathscr{T}_1$, then $\mathscr{T}_0 \leq \mathscr{T}_1$ and so $\mathscr{T}_0 \subseteq \mathscr{T}_1$. Thus $\mathscr{T}_0 \subseteq \bigcap\{\mathscr{T} : \mathrm{Q} \leq \mathscr{T}\}$. But for any family $\mathfrak{A}$ of topologies, we always have $\bigcap \mathfrak{A} \leq \bigwedge \mathfrak{A}$. In particular, $\bigcap\{\mathscr{T} : \mathrm{Q} \leq \mathscr{T}\}$ is coarser than Q. □

Sequences, that is functions defined on the positive integers $\mathbb{N}$, are special types of nets. Given two convergences Q and P on a set X, we will write $\mathrm{Q} \geq_{\mathrm{seq}} \mathrm{P}$ provided whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X and $a \in X$, then

$$\langle x_n \rangle_{n\in\mathbb{N}} \xrightarrow{\mathrm{Q}} a \Rightarrow \langle x_n \rangle_{n\in\mathbb{N}} \xrightarrow{\mathrm{P}} a.$$

We will write $\mathrm{Q} \approx_{\mathrm{seq}} \mathrm{P}$ to mean that Q and P have the same convergent sequences to the same limits. In this case, the convergences are deemed *sequentially equivalent*. As one or both of the convergences may arise from a topology $\mathscr{T}$, we will freely employ formulas such as $\mathrm{Q} \leq_{\mathrm{seq}} \mathscr{T}$ or $\mathscr{T}_1 \approx_{\mathrm{seq}} \mathscr{T}_2$ in the sequel. For example, it is well known [6, Theorem 5.2.10], [33, Theorem 9] that, in the context of first-countable spaces, we may write $\mathscr{T}_F \approx_{\mathrm{seq}} \mathrm{K}$ with respect to closed subsets.

Recall that a topology $\mathscr{T}$ on a set X is called *sequential* provided A is closed whenever A is stable under taking limits of sequences [27, 28, 29]. Intrinsic to sequential spaces is the *sequential modification* sQ of a convergence Q that yields the (sequential) topology on X whose closed sets consist of all subsets A of X such that whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in A which is Q-convergent to $x \in X$, then $x \in A$. It is obvious that $\tau\mathrm{Q} \leq \mathrm{sQ}$, and the two coincide if and only if $\mathrm{sQ} \leq \mathrm{Q}$.

While Q-convergence of sequences forces sQ-convergence [18], the reverse implication may fail.

Example 2.5. A real-valued net defined on a directed set $\langle \Lambda, \succeq \rangle$ is said to be *ultimately increasing* (resp. *ultimately decreasing*) [9] if $\forall \lambda_0 \in \Lambda$, $\exists \lambda_1 \in \Lambda$ such that $\lambda \succeq \lambda_1 \Rightarrow f(\lambda) \geq f(\lambda_0)$ (resp. $f(\lambda) \leq f(\lambda_0)$). Generically such functions are of course called *ultimately monotone*. While a subnet of a monotone net defined on a directed set need not be monotone, even if both directed sets are $\mathbb{N}$, ultimate monotonicity is preserved under passing to subnets.

We now declare a net $\langle a_\lambda \rangle_{\lambda\in\Lambda}$ in $\mathbb{R}$ Q-convergent provided $\langle a_\lambda \rangle_{\lambda\in\Lambda}$ is ultimately monotone and convergent in the usual topology. By the above remark, convergence so defined is a bona fide convergence. We claim that A is a closed set as determined by sQ if and only if A is closed with respect to the usual topology. Clearly, if A is closed in the usual sense, it is stable with respect to limits of ultimately monotone sequences in A. Conversely, suppose A is stable with respect to taking limits of ultimately monotone sequences in A. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence in A convergent to p in the usual sense. Then $\forall n \in \mathbb{N}$, $s_n := \sup\{a_k : k \geq n\}$ belongs to A, as s_n can be retrieved as a limit of an increasing sequence taken from $\{a_k : k \geq n\}$, and thus $p = \lim_{n\to\infty} s_n \in A$ because $\langle s_n \rangle_{n\in\mathbb{N}}$ is decreasing. Of course, there are convergent sequences in the usual topology that are not ultimately monotone, e.g., $\langle \frac{(-1)^n}{n} \rangle_{n\in\mathbb{N}}$.

The last example shows that if $\mathrm{sQ} = \mathrm{sP}$, we cannot conclude that $\mathrm{Q} \approx_{\mathrm{seq}} \mathrm{P}$ (but see bullet three immediately below). The following facts can be found at least implicitly in [18, 24]:

- A topology $\mathscr{T}$ is sequential if and only $\mathscr{T} = \mathrm{s}\mathscr{T}$.

- For convergences P_1 and P_2, $\mathrm{P}_1 \leq_{\mathrm{seq}} \mathrm{P}_2 \Rightarrow \mathrm{sP}_1 \subseteq \mathrm{sP}_2$.
- For topologies, $\mathscr{T}_1 \approx_{\mathrm{seq}} \mathscr{T}_2$ if and only if $\mathrm{s}\mathscr{T}_1 = \mathrm{s}\mathscr{T}_2$.
- The operator $\mathrm{P} \mapsto \mathrm{sP}$ on convergences is idempotent; as a result, for a convergence P, sP is the finest topology that is sequentially coarser than P.
- If $\{\mathscr{T}_i : i \in I\}$ is a family of sequential topologies, then $\cap_{i\in I}\mathscr{T}_i$ is sequential.

3. On the sequential modification of upper Kuratowski-Painlevé convergence

In their seminal paper [25, Corollary 3.2], Dolecki, Greco, and Lechicki characterized the closed subsets $\mathscr{F}$ of $\tau\mathrm{K}^+$ by the conjunction of the following two conditions: (1) $\mathscr{F} = \Uparrow \mathscr{F}$, and (2) for every family of open sets $\{G_i\}_{i\in I}$ such that $\cup_{i\in I}G_i \in \mathscr{F}^\sharp$ there exists a finite subset N of I such that $\cup_{i\in N}G_i \in \mathscr{F}^\sharp$. We begin by providing a parallel description of the closed sets of the sequential modification of the upper Kuratowski-Painlevé convergence.

Theorem 3.1 (cf. [25, Corollary 3.2]). *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space. Then a family of closed sets $\mathscr{F}$ is closed in* sK^+ *if and only if $\mathscr{F}$ verifies the following conditions:*

(*1*) $\mathscr{F} = \Uparrow \mathscr{F}$;

(*2*) *for every countable family of open sets $\{G_n\}_{n\in\mathbb{N}}$ such that $\cup_{n\in\mathbb{N}}G_n \in \mathscr{F}^\sharp$ there exists a finite subset N of $\mathbb{N}$ such that $\cup_{n\in N}G_n \in \mathscr{F}^\sharp$.*

Proof. Suppose that $\mathscr{F}$ is closed in sK^+. Given $F \in \mathscr{F}$ and $F_1 \in \mathscr{C}(X)$ with $F \subseteq F_1$ then it is clear that $F_1 \in \mathscr{F}$, since the constant sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ where for each n, $F_n = F$, satisfies $\mathrm{Ls}\ F_n = F \subseteq F_1$.

Now suppose that $\{G_n\}_{n\in\mathbb{N}}$ is a countable family of open sets such that $\cup_{n\in\mathbb{N}}G_n \in \mathscr{F}^\sharp$. This obviously implies that $A = X\setminus \cup_{n\in\mathbb{N}} G_n \notin \mathscr{F}$. Furthermore, if $\cup_{n\in N}G_n \notin \mathscr{F}^\sharp$ for every finite subset N of $\mathbb{N}$ then we deduce that $A_n = X\setminus \cup_{i=1}^{n} G_i \in \mathscr{F}$. It is clear that $\langle A_n\rangle_{n\in\mathbb{N}}$ is K^+-convergent to A, but $A \notin \mathscr{F}$ which contradicts the closedness of $\mathscr{F}$ in the sequential modification.

Conversely, let $\mathscr{F}$ be a family which verifies the two aforementioned properties. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$ which is K^+-convergent to A. We can suppose without loss of generality that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a decreasing sequence (otherwise construct the sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ where $B_n = \mathrm{cl}(\cup_{i=n}^{\infty}A_i)$ for all $n \in \mathbb{N}$). If $\cup_{n\in\mathbb{N}}X\setminus A_n \notin \mathscr{F}^\sharp$ then we can find $F \in \mathscr{F}$ such that $\cup_{n\in\mathbb{N}}X\setminus A_n \cap F = \varnothing$ so $F \subseteq A_n$ for all $n \in \mathbb{N}$. Hence, $F \subseteq \mathrm{Ls}\ A_n \subseteq A$ so by (*1*) $A \in \mathscr{F}$.

Alternatively we have $\cup_{n\in\mathbb{N}}X\setminus A_n \in \mathscr{F}^\sharp$. By (*2*), there exists a finite subset N of $\mathbb{N}$ such that $\cup_{n\in N}X\setminus A_n \in \mathscr{F}^\sharp$. If $n_0 = \max N$, then $X\setminus A_{n_0} = \cup_{n\in N}X\setminus A_n \in \mathscr{F}^\sharp$, i.e. $A_{n_0} \notin \mathscr{F}$, which is a contradiction. □

Corollary 3.2 (cf. [25, Theorem 3.1], [40, Lemma 2.2]). *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space. Then a family of closed sets $\mathscr{G}$ is open in* sK^+ *if and only if $\mathscr{G}$ verifies the following conditions:*

(*1*) $\mathscr{G} = \Downarrow \mathscr{G}$;

(2) *for every countable family of closed sets* $\{F_n\}_{n\in\mathbb{N}}$ *such that* $\cap_{n\in\mathbb{N}} F_n \in \mathscr{G}$ *there exists a finite subset* N *of* $\mathbb{N}$ *such that* $\cap_{n\in N} F_n \in \mathscr{G}$.

The next example shows that, in general, $\mathrm{sK}^+ \neq \tau\mathrm{K}^+$.

Example 3.3. Let us consider the interval $X = [0,1]$ endowed with the discrete topology. Direct $\mathscr{I} := \{I \subseteq [0,1] : |I| \leq \aleph_0\}$ by inclusion, and $\forall I \in \mathscr{I}$ put $F_I := X \backslash I$. It is easy to prove that $\mathrm{Ls}\ F_I = \varnothing$ so $\langle F_I \rangle_{I\in\mathscr{I}}$ is $\tau\mathrm{K}^+$-convergent to $\varnothing$. We claim $\mathscr{G} := \{A \subseteq [0,1] : X\backslash A \text{ is not countable}\}$ is sK^+-open. Clearly, $\mathscr{G} = \Downarrow \mathscr{G}$, and if $\{F_n\}_{n\in\mathbb{N}}$ is a family of closed sets such that $\cap_{n\in\mathbb{N}} F_n \in \mathscr{G}$ then $X\backslash \cap_{n\in\mathbb{N}} F_n$ is not countable. It is immediate to see that $X\backslash \cap_{n=1}^{k} F_n$ is not countable for some $k \in \mathbb{N}$. Consequently, we deduce from the above corollary that $\mathscr{G}$ is sK^+-open. Of course, $\varnothing \in \mathscr{G}$ but $F_I \notin \mathscr{G}$ for all $I \in \mathscr{I}$.

Coincidence of sK^+ and $\tau\mathrm{K}^+$ on $\mathscr{C}(X)$ has been implicitly characterized by Mynard [38] (see Condition 3 of Theorem 0.3 therein). We find it worthwhile to give a self-contained proof, based on Theorem 3.1 *supra*.

Theorem 3.4 (cf. [38, Theorem 0.3]). *Let* $\langle X, \mathscr{T} \rangle$ *be a Hausdorff space. Then* $\mathrm{sK}^+ = \tau\mathrm{K}^+$ *if and only if* X *is hereditarily Lindelöf.*

Proof. Suppose that $\mathrm{sK}^+ = \tau\mathrm{K}^+$. If X fails to be hereditarily Lindelöf, then there exists a family $\{G_i : i \in I\}$ of open subsets in X such that whenever $J \subseteq I$ is countable, $\cup_{j\in J} G_j \neq \cup_{i\in I} G_i$. Whenever $J \subseteq I$ with $|J| \leq \aleph_0$, put $F_J = X\backslash \cup_{j\in J} G_j$ and consider the family of closed subsets $\mathscr{F}$ of X defined by

$$\mathscr{F} := \Uparrow \{F_J : J \subseteq I \text{ and } |J| \leq \aleph_0\}.$$

We first show using Theorem 3.1 that $\mathscr{F}$ is sK^+-closed. Suppose $\{V_n : n \in \mathbb{N}\}$ is a countable family of open subsets where for each $n \in \mathbb{N}, \cup_{i=1}^{n} V_i \notin \mathscr{F}^\sharp$. For each n choose a countable $J_n \subseteq I$ with $\cup_{i=1}^{n} V_n \subseteq \cup_{j\in J_n} G_{j_n}$. Then with $J_\infty = \cup_{n\in\mathbb{N}} J_n$, we have

$$\cup_{n=1}^{\infty} V_n \cap F_{J_\infty} = \varnothing,$$

which shows that $\cup_{n=1}^{\infty} V_n$ fails to be in the grill of $\mathscr{F}$, as required. On the other hand, it is obvious that $\cup_{i\in I} G_i \in \mathscr{F}^\sharp$ while $\cup_{n\in N} G_n \notin \mathscr{F}^\sharp$ for every finite subset N of I since $X\backslash \cup_{n\in N} G_n \in \mathscr{F}$. From [25, Cor. 3.2], we deduce that $\mathscr{F}$ is not $\tau\mathrm{K}^+$-closed, which contradicts our assumption.

The converse is a direct consequence of Theorem 3.1 and [25, Cor. 3.2]. □

Coincidence of the modification with the sequential modification of course implies that the modification is sequential. That the hereditarily Lindelöf condition is both necessary and sufficient for $\tau\mathrm{K}^+$ to be sequential was discovered by Costantini, Holá, and Vitolo [16]. For completeness, we include a result characterizing when the upper Kuratowski-Painlevé convergence is sequentially topological.

Theorem 3.5. *Let* $\langle X, \mathscr{T} \rangle$ *be a Hausdorff space. Then* K^+ *is compatible with a sequential topology if and only if* X *is locally compact and hereditarily Lindelöf.*

Proof. If X is hereditarily Lindelöf then $\tau\mathrm{K}^+$ is sequential by Theorem 3.4. But if X is locally compact then, by [25, Theorem 1.1], K^+ is topological so it is compatible with $\tau\mathrm{K}^+$ which is sequential.

Conversely, if K^+ is topological then X must be locally compact by [25, Theorem 1.1]. As this topology must be the modification topology, $\tau\mathrm{K}^+$ is sequential and the result follows from [16, Prop. 3.6]. $\square$

Proposition 3.6. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space. Then $\mathscr{T}_{CC} \subseteq \mathrm{sK}^+$.*

Proof. Let $\mathscr{F}$ be nonempty and closed with respect to $\mathscr{T}_{CC}$. Suppose $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open sets such that $\cup_{n=1}^{\infty} G_n \in \mathscr{F}^\sharp$. Choose by Proposition 2.3 C countably compact within $\cup_{n=1}^{\infty} G_n$ such that $C \in \mathscr{F}^\sharp$. By countable compactness $\exists k \in \mathbb{N}$ with $C \subseteq \cup_{n=1}^{k} G_n$ and clearly $\cup_{n=1}^{k} G_n \in \mathscr{F}^\sharp$, too. Apply Theorem 3.1. $\square$

Definition 3.7. Let $\langle X, \mathscr{T}\rangle$ be a topological space. We say that a subset A of X is *σ-countably compact* if A is the union of a countably family of countably compact closed sets.

Proposition 3.8. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space such that the family of all closed countably compact sets is Urysohn. Then $\mathscr{T}_{CC} = \mathrm{sK}^+$ if and only if every open set is σ-countably compact.*

Proof. Let G be an open set which is not σ-countably compact. Let $\mathcal{C}$ be the family of all the countably compact closed sets contained in G. For each $C \in \mathcal{C}$, pick by the Urysohn condition $D_C \in \mathcal{C}$ with $C \subseteq \mathrm{int}(D_C)$. Next put

$$\mathscr{F} := \{X \setminus \cup_{i\in I} \mathrm{int}(D_{C_i}) : C_i \in \mathcal{C} \text{ and } |I| \leq \aleph_0\}.$$

Since $\cup_{i\in I}\mathrm{int}(D_{C_i}) \subseteq \cup_{i\in I} D_{C_i}$, each member of $\mathscr{F}$ hits G. Arguing as in the proof of Theorem 3.4, the family $\Uparrow \mathscr{F}$ is easily seen to verify the conditions of Theorem 3.1 so it is sK^+-closed. However, $G \in \mathscr{F}^\sharp = (\Uparrow \mathscr{F})^\sharp$ but if C is a countably compact closed subset of G, then $X\setminus\mathrm{int}(D_C) \in \Uparrow \mathscr{F}$ while $(X\setminus\mathrm{int}(D_C)) \cap C = \varnothing$. Therefore, by Proposition 2.3, $\Uparrow \mathscr{F}$ is not $\mathscr{T}_{CC}$-closed, and so $\mathscr{T}_{CC} \neq \mathrm{sK}^+$.

Conversely, assume each open subset is σ-countably compact. Suppose that $\mathscr{F}$ is closed in sK^+ and let G be an open set such that $G \in \mathscr{F}^\sharp$. Since G is σ-countably compact $G = \cup_{n\in\mathbb{N}} C_n$ where C_n is a countably compact closed set. By the Urysohn condition, we can find for each n, D_n both closed and countably compact such that $C_n \subseteq \mathrm{int}(D_n) \subseteq D_n \subseteq G$. Therefore, $\cup_{n\in\mathbb{N}}\mathrm{int}(D_n) \in \mathscr{F}^\sharp$ so by Theorem 3.1 there exists $k \in \mathbb{N}$ such that $\cup_{i=1}^{k} D_i \in \mathscr{F}^\sharp$ and the result follows from Proposition 2.3. $\square$

The last theorem can fail in both directions without the Urysohn condition, as the next two examples show.

Example 3.9. Consider the rationals $\mathbb{Q}$ as a metric subspace of the real line $\mathbb{R}$. Since each subset of $\mathbb{Q}$ is countable, each (open) subset is σ-countably compact. But $\mathscr{T}_C = \mathscr{T}_{CC}$ is properly coarser than sK^+; in fact, $\mathscr{T}_C$ is properly coarser than $\tau\mathrm{K}^+$, i.e., $\mathbb{Q}$ is dissonant [19].

Example 3.10. Now consider the irrationals $\mathbb{R}\backslash\mathbb{Q}$ as a metric subspace of the real line $\mathbb{R}$. Since $\mathbb{R}\backslash\mathbb{Q}$ is second countable, it is hereditarily Lindelöf, and since the space is Polish, it is consonant [19]. Thus we get

$$\mathscr{T}_{CC} = \mathscr{T}_C = \tau\mathrm{K}^+ = \mathrm{sK}^+.$$

But the set of irrationals itself fails to be σ-countably compact, as each compact subset must have empty interior, and so by Baire's theorem, $\mathbb{R}\backslash\mathbb{Q}$ cannot be a countable union of (countably) compact sets.

For completeness, we briefly look at τK^+ versus $\mathscr{T}_{CC}$. These topologies on $\mathscr{C}(X)$ are in general noncomparable: in the dissonant space $\mathbb{Q}$, we have $\mathscr{T}_{CC} = \mathscr{T}_C \subsetneq \tau K^+$, while in the ordinal space $[0, \omega_1)$ where ω_1 is the first uncountable ordinal, we have by local compactness $\mathscr{T}_C = \tau K^+$ whereas $\mathscr{T}_C \subsetneq \mathscr{T}_{CC}$, as the space is countably compact but not compact (see, e.g., [41, pp. 68–70] and Theorem 2.1 *supra*).

Proposition 3.11. *Let X be Hausdorff. Suppose the family of closed countably compact sets is local. Then $\tau K^+ \subseteq \mathscr{T}_{CC}$.*

Proof. Let $\mathscr{F}$ be a closed set with respect to τK^+, and let G be open in X which belongs to the grill of $\mathscr{F}$. For each $x \in X$ choose an open neighborhood V_x of x such that $\mathrm{cl}(V_x)$ is countably compact and $\mathrm{cl}(V_x) \subseteq G$. As $\cup_{x\in G} V_x = G$, by [25, Corollary 3.2], there exists $\{x_1, x_2, \ldots, x_n\}$ with $\cup_{j=1}^n V_{x_j} \in \mathscr{F}^\sharp$. Then $\cup_{j=1}^n \mathrm{cl}(V_{x_j})$ is a countably compact subset of G in the grill of $\mathscr{F}$. $\square$

In particular, we see that with the Urysohn condition, $\tau K^+ \subseteq \mathscr{T}_{CC}$ holds, whereas $\mathrm{sK}^+ \subseteq \mathscr{T}_{CC}$ can fail. Unfortunately, localness is not necessary for even $\tau K^+ = \mathscr{T}_{CC}$; consider again $\mathbb{R}\backslash\mathbb{Q}$.

In the following, we obtain analogues for the convergence $\mathrm{K} = \mathrm{K}^- \vee \mathrm{K}^+$ of some results for the convergence K^+ obtained by Costantini and Vitolo [18] in the context of a metrizable space (see more generally [16, 38]).

As is well-known, in a first countable space and thus in a metrizable space, K agrees with the $\mathscr{T}_F$-convergence for sequences, so $\mathrm{sK} = \mathrm{s}\mathscr{T}_F$. In unpublished notes, Fremlin [30] proved in the context of metrizable spaces that $\tau\mathrm{K} = \mathscr{T}_F$ if and only if X has at most one point that has no compact neighborhood (see more generally [2]). In an arbitrary Hausdorff space, the property $\tau\mathrm{K} = \mathscr{T}_F$ is called *hyperconsonance* [1, 2, 3, 12], adapted from the terminology *consonance* [1, 12, 13, 19, 25, 40] used to describe when $\tau\mathrm{K}^+$ agrees with the co-compact topology. Of course, $\tau\mathrm{K}^- = \mathscr{T}_{LV}$ and it would be natural to guess, assuming the modification operator distributes over join in the space of convergences on $\mathscr{C}(X)$, that

$$\tau\mathrm{K} = \tau(\mathrm{K}^- \vee \mathrm{K}^+) = \tau\mathrm{K}^- \vee \tau\mathrm{K}^+ = \mathscr{T}_{LV} \vee \tau\mathrm{K}^+.$$

But this distributivity fails: it has been proved [17, 25] that for completely metrizable spaces, $\tau\mathrm{K}^+ = \mathscr{T}_C$, so that in this case, $\mathscr{T}_{LV} \vee \tau\mathrm{K}^+$ reduces to Fell topology, independent of local compactness considerations (see also [17, Example 4.4]).

Theorem 3.12. *Let $\langle X, \mathscr{T}\rangle$ be a metrizable space. The following conditions are equivalent:*

(*1*) X *is separable;*
(*2*) τK *is sequential.*

Proof. (*1*) $\Rightarrow$ (*2*). Costantini, Levi and Pelant [17] have shown that

$$\mathrm{K} = \bigwedge\{\mathscr{T}_{W_d} : d \text{ is a metric compatible with } \mathscr{T}\},$$

and by Proposition 2.4, we get $\tau\mathrm{K} = \bigcap\{\mathscr{T}_{W_d} : d$ is a metric compatible with $\mathscr{T}\}$. As is well-known (see, e.g., [6]), each Wijsman topology is metrizable for a separable metrizable space and is thus sequential. Since the intersection of a family of sequential topologies is sequential [18], (*2*) follows from (*1*).

(*2*) $\Rightarrow$ (*1*). Suppose (*1*) fails; let d be a fixed compatible metric. Then we can find for some $\delta > 0$ an uncountable subset E of X such that $x \neq y$ in $E \Rightarrow d(x,y) > \delta$. So by passing to a subset we can write $E = \{x_\alpha : \alpha < \omega_1\}$ where ω_1 is the first uncountable ordinal. Note that by the uniform discreteness of E, for a net of (closed) subsets $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ in E, we have $A = \mathrm{K} - \lim A_\lambda \Rightarrow A = \cap_{\mu\in\Lambda} \cup_{\lambda\succeq\mu} A_\lambda$. Clearly, $\mathscr{A} := \{A \subseteq E : A \text{ is countable}\}$ is closed under taking K-limits of sequences, because if each A_n is countable, so is $\cap_{n=1}^\infty \cup_{k=n}^\infty A_k$. On the other hand, if for each ordinal $\alpha < \omega_1$ we put $B_\alpha := \{x_\beta : \beta < \alpha\} \in \mathscr{A}$, then $E = \mathrm{K} - \lim B_\alpha$. Since $E \notin \mathscr{A}$, it follows that τK is not sequential. □

Theorem 3.13. *Let $\langle X, \mathscr{T}\rangle$ be a metrizable space. Then $\mathscr{T}_F$ is sequential if and only if X is separable and X has at most one point having no compact neighborhood.*

Proof. For sufficiency, by Fremlin's theorem, the condition regarding points of non-local compactness is equivalent to $\tau\mathrm{K} = \mathscr{T}_F$, so by the last theorem, separability of X ensures that $\mathscr{T}_F$ is sequential.

Conversely, if $\mathscr{T}_F$ is sequential, then $\mathrm{s}\mathscr{T}_F = \mathscr{T}_F$. Using the fact that a sequence in a first countable space is K-convergent if and only if it is $\mathscr{T}_F$-convergent, and that τK is the strongest topology coarser than K-convergence, we obtain

$$\mathrm{s}\tau\mathrm{K} \leq \mathrm{sK} = \mathrm{s}\mathscr{T}_F = \mathscr{T}_F \leq \tau\mathrm{K}.$$

As in general $\mathrm{s}\tau\mathrm{K} \geq \tau\mathrm{K}$, all inequalities become equalities. In particular, τK is sequential, whence X is separable, and $\mathscr{T}_F = \tau\mathrm{K}$, whence at most one point of X fails to have a compact neighborhood. □

Example 3.14. Our favorite nontrivial example of a metric space for which $\mathscr{T}_F$ is sequential is the following metric subspace of the sequence space ℓ_2:

$$\left\{\frac{1}{j}e_n : j \in \mathbb{N}, n \in \mathbb{N}\right\} \cup \{\theta\},$$

where θ is the origin of the space and $e_1, e_2, e_3, \ldots$ is the standard orthonormal basis. This space is also an example of a nonlocally compact space on which each continuous function with values in an arbitrary metric space is uniformly continuous [6].

We next give a curious characterization of those regular Hausdorff spaces $\langle X, \mathscr{T}\rangle$ that satisfy the conditions of the last theorem.

Theorem 3.15. *Let $\langle X, \mathscr{T}\rangle$ be a regular Hausdorff space. The following conditions are equivalent:*

(*1*) *X is separable and metrizable and X has at most one point having no compact neighborhood;*

(*2*) *for some point $p \in X$, $\mathscr{C}(X\backslash\{p\})$ equipped with the Fell topology is metrizable and $\mathscr{T}$ has a countable local base at p.*

Proof. (*1*) $\Rightarrow$ (*2*). Assume (*1*) holds. Choose $p \in X$ such that $\forall x \in X\backslash\{p\}$, x has a compact neighborhood in X. Clearly, each point of $X\backslash\{p\}$ must have a compact neighborhood in the relative topology. Also, the relative topology is second countable. It is well-known that the Fell topology for such a space is compact, Hausdorff, and second countable and thus is metrizable [6, 35].

(*2*) $\Rightarrow$ (*1*). Assume (*2*) holds. Since the Fell topology for $\mathscr{C}(X\backslash\{p\})$ is metrizable, as it is always compact, it must be second countable [5, 6]. Now $x \to \{x\}$ is an embedding of $X\backslash\{p\}$ with the relative topology into $\langle\mathscr{C}(X\backslash\{p\}), \mathscr{T}_F\rangle$. As a result, $\langle X\backslash\{p\}, \mathscr{T}\rangle$ is second countable and Tychonoff, and so by the Urysohn Metrization Theorem [44], it is metrizable. But any one-point regular extension of a metrizable space where the ideal point has a countable local base is metrizable (see, e.g., [8, Theorem 4]), and since $X\backslash\{p\}$ is separable, so is the extension. Finally, since the Fell topology on $\mathscr{C}(X\backslash\{p\})$ being assumed metrizable is Hausdorff, $X\backslash\{p\}$ is locally compact in the relative topology, and thus each point of $X\backslash\{p\}$ has a compact neighborhood with respect to $\langle X, \mathscr{T}\rangle$ because $\{p\}$ is closed. $\square$

We note that any space that satisfies these conditions must also be Polish, as any one-point metrizable extension of a Polish space space is Polish [7], and each locally compact separable metrizable space is Polish (in fact, there is a compatible metric for which closed and bounded sets are compact).

4. Hit-and-miss topologies compatible with sequential K-convergence

It is the purpose of this section to display miss topologies that are compatible with upper Kuratowski-Painlevé convergence for sequences of closed sets. Taking the supremum of such a topology with the lower Vietoris topology yields topologies that are sequentially equivalent with Kuratowski-Painlevé convergence. We begin giving the largest topology sequentially coarser than K^+.

Proposition 4.1. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff topological space. Then the co-countably compact topology $\mathscr{T}_{CC}$ is the finest miss topology sequentially coarser than* K^+.

Proof. From Proposition 3.6 and $\mathrm{sK}^+ \leq_{\mathrm{seq}} \mathrm{K}^+$, we have $\mathscr{T}_{CC} \leq_{\mathrm{seq}} \mathrm{K}^+$. By Theorem 2.1 each miss topology determined by a co-countably compact cobase will be coarser than $\mathscr{T}_{CC}$, and thus sequentially coarser than K^+. We show that if Δ is a cobase that contains a non-countably compact closed set F, then it fails to be sequentially coarser than K^+. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence of distinct points in F with no accumulation point (see, e.g., [44, p. 125]). It is obvious that $\langle\{x_n\}\rangle_{n\in\mathbb{N}}$ is K^+-convergent to $\varnothing$ but it is not $\mathscr{T}_\Delta^+$-convergent to $\varnothing$. $\square$

Remark 4.2. The proof of the last proposition apparently shows something more, namely that $\mathscr{T}_{CC}$ is the finest miss topology sequentially coarser than sK^+. But we know of no example showing that sK^+ can be properly sequentially coarser than K^+.

Remark 4.3. In the same manner we can see that the co-compact topology $\mathscr{T}_C$ is the finest miss topology coarser than $\tau\mathrm{K}$.

The above result shows that if $\mathscr{T}_\Delta^+ \approx_{\text{seq}} \mathrm{K}^+$, then the sets of the cobase Δ must be countably compact. Since $\mathscr{T}_{CC} \leq_{\text{seq}} \mathrm{K}^+$, if Δ is a countably compact cobase then $\mathscr{T}_\Delta^+ \approx_{\text{seq}} \mathrm{K}^+$ if and only if $\mathrm{K}^+ \leq_{\text{seq}} \mathscr{T}_\Delta^+$.

Theorem 4.4 (cf. [16, Theorem 1.8]). *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff topological space and Δ a cobase. The following conditions are equivalent:*

(*1*) $\mathrm{K}^+ \approx_{seq} \mathscr{T}_\Delta^+$;

(*2*) *Δ is a countably compact cobase, and whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of closed sets with $x \in Ls\ A_n$, every neighbourhood G of x contains some $D \in \Delta$ that intersects infinitely many A_n.*

Proof. (*2*) $\Rightarrow$ (*1*). Suppose $\langle A_n\rangle_{n\in\mathbb{N}}$ converges to A in $\mathscr{T}_\Delta^+$. We must show that $\mathrm{Ls}\ A_n \subseteq A$. Suppose to the contrary $x \in \mathrm{Ls}\ A_n \backslash A$. By assumption, $X\backslash A$ must contain $D \in \Delta$ that hits infinitely many A_n. Then $\{F \in \mathscr{C}(X) : F \subseteq X\backslash D\}$ is a $\mathscr{T}_\Delta^+$-neighborhood of A that fails to contain A_n eventually, and we have a contradiction.

(*1*) $\Rightarrow$ (*2*). We already know that the cobase must be countably compact. For the other condition, we argue exactly as in [16]. If $x \in \mathrm{Ls}\ A_n$ and G is a neighborhood of x, then $\langle A_n\rangle_{n\in\mathbb{N}}$ is not K^+-convergent and hence not $\mathscr{T}_\Delta^+$-convergent to $X\backslash G$, and so there must exist $D \in \Delta$ disjoint from $X\backslash G$ but that hits infinitely many A_n. □

Remark 4.5. The proof of the last proposition shows that the second part of (*2*) is both necessary and sufficient for $\mathrm{K}^+ \leq_{seq} \mathscr{T}_\Delta^+$ for an arbitrary cobase Δ.

Example 4.6. In any (real) infinite dimensional Banach space X equipped with the weak topology, the countably co-compact topology on $\mathscr{C}(X)$ fails to be sequentially equivalent to K^+. Thus, there can be no miss topology in this setting compatible with upper Kuratowski-Painlevé convergence of sequences of sets. Of course, by the Eberlein-Šmulian Theorem, for a weakly closed subset, compactness, countable compactness, and sequential compactness all agree (see, e.g. [32]), so $\mathscr{T}_C = \mathscr{T}_{CC}$.

We will employ the above characterization theorem to verify our claim.

Let W be a closed separable infinite dimensional linear subspace of X. By a theorem of Kadets [34], we can find a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ having the origin θ as a weak cluster point with respect to the weak topology of W, but such that $\lim_{n\to\infty} ||x_n|| = \infty$ (in fact, by recent results of Aron, García, and Maestre [4], we can construct the sequence to be weakly dense in W!). By the Hahn-Banach Theorem, the weak topology of W is the relative weak topology, so in the space X equipped with the weak topology, we have $\theta \in \mathrm{Ls}\ \{x_n\}_{n\in\mathbb{N}}$. But $\langle x_n\rangle_{n\in\mathbb{N}}$ eventually lies outside each (closed countably) compact set, as by the uniform boundedness principle, each such set is norm bounded. Thus, condition (*2*) of the above theorem fails.

Our next goal is to show that in a first countable Hausdorff space, where the closed countably compact subsets reduce to the sequentially compact subsets, there is in general a rich supply of cobases satisfying the conditions of the last result.

Proposition 4.7. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff first countable topological space and Δ a cobase. Then $\mathrm{K}^+ \leq_{seq} \mathscr{T}^+_\Delta$ if and only if whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent to x and $x \in G \in \mathscr{T}$ there exists $F \in \Delta$ with $F \subseteq G$ such that F contains a subsequence of $\langle x_n\rangle_{n\in\mathbb{N}}$.*

Proof. For necessity, suppose that the sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to x but we can find $x \in G \in \mathscr{T}$ such that for all $F \in \Delta$ with $F \subseteq G$ then $x_n \in X\backslash F$ residually. This means that $\langle\{x_n\}\rangle_{n\in\mathbb{N}}$ is $\mathscr{T}^+_\Delta$-convergent to $X\backslash G$. However $\mathrm{Ls}\{x_n\} \nsubseteq X\backslash G$ since $x \in \mathrm{Ls}\{x_n\}$.

For sufficiency, we can apply Remark 4.5, for if $x \in \mathrm{Ls}\, A_n$, then $\exists n_1 < n_2 < n_3 < \cdots$ and $x_{n_k} \in A_{n_k}$ with $\langle x_{n_k}\rangle_{k\in\mathbb{N}}$ convergent to x [6, Lemma 5.2.8]. □

Example 12 in [33] shows that the first countability assumption in the above theorem is essential for proving the sufficiency. Combining Proposition 4.1 and the last result, we get this refinement of Theorem 4.4.

Theorem 4.8. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff first countable topological space and Δ a cobase. Then $\mathscr{T}^+_\Delta$ is sequentially equivalent to K^+ if and only if Δ is countably compact and whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent to x and $x \in G \in \mathscr{T}$ there exists $F \in \Delta$ with $F \subseteq G$ such that F contains a subsequence of $\langle x_n\rangle_{n\in\mathbb{N}}$.*

The next result says that we can extend a cobase on a closed subset E of X that induces a miss topology sequentially equivalent to K^+ restricted to E to a cobase for X with the same property by adding certain compact subsets and taking the generated cobase.

Proposition 4.9. *Let E be a nonempty closed subset of a Hausdorff first countable topological space $\langle X, \mathscr{T}\rangle$, and suppose $\triangle_E$ is a cobase of (closed countably compact) subsets of E such that $\mathrm{K}^+ \approx_{seq} \mathscr{T}^+_{\triangle_E}$ on E. Let $\triangle = \Sigma(\triangle_E \cup \{\widehat{\alpha} : \alpha \in seq(X)$ and $\forall n \in \mathbb{N}, \alpha(n) \notin E\})$. Then $\mathscr{T}^+_\triangle \approx_{seq} \mathrm{K}^+$ on X.*

Proof. Since the cobase for X consists of countably compact sets, we only need to show that $\mathrm{K}^+ \leq_{\mathrm{seq}} \mathscr{T}^+_\triangle$. To this end, suppose $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X convergent to x in X. We consider two cases: (i) $\langle x_n\rangle_{n\in\mathbb{N}}$ is frequently in E; (ii) $\langle x_n\rangle_{n\in\mathbb{N}}$ is frequently in $X\backslash E$. In the first case, $x \in E$, and if G is an open neighborhood of x, then $G \cap E$ contains some element of $\triangle_E$ containing some subsequence of $\langle x_n\rangle_{n\in\mathbb{N}}$ because $\mathrm{K}^+ \leq_{\mathrm{seq}} \mathscr{T}^+_{\triangle_E}$. In the second case, if G is a open neighborhood of x, then G contains some subsequence α of $\langle x_n\rangle_{n\in\mathbb{N}}$ lying in $X\backslash E$ and thus contains $\widehat{\alpha}$. □

Definition 4.10. Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space. By a *subsequential selector* for $\mathrm{seq}(X)$, we mean a function $f : \mathrm{seq}(X) \to \mathrm{seq}(X)$ such that $\forall\alpha \in \mathrm{seq}(X)$, $f(\alpha)$ is a subsequence of α.

Given a subsequential selector f in a Hausdorff topological space, put $\Delta^f := \Sigma(\{\widehat{f(\alpha)} : \alpha \in \mathrm{seq}(X)\})$. Since the complete range of a convergent sequence is compact and constant sequences are convergent, we see that Δ^f is a compact cobase. Note that if f is the identity function on $\mathrm{seq}(X)$, then we get $\Delta^f = \Delta^{\mathrm{seq}}$.

The next result is a direct consequence of Theorem 4.8.

Proposition 4.11. *Let $\langle X, \mathscr{T}\rangle$ be a first countable Hausdorff topological space and let f be a subsequential selector. Then $\mathscr{T}^+_{\Delta^f}$ is sequentially equivalent to* K^+.

In view of Proposition 4.1, since Δ^{seq} is a compact cobase, we get

Corollary 4.12. *Let $\langle X, \mathscr{T}\rangle$ be a first countable Hausdorff topological space. Then $\mathscr{T}_{\Delta^{seq}}$, $\mathscr{T}_C$, $\mathscr{T}_{CC}$ and K^+ are all sequentially equivalent.*

Theorem 4.13. *Let $\langle X, \mathscr{T}\rangle$ be a first countable Hausdorff topological space and suppose Δ is a countably compact cobase such that $\Downarrow \Delta = \Delta \cup \{\varnothing\}$. Then $\mathscr{T}^+_\Delta$ is sequentially equivalent to K^+ if and only if there exists a subsequential selector f for which $\Delta^f \subseteq \Downarrow \Delta$.*

Proof. Suppose first that $\mathscr{T}^+_\Delta \approx_{seq} \mathrm{K}^+$, and $\alpha \in \mathrm{seq}(X)$ is convergent to x. By Theorem 4.8, X itself contains some $D_\alpha \in \Delta$ that in turn contains a subsequence of α which we denote by $f(\alpha)$. As

$$\widehat{f(\alpha)} \subseteq \{x\} \cup D_\alpha \in \Delta,$$

we have $\widehat{f(\alpha)} \in \Downarrow \Delta$. Since $\alpha \in \mathrm{seq}(X)$ was arbitrary, this proves $\Delta^f \subseteq \Downarrow \Delta$.

Conversely, suppose for some subsequential selector that $\Delta^f \subseteq \Downarrow \Delta$. Then actually $\Delta^f \subseteq \Delta$ and by Proposition 4.11,

$$\mathscr{T}^+_{\Delta^f} \leq_{seq} \mathscr{T}^+_\Delta \leq_{seq} \mathrm{K}^+ \approx_{seq} \mathscr{T}^+_{\Delta^f},$$

and so $\mathscr{T}^+_\Delta \approx_{seq} \mathrm{K}^+$. $\square$

The following example shows that in the previous theorem, we cannot delete the assumption of considering a cobase stable under closed subsets.

Example 4.14. Consider $X = [0,1] \times [0,1]$, equipped with the cobase Δ generated by $\{[0, \frac{1}{2}] \times [0,1], [\frac{1}{2}, 1] \times [0,1]\}$. Thus, a set B is in the cobase if and only if B satisfies one of these four conditions: (i) $B = X$; (ii) B is a nonempty finite subset of X; (iii) $B = [0, \frac{1}{2}] \times [0,1] \cup F$ where F is finite; (iv) $B = [\frac{1}{2}, 1] \times [0,1] \cup F$ where F is finite. Notice that Δ is not stable under taking nonempty closed subsets of its members. Since $X \in \Delta$, we in fact have $\Delta^{\mathrm{seq}} \subseteq \Downarrow \Delta$. Evidently the sequence of segments with nth term $A_n = \{(\frac{n}{2n+1}, y) : 0 \leq y \leq 1\}$ is $\mathscr{T}^+_\Delta$-convergent to $\{(\frac{1}{2}, \frac{1}{2})\}$ because if the singleton failed to hit a member B of the cobase, then B must be a nonempty finite set. On the other hand, it is clear that the sequence is only upper Kuratowski-Painlevé convergent to supersets of $\{(\frac{1}{2}, y) : 0 \leq y \leq 1\}$.

The next result shows that in the case that X has some convergent sequence with distinct terms, there is no minimal topology of the form $\mathscr{T}^+_{\Delta^f}$.

Proposition 4.15. *Let $\langle X, \mathscr{T}\rangle$ be a first countable Hausdorff topological space such that X' is nonempty and let f be a subsequential selector. Then there exists a subsequential selector g such that $\mathscr{T}^+_{\triangle^g}$ is strictly coarser than $\mathscr{T}^+_{\triangle^f}$.*

Proof. There exists a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in the range of f with distinct terms convergent to some $p \in X$. To form our new subsequential selector g, we consider two cases for $f(\alpha)$ where $\alpha \in \operatorname{seq}(X)$:

(1) $\widehat{f(\alpha)} \cap \{x_n : n \in \mathbb{N}\}$ is finite;

(2) $\widehat{f(\alpha)} \cap \{x_n : n \in \mathbb{N}\}$ is infinite.

In the first case, put $g(\alpha) = f(\alpha)$. In the second case, let $\widetilde{\alpha}$ be a subsequence of α with distinct terms containing infinitely many x_n yet for each $n \in \mathbb{N}$ at most one term from $\{x_k : 10^{n-1} \leq k < 10^n\}$. We then put $g(\alpha) = f(\widetilde{\alpha})$. By construction, $\triangle^g \subseteq \triangle^f$ which ensures that $\mathscr{T}^+_{\triangle^g}$ is coarser than $\mathscr{T}^+_{\triangle^f}$. That it is strictly coarser follows from the fact that $\{p, x_1, x_2, x_3, \ldots\} \in \Delta^f$ cannot be residually covered by a finite union of completed ranges of the form $\widehat{g(\alpha)}$ due to the sparseness of the sequences $g(\alpha)$ whose terms overlap the terms of $\langle x_n\rangle_{n\in\mathbb{N}}$ cofinally. By Theorem 2.1, $\mathscr{T}^+_{\triangle^g}$ is a proper subset of $\mathscr{T}^+_{\triangle^f}$. □

Theorem 4.16. *Let $\langle X, \mathscr{T}\rangle$ be a first countable Hausdorff topological space. The following conditions are equivalent:*

(1) $X' \neq \varnothing$;

(2) there exists a miss topology strictly coarser than the co-countably compact topology sequentially equivalent to K$^+$*-convergence;*

(3) there is an infinite descending chain of miss topologies each sequentially equivalent to K$^+$*-convergence.*

Proof. *(1)* ⇒ *(3)*. This is an immediate consequence of Propositions 4.11 and 4.15.

(3) ⇒ *(2)*. This follows from the fact that each cobase that produces a miss topology sequentially equivalent to K$^+$-convergence must be countably compact (see Proposition 4.1 *supra*).

(2) ⇒ *(1)*. Suppose that *(1)* fails, i.e., that X has no limit points. Then the only closed countably compact sets are the finite sets, and thus the only countably compact cobase is $\mathscr{F}_0(X)$. Thus *(2)* fails. □

Remark 4.17. From the above results, we can deduce that the family of all miss topologies sequentially equivalent to the upper Kuratowski-Painlevé convergence which are determined by a cobase Δ satisfying $\Downarrow \Delta = \Delta \cup \{\varnothing\}$ has a minimum if and only if $X' = \varnothing$. In this case, the minimum topology is $\mathscr{T}^+_{\mathscr{F}_0(X)}$.

The identity subsequential selector produces the cobase Δ^{seq} which contains Δ^f for any subsequential selector f. As a result, $\mathscr{T}^+_{\Delta^{\text{seq}}}$ is the largest topology of the form $\mathscr{T}^+_{\Delta^f}$. In a metric space, $\mathscr{T}_C = \mathscr{T}_{CC}$ as the countably compact sets reduce to the compact sets. The next result in this setting answers the question: when does $\mathscr{T}^+_{\Delta^{\text{seq}}}$ agree with the co-compact topology?

Theorem 4.18. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent:*

(*1*) *There exists a compact subset C of X whose set of limit points C' is infinite;*
(*2*) $\Delta^{seq} \neq \mathscr{K}_0(X)$;
(*3*) $\mathscr{T}^+_{\Delta^{seq}} \neq \mathscr{T}_C$;
(*4*) *there exist uncountably many miss topologies between $\mathscr{T}^+_{\Delta^{seq}}$ and $\mathscr{T}_C$ each sequentially equivalent to the upper Kuratowski-Painlevé convergence.*

Proof. (*1*) $\Rightarrow$ (*4*). Assuming (*1*), by the compactness of C', we can find a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in C' with distinct terms convergent to some $p \notin \{x_n : n \in \mathbb{N}\}$. Now for each $j \in \mathbb{N}$ the set $\{x_n : n \neq j\} \cup \{p\}$ is compact so we can find $\delta_j > 0$ such that $\forall j$

$$S_{2\delta_j}(x_j) \cap (\{x_n : n \neq j\} \cup \{p\}) = \varnothing.$$

Note that the balls $S_{\delta_j}(x_j)$ we have constructed are pairwise disjoint and that by the convergence of $\langle x_n\rangle_{n\in\mathbb{N}}$ we have $\lim_{j\to\infty}\delta_j = 0$.

Next, $\forall j \in \mathbb{N}$, let $\widehat{\alpha_j}$ be the complete range of a sequence in C with distinct terms inside $S_{\delta_j}(x_j)$ convergent to x_j. Define an equivalence relation $\equiv$ on the infinite subsets of $\mathbb{N}$ as follows:

$$N \equiv M \ \text{ provided } (N\backslash M) \cup (M\backslash N) \text{ is a finite set.}$$

As each equivalence class is countable, if $\{N_i : i \text{ in } I\}$ selects a representative from each, then the index set I must be uncountable. For each index $i \in I$, put

$$B_i := \{p\} \cup \ \cup_{k\in N_i}\widehat{\alpha_k},$$

and then let Δ_i be the compact cobase generated by $\Delta^{\text{seq}} \cup \{B_i\}$. If i_1 and i_2 are distinct indices in I, then it is clear that $B_{i_1} \notin\Downarrow \Delta_{i_2}$. By Theorem 2.1, they determine distinct miss topologies on X. As each is trapped between $\mathscr{T}^+_{\Delta^{\text{seq}}}$ and $\mathscr{T}_C$, each is sequentially equivalent to upper Kuratowski-Painlevé convergence.

(*4*) $\Rightarrow$ (*3*). This is trivial.

(*3*) $\Rightarrow$ (*2*). This follows from $\mathscr{T}_C := \mathscr{T}^+_{\mathscr{K}_0(X)}$.

(*2*) $\Rightarrow$ (*1*). Suppose (*1*) fails, that is, suppose each $C \in \mathscr{K}_0(X)$ has at most finitely many limit points. Let C be a nonempty compact subset of X. We intend to show that $C \in \Delta^{\text{seq}}$.

If C has no limit points whatsoever, then C must be finite and is obviously the complete range of a convergent sequence. Otherwise let $\{c_1, c_2, \ldots, c_n\}$ be the limit points of C. Choose $\varepsilon > 0$ such that $\{S_\varepsilon(c_j) : j = 1, 2, \cdots, n\}$ is a pairwise disjoint family of balls. By the compactness of $C\backslash \cup_{j=1}^n S_\varepsilon(c_j)$ and the location of the limit points of C, this set must be finite. Now for each $j \leq n$ and $k \in \mathbb{N}$, similar considerations ensure that the annulus

$$A_{j,k} := \left\{x \in C : \frac{1}{k+1}\varepsilon \leq d(x, c_j) < \frac{1}{k}\varepsilon\right\}$$

must be finite. As a result, $\forall j \leq n, \{x \in C : 0 < d(x, c_j) < \varepsilon\}$ can be listed in a manner to produce a sequence convergent to c_j. As a result,

$$C = \left(C\backslash \bigcup_{j=1}^{n} S_\varepsilon(c_j)\right) \cup \left(C \cap \bigcup_{j=1}^{n} S_\varepsilon(c_j)\right)$$

belongs to Δ^{seq}, and so $\Delta^{\mathrm{seq}} = \mathscr{K}_0(X)$. $\square$

5. On Mrowka's Theorem

More than 30 years ago S. Mrowka [37] produced a short, elegant proof of compactness of Kuratowski-Painlevé convergence in an arbitrary topological space $\langle X, \mathscr{T}\rangle$ as an application of the Tychonoff Theorem. Since $\mathscr{T}_F \leq \mathrm{K}$ [6], this of course yields as a corollary compactness of the Fell topology, which can be proved directly by using the Alexander subbase theorem [5, 6, 35]. That said, Mrowka's construction is high-level legerdemain: while it produces a convergent subnet of a given net of closed sets, the limit of the subnet is nowhere to be found! In this section we modify his proof so that the limit of the convergent subnet is explicit. Effectively this involves embedding $\mathscr{C}(X)$ in a compact Hausdorff space producing a Tychonoff topology on $\mathscr{C}(X)$ that is stronger than Kuratowski-Painlevé convergence. We show this topology is properly stronger unless X is a finite set. In particular, it does not collapse to the Fell topology when X is in addition locally compact. This product involves copies of $\{0,1\}$ equipped with the discrete topology. Our index set Γ is the subset of $\mathscr{T} \times X$ defined as follows:

$$\Gamma := \{(V, x) : x \in V \in \mathscr{T}\}.$$

For each closed subset A we define $f_A \in \{0,1\}^\Gamma$ by

$$f_A(V,x) = \begin{cases} 1 & \text{if } A \cap V \neq \varnothing \\ 0 & \text{if } A \cap V = \varnothing. \end{cases}$$

It is easy to see that the assignment is one-to-one, for if A_1 and A_2 are closed sets and $x \in A_1 \backslash A_2$, then $f_{A_1}(X\backslash A_2, x) = 1$ while $f_{A_2}(X\backslash A_2, x) = 0$.

Proposition 5.1. *Let $\langle X, \mathscr{T}\rangle$ be an arbitrary topological space and let A be a closed subset. Then $A = \{x \in X : \text{ whenever } (V,x) \in \Gamma,\ f_A(V,x) = 1\}$.*

Proof. Suppose $a \in A$ is arbitrary. Then for each open neighborhood V of a we have $A \cap V \neq \varnothing$ and so $f_A(V,a) = 1$. For the reverse inclusion, suppose $x \notin A$. Then $f_A(X\backslash A, x) = 0$. $\square$

Theorem 5.2. *Let $\langle X, \mathscr{T}\rangle$ be a topological space and let $\langle A_\lambda\rangle$ be a net of closed subsets of X. Suppose that the associated net $\langle f_{A_\lambda}\rangle$ converges in the product topology to f. Then $\langle A_\lambda\rangle$ is* K*-convergent to $\{a \in X : \text{ whenever } (V,a) \in \Gamma,\ f(V,a) = 1\}$.*

Proof. First put $A := \{a \in X : \text{ whenever } (V,a) \in \Gamma,\ f(V,a) = 1\}$, which could be empty. We first show that A is closed. To this end, suppose $x \notin A$. Choose $V \in \mathscr{T}$ with $f(V,x) = 0$. Then eventually $f_{A_\lambda}(V,x) = 0$ which means that eventually $\forall v \in V,\ f_{A_\lambda}(V,v) = 0$, and so $f(V,v) = 0$. This shows that $V \cap A = \varnothing$, as required.

We next show that $A \subseteq \mathrm{Li}\ A_\lambda$ and $\mathrm{Ls}\ A_\lambda \subseteq A$. For the first inclusion, let $a \in A$ and let V be an arbitrary open neighborhood of a. Since $f(V,a) = 1$, eventually $f_{A_\lambda}(V,a) = 1$ which means eventually that $A_\lambda \cap V \neq \varnothing$. This shows $a \in \mathrm{Li}\ A_\lambda$. On the other hand let $x \in \mathrm{Ls}\ A_\lambda$ be arbitrary. We must show that whenever $(V,x) \in \Gamma,$ we have $f(V,x) = 1$. So suppose $x \in V \in \mathscr{T}$. Since $A_\lambda \cap V \neq \varnothing$ frequently, and since $\lim f_{A_\lambda}(V,x) = f(V,x)$, we conclude $f(V,x) = 1$. Thus, $x \in A$, completing the proof. $\square$

Corollary 5.3. *Let $\langle X, \mathscr{T}\rangle$ be a topological space and let $\langle A_\lambda\rangle$ be a net of closed subsets of X. Then $\langle A_\lambda\rangle$ has a subnet Kuratowski-Painlevé convergent to a closed subset A.*

Proof. This is immediate from the last theorem and the Tychonoff Theorem. □

Applying Proposition 5.1 we obtain

Corollary 5.4. *Let $\langle X, \mathscr{T}\rangle$ be a topological space and let $\langle A_\lambda\rangle$ be a net of closed subsets of X. Suppose $\langle A_\lambda\rangle$ converges to A in the product topology under the identification $A \leftrightarrow f_A$. Then $A = \mathrm{K} - lim\ A_\lambda$.*

While the set A in the proof is determined by the limit function f for $\langle f_{A_\lambda}\rangle$, it is not in general true that $f = f_A$. Put differently, $\mathscr{C}(X)$ need not be closed in $\{0,1\}^\Gamma$ under the identification $A \leftrightarrow f_A$. We intend to show that if $\langle X, \mathscr{T}\rangle$ is a Hausdorff space with infinitely many points, then there is a sequence $\langle A_n\rangle$ in $\mathscr{C}(X)$ for which $\langle f_{A_n}\rangle$ is convergent in the product to a limit that is not f_A for any closed set A.

The next fact immediately follows from $A \subseteq B \Rightarrow f_A \leq f_B$.

Lemma 5.5. *Let $\langle X, \mathscr{T}\rangle$ be an arbitrary topological space and let $\langle A_n\rangle$ be a decreasing sequence of closed subsets of X. Then $\langle f_{A_n}\rangle$ is convergent in $\{0,1\}^\Gamma$.*

Lemma 5.6. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space with infinitely many points. Then there is a strictly decreasing sequence $\langle A_n\rangle$ of nonempty closed subsets of X.*

Proof. Let $\langle x_n\rangle$ be a sequence in X with distinct terms. We consider two cases: (i) $\langle x_n\rangle$ has a convergent subsequence to a (unique) point $p \in X$; (ii) $\langle x_n\rangle$ has no convergent subsequence. In case (i), by passing to a subsequence, we can assume that no term of the sequence is p and the sequence converges to p. For each $n \in \mathbb{N}$, put $A_n := \{p\} \cup \{x_k : k \geq n\}$, a compact set. Since the topology is Hausdorff, each A_n is closed, and the desired sequence is $\langle A_n\rangle$. In case (ii), put $b_1 = x_1$, and let V_1 be an open neighborhood of b_1 for which $X\backslash V_1$ contains an infinite set of terms B_1 of the original sequence. Let b_2 be that term of B_1 of lowest index, and choose an open neighborhood V_2 of b_2 such that $B_1\backslash V_2$ is an infinite set B_2. Producing additional b_n and then V_n in this manner, the desired strictly decreasing sequence of closed sets is $X\backslash V_1, X\backslash(V_1 \cup V_2), X\backslash(V_1 \cup V_2 \cup V_3), \dots$. □

We note that the assertion of last lemma can fail in a general T_1 space: consider any infinite set equipped with the cofinite topology.

Theorem 5.7. *Let $\langle X, \mathscr{T}\rangle$ be a Hausdorff space. The following conditions are equivalent:*

(*1*) *X is a finite set;*
(*2*) *The topology that $\mathscr{C}(X)$ inherits from $\{0,1\}^\Gamma$ under the identification $A \leftrightarrow f_A$ agrees with the Fell topology;*
(*3*) *Kuratowski-Painlevé convergence of each net $\langle A_\lambda\rangle$ in $\mathscr{C}(X)$ to a closed set A ensures the convergence of $\langle f_{A_\lambda}\rangle$ to f_A in $\{0,1\}^\Gamma$;*
(*4*) *$\mathscr{C}(X)$ is closed in $\{0,1\}^\Gamma$ under the identification $A \leftrightarrow f_A$.*

Proof. *(1)* $\Rightarrow$ *(2)*. As the Fell topology is always coarser than the relative product topology, we need only show that each subbasic open set in the relative product topology is open in the Fell topology. Since $\{\{0\},\{1\}\}$ is a basis for the topology of $\{0,1\}$, a subbase for the relative product topology consists of all sets of the form $\{f_A : f_A(V,x) = 1\}$ and $\{f_A : f_A(V,x) = 0\}$ where (V,x) runs over the index set Γ. Evidently,

$$\{A : f_A(V,x) = 1\} = \{A : A \cap V \neq \varnothing\}$$

and

$$\{A : f_A(V,x) = 0\} = \{A : A \cap V = \varnothing\},$$

and since each open subset of X is compact, these both lie in the Fell topology.

(2) $\Rightarrow$ *(3)*. By *(2)*, convergence in the Fell topology ensures convergence in the product topology, and since K-convergence is of intermediate strength, condition *(3)* immediately follows.

(3) $\Rightarrow$ *(4)*. Suppose $\langle f_{A_\lambda}\rangle$ is convergent in the product topology to some f. We know that the net $\langle A_\lambda\rangle$ is Kuratowski-Painlevé convergent to $A := \{a \in X :$ whenever $(V,a) \in \Gamma,\ f(V,a) = 1\}$. By *(3)*, $\langle f_{A_\lambda}\rangle$ is convergent in the product topology to f_A. Since the product topology is Hausdorff, limits are unique, and we get $f = f_A$. This proves that $\{f_A : A \in \mathscr{C}(X)\}$ is closed in the product.

(4) $\Rightarrow$ *(1)*. If *(1)* fails, let $\langle A_n\rangle$ be a strictly decreasing sequence of nonempty closed subsets of X. Then $\langle f_{A_n}\rangle$ converges in the product topology to some f. Now if $f = f_A$ for some closed A, then $A = \cap_{n=1}^{\infty} A_n$ must hold by Corollary 5.4 and uniqueness of K-limits. Let $p \in A_1\backslash A$ be arbitrary; then $f_A(X\backslash A, p) = 0$ while $\forall n \in \mathbb{N},\ f_{A_n}(X\backslash A, p) = 1$. Thus *(4)* fails. $\square$

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