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Guarding a Line Segment

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A line segment on the plane is guarded by the defender **D**. The invader **I** wants to pass through the line segment but he has to keep the distance from the defender no less than a given constant $\varrho > 0$. The defender can move on the whole plane with maximal speed 1. The invader can move on the whole plane with maximal speed θ, greater than 1. No further kinematic or dynamic constraints are imposed on the defender and the invader motions. The maximal length of the line segment which can be guarded by the defender is established in this paper.

Keywords: Differential games, Strategies, Guarding territory problems

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1. Introduction

Initial description of the game. We consider a guarding territory game on the plane $\mathbb{R}^2$ with two players: the defender **D** and the faster invader **I.** Both players can move on the whole plane $\mathbb{R}^2$. The defender guards a line segment Δ of the form

$$\Delta(\Lambda) = [-\Lambda, \Lambda] \times \{0\} \subset \mathbb{R}^2,$$

where $\Lambda > 0$, and he is equipped in an arm with destruction radius $\varrho > 0$. In this connection invader **I** wins the game if he can reach the line segment $\Delta(\Lambda)$ without being captured, i.e. keeping the distance to the defender no less than ϱ, for all $t \geq 0$. The defender **D** wins the game when he approaches the invader closely than ϱ before **I** enters the line segment guarded or **I** does not enter $\Delta(\Lambda)$ at all. Both the invader and the defender know each other's position.

Trajectories. Suppose that players **D**, **I** are located at the initial positions $a, b \in \mathbb{R}^2$ respectively. Let $\|\cdot\|$ stand for the Euclidean norm in $\mathbb{R}^2$. Admissible trajectory of the defender is then represented by a function $x : [0, \infty) \to \mathbb{R}^2$, with the initial condition

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$x(0) = a$, and satisfying the Lipschitz condition with constant 1. We shall denote by $X(a)$ the set of all admissible defender's trajectories satisfying the initial condition $x(0) = a$. Analogously, we shall denote by $Y(b)$ the set of all $y : [0, \infty) \to \mathbb{R}^2$, with the initial condition $y(0) = b$, satisfying the Lipschitz condition with a fixed constant $\theta > 1$. Each $y \in Y(b)$ represents an admissible trajectory of the invader $\mathbf{I}$ at the initial position b. By the known Rademacher Theorem trajectories of both players are differentiable almost everywhere in $[0, \infty)$.

Strategies. Similarly as it was done in [8], see Chapter II, §6, the game will be considered from both invader's and defender's points of view. If we consider the game from invader's point of view we will assume that, if $a, b \in \mathbb{R}^2$ stand for initial positions of $\mathbf{D}$ and $\mathbf{I}$ respectively, then $\mathbf{I}$ can apply any non-anticipating function

$$\sigma_{\mathbf{I}} : X(a) \to Y(b)$$

as his strategy. A function $\sigma_{\mathbf{I}} : X(a) \to Y(b)$ is called non-anticipating if, for each $t \geq 0$ and all $x, \widehat{x} \in X(a)$, with

$$x(s) = \widehat{x}(s), \quad \text{for all } s \in [0, t],$$

we have

$$\sigma_{\mathbf{I}}(x)(s) = \sigma_{\mathbf{I}}(\widehat{x})(s), \quad \text{for all } s \in [0, t].$$

The set of all non-anticipating functions $\sigma_{\mathbf{I}} : X(a) \to Y(b)$ will be denoted by $\Sigma_{\mathbf{I}}(a, b)$. Similarly, if we consider the game from defender's point of view we will assume that he can apply any non-anticipating function

$$\sigma_{\mathbf{D}} : Y(b) \to X(a)$$

as his strategy and $\Sigma_{\mathbf{D}}(a, b)$ will stand then for the set of all such functions. Strategies of this kind were introduced (without name) in [12], see also [3].

Organization. Basic notation and definitions together with main result (Theorem 2.7) giving an explicit formula on the maximal length of the interval guarded are given in Section 2. The proof of Theorem 2.7 is given in Sections 3 and 4. In short Section 5 the problem of patrolling a channel (see Example 9.6.4 of [5]) is considered. Both players $\mathbf{D}$ and $\mathbf{I}$ are confined between two parallel lines of distance apart L there. The invader's objective is to get past $\mathbf{D}$ without being captured. The critical width L_c of channel patrolled is established in Section 5.

Related Works. The most of works concerning the problem of guarding territory study problems involving the visibility of geometrical shapes, the visibility of a moving object or capturing an evader in an environment, see e.g. [1], [6], [7], [9], [11]. Our paper is related to the problem of capturing the evader in an environment. It occurs, for example, that the pursuit strategy in "wall pursuit" game (see Example 9.5.2 of [5]) coincides in a suitable region with our strategy of guarding a line segment. It gives an answer to the question raised in *Research Problem 9.5.1* on page 264 of [5]. The result of Section 5 can be used to capture the evader while protecting a door when he may leave a polygonal area through the door and win the game. Moreover, a vaguely reminiscent problems are discussed now as a part of the RoboFlag game, see [7] for example. A game of guarding line segment by n defenders moving along a fixed straight line was considered in [13]. A problem of guarding a region with maximal area was solved in [14]. In this and in the last two papers similar methods are used.

A similar problem of guarding a set $\Omega \subset \mathbb{R}^2$ was investigated in [10]. Among others it was assumed there that the motion of the invader and the defender is controlled by its compact set of permissible velocities $C', C \subset \mathbb{R}^2$, with $C' \subset \operatorname{int}(\operatorname{conv} C)$. It follows from this assumption that slower invader can be captured by faster defender. This does an essential difference between our game and game considered in [10]. With such an assumption in our game of guarding a bounded set the defender would always be a winner.

2. Notation. Main result

Throughout this paper the symbols $\|a\|$ and $\langle a, b\rangle$ will stand for the euclidean norm of a vector $a \in \mathbb{R}^2$ and the euclidean scalar product of vectors $a, b \in \mathbb{R}^2$, respectively. Moreover, the closed and open balls of radius $r > 0$, centered at a point $c \in \mathbb{R}^2$, will be defined by the formulae:

$$B[c, r] = \left\{a \in \mathbb{R}^2 : \|a - c\| \leq r\right\} \quad \text{and} \quad B(c, r) = \left\{a \in \mathbb{R}^2 : \|a - c\| < r\right\}.$$

For a non-empty set $Z \subset \mathbb{R}^2$ symbols $\overline{Z}$, ∂Z and $\operatorname{conv} Z$ will stand for the closure, the boundary and the convex hull of the set Z, respectively. For each non-empty set $Z \subset \mathbb{R}^2$, we define

$$\operatorname{dist}(b, Z) = \inf_{z \in Z} \|b - z\|, \quad b \in \mathbb{R}^2.$$

If Z is a closed and convex set then, for each $b \in \mathbb{R}^2$, there exists exactly one a $\Pi_Z(b) \in Z$ such that

$$\|b - \Pi_Z(b)\| = \operatorname{dist}(b, Z).$$

Moreover, it is known that

$$\|\Pi_Z(b) - \Pi_Z(\widetilde{b})\| \leq \|b - \widetilde{b}\|, \quad \text{for all } b, \widetilde{b} \in \mathbb{R}^2,$$

and

$$\Pi_Z(b) \in \partial Z, \quad \text{for all } b \in \overline{\mathbb{R}^2 \setminus Z}.$$

Let us define

$$e(\alpha) = \begin{bmatrix} -\cos\alpha \\ \sin\alpha \end{bmatrix}, \quad \text{for all } \alpha \in \mathbb{R},$$

$$\mathbf{P}(z) = \{\lambda z : \ \lambda \geq 0\}, \quad \text{for all } z \in \mathbb{R}^2,$$

and

$$Rz = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}, \quad \text{for all } z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2.$$

Note that

$$e'(\alpha) \overset{\text{def}}{=} \frac{de}{d\alpha}(\alpha) = Re(\alpha), \quad \text{for all } \alpha \in \mathbb{R}.$$

Finally, for each $r > 0$, set

$$\Lambda_r = \frac{\theta^2 r}{\theta^2 - 1} + \frac{\theta r}{\theta^2 - 1}\frac{\pi}{2} \tag{1}$$

and define, for all $\Lambda > 0$, a line segment $\Delta(\Lambda) \subset \mathbb{R}^2$, by the formula

$$\Delta(\Lambda) = [-\Lambda, \Lambda] \times \{0\}.$$

Clearly,

$$(0, \infty) \ni r \mapsto \Lambda_r$$

is a strictly increasing function. The constants $\theta > 1$ and $\varrho > 0$ will be fixed in the sequel.

Definition 2.1. Given a non-empty set $Z \subset \mathbb{R}^2$ and initial positions $a, b \in \mathbb{R}^2$, with

$$b \notin Z \quad \text{and} \quad \|b - a\| \geq \varrho.$$

We say that a strategy $\sigma_{\mathbf{D}} \in \Sigma_{\mathbf{D}}(a, b)$ guards the set Z if, for each $y \in Y(b)$, it follows from the relation $y(t) \in Z$ that there exists an $s \geq 0$ such that

$$s < t \quad \text{and} \quad \|y(s) - \sigma_{\mathbf{D}}(y)(s)\| < \varrho.$$

Definition 2.2. Given a non-empty bounded set $Z \subset \mathbb{R}^2$. We say that the player **D** guards the set Z successfully if there exists an open ball $B \subset \mathbb{R}^2$ such that $Z \subset \overline{B}$ and if $b \notin B$ then, for each initial position $a \in Z$, there exists a strategy $\sigma_{\mathbf{D}} \in \Sigma_{\mathbf{D}}(a, b)$ guarding the set Z.

Clearly, if **D** guards a set Z successfully then he guards successfully each subset of Z as well.

Definition 2.3. Given a non-empty set $Z \subset \mathbb{R}^2$ and initial positions $a, b \in \mathbb{R}^2$, with

$$b \notin Z \quad \text{and} \quad \|b - a\| \geq \varrho.$$

We say that a strategy $\sigma_{\mathbf{I}} \in \Sigma_{\mathbf{I}}(a, b)$ rushes for the set Z if, for each $x \in X(a)$, we have

$$\|\sigma_{\mathbf{I}}(y)(t) - x(t)\| \geq \varrho, \quad t \geq 0,$$

and there exists an $s \geq 0$ such that

$$\sigma_{\mathbf{I}}(y)(s) \in Z.$$

Definition 2.4. Given a non-empty set $Z \subset \mathbb{R}^2$. We say that the player **I** rushes for the set Z successfully if, for all initial positions $a, b \in \mathbb{R}^2$, with

$$b \notin Z \quad \text{and} \quad \|b - a\| \geq \varrho,$$

there exists a strategy $\sigma_{\mathbf{I}} \in \Sigma_{\mathbf{I}}(a, b)$ such that $\sigma_{\mathbf{I}}$ rushes for the set Z.

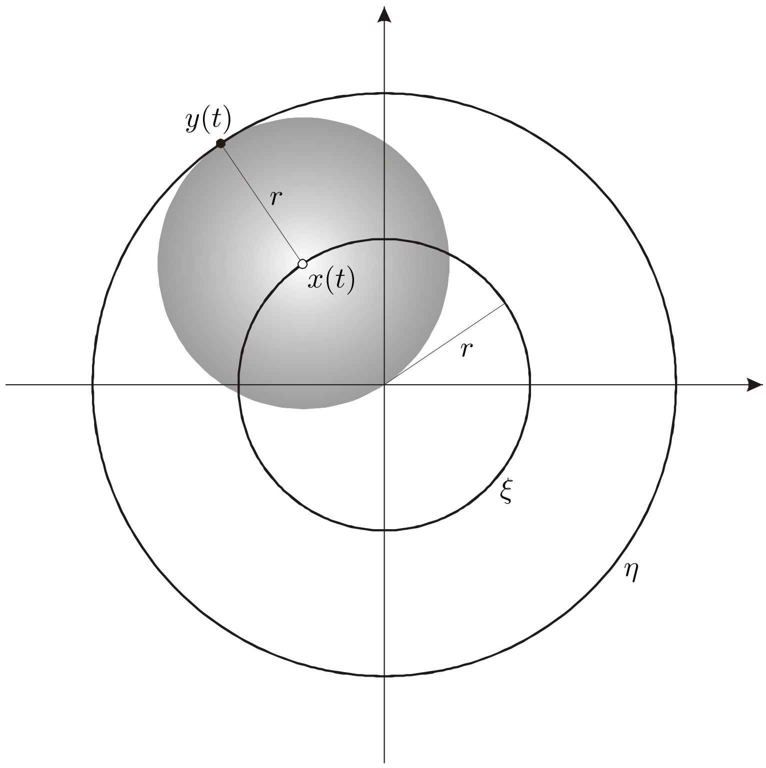

Figure 2.1: Defender guards a set of maximal area, see [14].

Clearly, if **I** rushes for the set Z successfully and $Z \subset Z' \subset \mathbb{R}^2$ then he rushes for the set Z' successfully as well.

It seems that in all problems of guarding a set with an extremal property a successful guarding strategy can be obtained with the aid of two appropriate closed convex curves (or two corresponding compact convex sets). Let $A \subset \mathbb{R}^2$ be a compact convex set. Suppose that the invader moves along the boundary of the set A (curve η) with his maximal velocity θ. Assume now that a compact convex set B is contained in the interior of A and the defender is able to move along the boundary of B (curve ξ) keeping a constant distance $r < \rho$ to the invader, see Fig. 2.1, where

$$\theta = 2, \qquad A = B\,[0, 2r]\,, \qquad B = B\,[0, r]\,.$$

It is rather clear intuitively that **D** can guard the set A successfully. Obviously, the construction of a suitable pair of curves depends on the problem considered. In this paper we use the above approach in order to solve the problem of guarding a line segment of a maximal length. Similar approaches were used in papers [13] and [14] but, in fact, all of them are different.

Definition 2.5. Given two Lipschitz functions $\eta, \xi : \mathbb{R} \to \mathbb{R}^2$ of period 2π. We say that (η, ξ) is a guarding pair if the following conditions hold true:

(a) $\eta, \xi : [0, 2\pi) \to \mathbb{R}^2$ are one-to-one mappings,

(b) $\|\eta(\alpha) - \xi(\alpha)\| = \|\eta(0) - \xi(0)\| < \varrho$, for all $\alpha \in \mathbb{R}$,

(c) $\|\eta'(\alpha)\| \geq \theta\,\|\xi'(\alpha)\|$, almost everywhere in $(0, 2\pi)$,

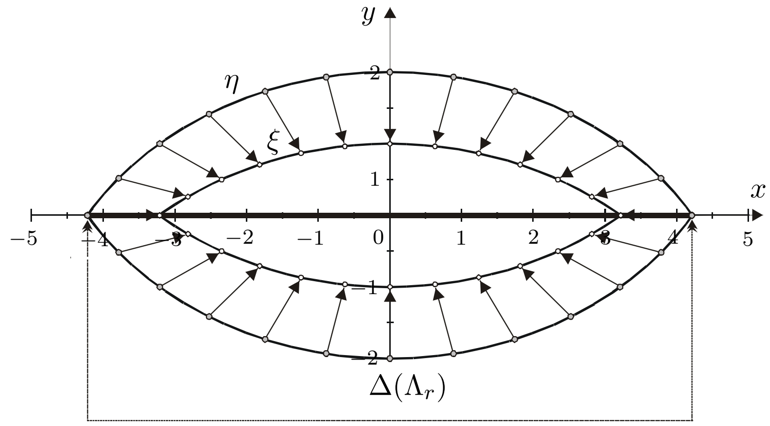

Figure 2.2: Curves $\eta(r,\cdot)$ and $\xi(r,\cdot)$ together with mapping $\xi \circ \eta^{-1}$ used in next Section.

(d) η is a closed, convex curve, i.e.

$$\eta([0,2\pi)) = \partial(\operatorname{conv} \eta([0,2\pi))).$$

As it was mentioned above we will need here special guarding pairs. It seems (we omit several technical details) that in games considered the invader should move along the boundary of the capture zone (which is also moving according to the defender's action) turning around the defender and looking for the chance to enter the set guarded. Consequently, in our game the defender should move along the boundary of a convex set (curve ξ) with maximal diameter. This leads to a corresponding problem from control theory. Solving this problem we have obtained needed guarding pairs.

Definition 2.6. For each $r > 0$ and for each $\alpha \in \mathbb{R}$, we put (see (1))

$$\begin{aligned}\eta(r,\alpha) &= \begin{bmatrix}\eta_1(r,\alpha)\\ \eta_2(r,\alpha)\end{bmatrix} \\ &= \begin{bmatrix}-\Lambda_r\\ 0\end{bmatrix} + \frac{\theta^2 r}{\theta^2-1}\begin{bmatrix}1-\cos\alpha\\ \sin\alpha\end{bmatrix} + \frac{\theta r}{\theta^2-1}\begin{bmatrix}\pi-|\pi-\alpha|\\ 0\end{bmatrix}\end{aligned} \tag{2}$$

$$\xi(r,\alpha) = \begin{bmatrix}\xi_1(r,\alpha)\\ \xi_2(r,\alpha)\end{bmatrix} = \eta(r,\alpha) - re(\alpha) \tag{3}$$

and

$$\Omega_r = \operatorname{conv} \eta(r,[0,2\pi)). \tag{4}$$

See Fig. 2.2, where $\theta = \sqrt{2}$ and $r = 1$.

The curve $\eta(\alpha)$ resembles a lens in shape but an arc of cycloid in a way of construction. It is easy to observe that

$$\Delta(\Lambda_r) \subset \Omega_r, \quad \text{for } r > 0,$$

so that **D** guards the line segment $\Delta(\Lambda_r)$ if he guards the set Ω_r.

Theorem 2.7.

(*a*) *If* $r \in (0, \varrho)$ *then the player* ***D*** *guards the set* Ω_r *successfully.*

(*b*) *If* $\Lambda > \Lambda_\varrho$ *then the player* ***I*** *rushes for the line segment* $\Delta(\Lambda)$ *successfully.*

The proof of the Theorem 2.7 will be given in Sections 3 and 4.

3. Defense

We shall prove here that, for each $r \in (0, \varrho)$, the player **D** guards successfully the set Ω_r. First of all we are going to check that formulas (2) and (3) define guarding pairs.

Lemma 3.1. $(\eta(r, \cdot), \xi(r, \cdot))$ *is a guarding pair, for each* $r > 0$.

Proof. Let us fix an $r > 0$ and put

$$\eta(r, \cdot) = \eta, \qquad \xi(r, \cdot) = \xi,$$

for the simplicity. Clearly, $\eta, \xi : \mathbb{R} \to \mathbb{R}^2$ are Lipschitz functions of period 2π.

Claim 1. $\eta, \xi : [0, 2\pi) \to \mathbb{R}^2$ are one-to-one mappings.

Proof of Claim 1. It follows from the formulas:

$$\eta_1'(\alpha) = \frac{\theta^2 r}{\theta^2 - 1}\sin\alpha + \frac{\theta r}{\theta^2 - 1}, \qquad \xi_1'(\alpha) = \frac{r}{\theta^2 - 1}\sin\alpha + \frac{\theta r}{\theta^2 - 1}, \quad \text{in } (0, \pi),$$

and

$$\eta_2'(\alpha) = \frac{\theta^2 r}{\theta^2 - 1}\sin\alpha - \frac{\theta r}{\theta^2 - 1}, \qquad \xi_2'(\alpha) = \frac{r}{\theta^2 - 1}\sin\alpha - \frac{\theta r}{\theta^2 - 1}, \quad \text{in } (\pi, 2\pi),$$

that both η_1 and ξ_1 are strictly increasing in $[0, \pi]$ and strictly decreasing in $[\pi, 2\pi]$. Since both $\eta_2(\alpha)$ and $\xi_2(\alpha)$ are positive in $(0, \pi)$ and negative in $(\pi, 2\pi)$, $\eta, \xi : [0, 2\pi) \to \mathbb{R}^2$ are one-to-one mappings.

Claim 2. $\|\eta'(\alpha)\| = \theta \|\xi'(\alpha)\|$, almost everywhere in $(0, 2\pi)$.

Proof of Claim 2. Set

$$b_0 = \begin{bmatrix} -\Lambda_r \\ 0 \end{bmatrix} + \frac{\theta^2 r}{\theta^2 - 1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega(\alpha) = \begin{bmatrix} \pi - |\pi - \alpha| \\ 0 \end{bmatrix}, \quad \alpha \in \mathbb{R},$$

and note that

$$\eta(\alpha) = b_0 + \frac{\theta^2 r}{\theta^2 - 1}e(\alpha) + \frac{\theta r}{\theta^2 - 1}\omega(\alpha), \qquad \xi(\alpha) = b_0 + \frac{r}{\theta^2 - 1}e(\alpha) + \frac{\theta r}{\theta^2 - 1}\omega(\alpha).$$

Hence

$$\eta'(\alpha) = \frac{\theta^2 r}{\theta^2 - 1}Re(\alpha) + \frac{\theta r}{\theta^2 - 1}\omega'(\alpha), \qquad \xi'(\alpha) = \frac{r}{\theta^2 - 1}Re(\alpha) + \frac{\theta r}{\theta^2 - 1}\omega'(\alpha),$$

and $\|\omega'(\alpha)\| = 1$, for $\alpha \in (0, 2\pi) \setminus \{\pi\}$. Consequently

$$\begin{aligned}
\|\eta'(\alpha)\|^2 &= \frac{\theta^4 r^2}{(\theta^2-1)^2} + \frac{\theta^2 r^2}{(\theta^2-1)^2} + \frac{2\theta^3 r^2}{(\theta^2-1)^2} \langle Re(\alpha), \omega'(\alpha)\rangle \\
&= \theta^2 \left(\frac{\theta^2 r^2}{(\theta^2-1)^2} + \frac{r^2}{(\theta^2-1)^2} + \frac{2\theta r^2}{(\theta^2-1)^2} \langle Re(\alpha), \omega'(\alpha)\rangle \right) \\
&= \theta^2 \|\xi'(\alpha)\|^2,
\end{aligned}$$

in $(0, 2\pi) \setminus \{\pi\}$.

Claim 3. $\eta([0, 2\pi)) = \partial(\operatorname{conv} \eta([0, 2\pi)))$.

Proof of Claim 3. The image $\eta([0, 2\pi])$ is symmetric with respect to the x-axis, because of

$$\begin{bmatrix} \eta_1(\alpha) \\ -\eta_2(\alpha) \end{bmatrix} = \eta(2\pi - \alpha) \in \eta([\pi, 2\pi]), \quad \alpha \in [0, \pi],$$

$$\begin{bmatrix} \eta_1(\alpha) \\ -\eta_2(\alpha) \end{bmatrix} = \eta(2\pi - \alpha) \in \eta([0, \pi]), \quad \alpha \in [\pi, 2\pi].$$

Moreover

$$\min_{\alpha \in [0, 2\pi)} \eta_1(\alpha) = \eta_1(0) = -\Lambda_r, \qquad \max_{\alpha \in [0, 2\pi)} \eta_1(\alpha) = \eta_1(\pi) = \Lambda_r$$

and

$$\eta_2(0) = \eta_2(\pi) = 0.$$

It is enough to observe now that

$$\eta_1'(\alpha)\eta_2''(\alpha) - \eta_1''(\alpha)\eta_2'(\alpha) = -\frac{\theta^4 r^2}{(\theta^2-1)^2} - \frac{\theta^3 r^2}{(\theta^2-1)^2} \sin\alpha < 0,$$

for $\alpha \in (0, \pi)$, which means that $\eta : [0, \pi] \to \mathbb{R}^2$ is a convex arc.

We have just proved that the pair (η, ξ) has properties (a), (c) and (d) of Definition 2.5. Clearly, by the definition, it has the property (b) as well. This completes the proof of Lemma 3.1.

We are going now to prove next two auxiliary lemmas concerned with properties of Lipschitz functions. We will follow Chapter 3 of [4] together with notation concerning Hausdorff measure and Lipschitz constant of a mapping used there. Recall that one-dimensional Hausdorff measure $\mathcal{H}^1$ generalizes the classical notion of the length of a curve.

Lemma 3.2. *If $\zeta : [\gamma_1, \gamma_2] \to \mathbb{R}^2$ is a continuous function then*

$$\|\zeta(\gamma_1) - \zeta(\gamma_2)\| \le \mathcal{H}^1(\zeta([\gamma_1, \gamma_2])).$$

Proof. Define the line segment

$$[\zeta(\gamma_1), \zeta(\gamma_2)] = \{(1-\lambda)\zeta(\gamma_1) + \lambda\zeta(\gamma_2) : \lambda \in [0,1]\}.$$

Applying the formula on the length of a curve, see Section 3.3.4 A of [4], we obtain

$$\mathcal{H}^1([\zeta(\gamma_1), \zeta(\gamma_2)]) = \int_0^1 \|\zeta(\gamma_2) - \zeta(\gamma_1)\| \, d\lambda = \|\zeta(\gamma_1) - \zeta(\gamma_2)\|.$$

Since

$$\left(\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]} \circ \zeta\right)(\gamma_i) = \zeta(\gamma_i), \quad i = 1, 2,$$

the inclusion

$$[\zeta(\gamma_1), \zeta(\gamma_2)] \subset \left(\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]} \circ \zeta\right)([\gamma_1, \gamma_2])$$

follows from the continuity of the composition $\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]} \circ \zeta$. The projection

$$\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]} : \mathbb{R}^2 \to [\zeta(\gamma_1), \zeta(\gamma_2)]$$

satisfies the Lipschitz condition with constant 1, thus it follows from Theorem 1, p. 75, of [4] that

$$\begin{aligned}\|\zeta(\gamma_1) - \zeta(\gamma_2)\| &= \mathcal{H}^1([\zeta(\gamma_1), \zeta(\gamma_2)]) \le \mathcal{H}^1\left(\left(\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]} \circ \zeta\right)([\gamma_1, \gamma_2])\right) \\ &\le \operatorname{Lip}\left(\Pi_{[\zeta(\gamma_1),\zeta(\gamma_2)]}\right) \mathcal{H}^1(\zeta([\gamma_1, \gamma_2])) \le \mathcal{H}^1(\zeta([\gamma_1, \gamma_2])),\end{aligned}$$

as claimed.

Lemma 3.3. *Suppose (η, ξ) to be a guarding pair, take*

$$\Omega = \operatorname{conv} \eta([0, 2\pi)), \qquad \Xi = \xi([0, 2\pi])$$

and define a mapping $F : \overline{\mathbb{R}^2 \setminus \Omega} \to \Xi$, by the formula

$$F = \xi \circ \eta^{-1} \circ \Pi_\Omega,$$

where η^{-1} is the inverse to $\eta : [0, 2\pi) \to \mathbb{R}^2$. Then

$$\operatorname{Lip}(F \circ z) \le \frac{1}{\theta} \operatorname{Lip}(z),$$

for every Lipschitz function $z : [0, \tau] \to \overline{\mathbb{R}^2 \setminus \Omega}$.

Proof. Take,

$$\mu \overset{\text{def}}{=} \mathcal{H}^1(\partial\Omega) = \int_0^{2\pi} \|\eta'(\alpha)\| \, d\alpha,$$

where $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure, see Section 3.3.4 A of [4] for details. Note that $\mu > 0$, by assertion (a) of Definition 2.5. It is enough to show that

$$\|(F \circ z)(s) - (F \circ z)(t)\| \le \frac{1}{\theta} \operatorname{Lip}(z) |s - t|,$$

for all $s, t \in [0, \tau]$, with

$$0 \leq s \leq t \leq \tau \quad \text{and} \quad \operatorname{Lip}(z)|s-t| < \mu.$$

Let us fix arbitrarily $s, t \in [0, \tau]$ satisfying the above conditions. The image $(\Pi_\Omega \circ z)([s,t])$ is a connected arc of $\partial\Omega$ and, by Theorem 1, p. 75, of [4], we have

$$\begin{aligned}\mathcal{H}^1\left((\Pi_\Omega \circ z)([s,t])\right) &\leq \operatorname{Lip}(\Pi_\Omega \circ z)\,\mathcal{H}^1([s,t]) \leq \operatorname{Lip}(z)\,\mathcal{H}^1([s,t]) \\ &= \operatorname{Lip}(z)|s-t| < \mu = \mathcal{H}^1(\partial\Omega).\end{aligned}$$

Thus, there exist γ_1, γ_2 such that

$$0 \leq \gamma_1 \leq \gamma_2 < \gamma_1 + 2\pi \quad \text{and} \quad (\Pi_\Omega \circ z)([s,t]) = \eta([\gamma_1, \gamma_2]).$$

Consequently, since η is a function of period 2π, we obtain

$$(F \circ z)([s,t]) = \left(\xi \circ \eta^{-1} \circ \Pi_\Omega \circ z\right)([s,t]) = \xi([\gamma_1, \gamma_2]).$$

It follows now from Lemma 3.2 that

$$\|(F \circ z)(s) - (F \circ z)(t)\| \leq \mathcal{H}^1\left((F \circ z)([s,t])\right) = \mathcal{H}^1\left(\xi([\gamma_1, \gamma_2])\right).$$

Both functions $\xi, \eta : [\gamma_1, \gamma_2] \to \mathbb{R}^2$ are Lipschitz, one-to-one and satisfy condition (c) of Definition 2.5, so

$$\begin{aligned}\mathcal{H}^1\left(\xi([\gamma_1, \gamma_2])\right) &= \int_{\gamma_1}^{\gamma_2} \|\xi'(\alpha)\|\,d\alpha \leq \frac{1}{\theta}\int_{\gamma_1}^{\gamma_2} \|\eta'(\alpha)\|\,d\alpha = \frac{1}{\theta}\mathcal{H}^1\left(\eta([\gamma_1, \gamma_2])\right) \\ &= \frac{1}{\theta}\mathcal{H}^1\left((\Pi_\Omega \circ z)([s,t])\right) \leq \frac{1}{\theta}\operatorname{Lip}(z)|s-t|.\end{aligned}$$

This finishes the proof of Lemma 3.3.

In view of Lemma 3.1 and the definition of the set Ω_r, see formula (4), the part (a) of Theorem 2.7 follows from

Lemma 3.4. *If (η, ξ) is a guarding pair then the player **D** guards the set*

$$\Omega \stackrel{def}{=} \operatorname{conv}\eta([0, 2\pi))$$

successfully.

Proof. Let us fix a $c \in \Omega$ and an $r_0 > 0$ such that

$$\Omega \subset B[c, r_0],$$

Set

$$\Xi = \xi([0, 2\pi]), \qquad d = \max_{b \in \Omega}\max_{a \in \Xi}\|b-a\|$$

and

$$r_1 = 2\theta r_0, \qquad r_2 = 2\theta(d + r_0).$$

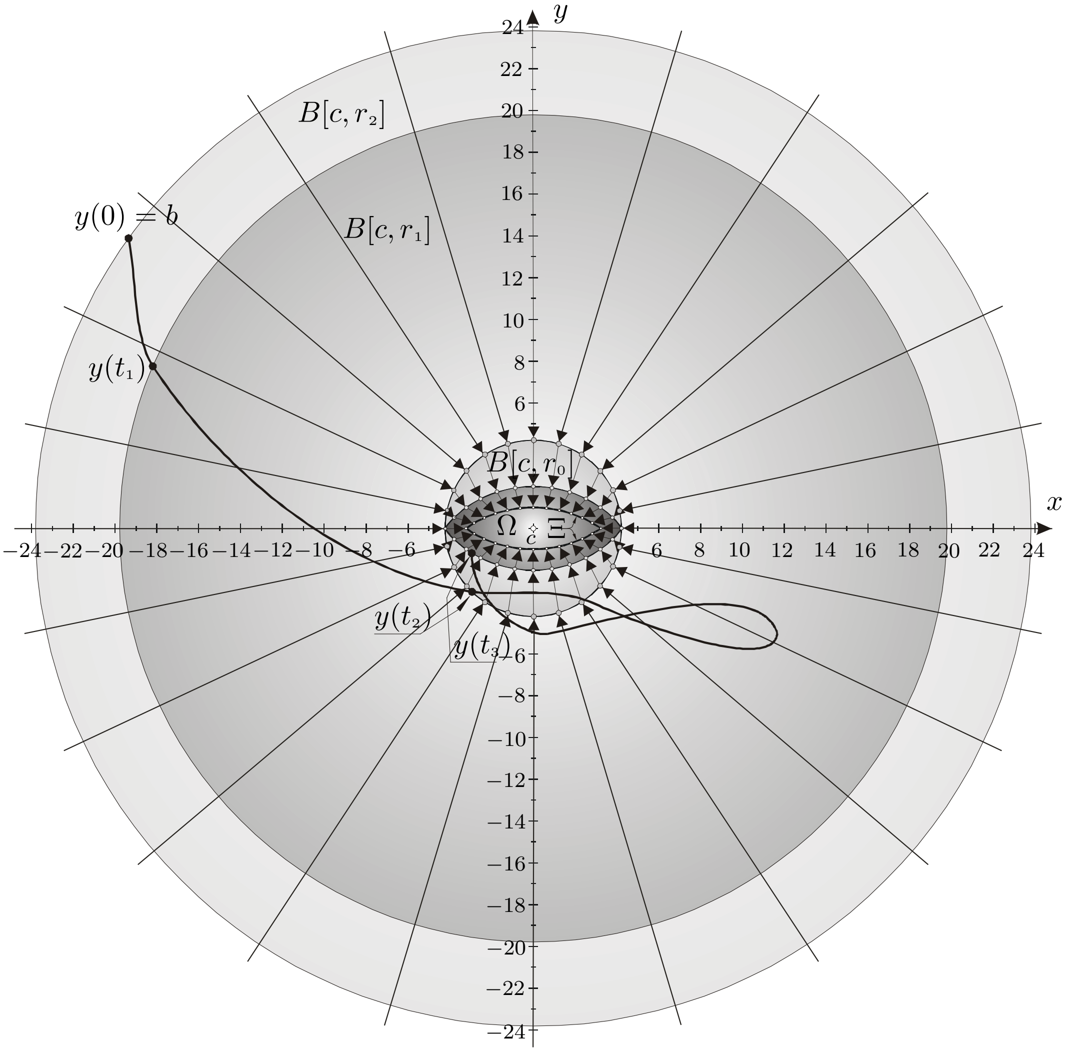

Figure 3.1: Sets Ξ, Ω, $B\,[c, r_0]$, $B\,[c, r_1]$, $B\,[c, r_2]$, mapping $\xi \circ \eta^{-1} \circ \Pi_\Omega \circ \Pi_{B[c,r_0]}$ and possible invader's trajectory, where $c = 0$ and $t_i = \tau_i\,(y)$, $i = 1, 2, 3$.

Clearly, $r_0 < r_1 < r_2$ and consequently

$$\Xi \subset \Omega \subset B\,[c, r_0] \subset B\,[c, r_1] \subset B\,[c, r_2]\,,$$

see Fig. 3.1, where the guarding pair (η, ξ) is defined by (2) and (3), with $\theta = \sqrt{2}$, $r = 1$.

Given $a \in \Omega$ and $b \in \mathbb{R}^2$, with $b \notin B\,(c, r_2)$ We are going to describe, for each $y \in Y\,(b)\,$, a trajectory $\sigma_{\mathbf{D}}\,(y) \in X\,(a)\,$. Let $y \in Y\,(b)$ be fixed arbitrarily. Set

$$T_1\,(y) = \{t \geq 0 : \; \|y\,(t) - c\| = r_1\}\,, \qquad T_2\,(y) = \{t \geq 0 : \; \|y\,(t) - c\| = r_0\}\,,$$
$$T_3\,(y) = \{t \geq 0 : \; y\,(t) \in \Omega\}$$

and

$$\tau_i\,(y) = \begin{cases} \min T_i\,(y)\,, & \text{when } T_i\,(y) \neq \varnothing, \\ \infty, & \text{when } T_i\,(y) = \varnothing, \end{cases}$$

for $i = 1, 2, 3$.

Clearly, $\tau_1(y) \le \tau_2(y) \le \tau_3(y)$ and

$$\tau_1(y) < \tau_2(y) \le \tau_3(y),$$

when $\tau_3(y) < \infty$. Next, set $\psi_y(t) = \Pi_{B[c,r_0]}(y(t))$, for all $t \ge 0$, i.e.

$$\psi_y(t) = c + \frac{r}{\|y(t) - c\|}(y(t) - c), \quad \text{if } \|y(t-c)\| \ge r_0,$$
$$\psi_y(t) = y(t), \quad \text{otherwise,}$$

and

$$\varphi_y(t) = \left(\xi \circ \eta^{-1} \circ \Pi_\Omega \circ \psi_y\right)(t), \quad \text{for all } t \in [0, \tau_3(y)), \text{ when } \tau_3(y) = \infty,$$
$$\varphi_y(t) = \left(\xi \circ \eta^{-1} \circ \Pi_\Omega \circ \psi_y\right)(t), \quad \text{for all } t \in [0, \tau_3(y)], \text{ when } \tau_3(y) < \infty,$$

where η^{-1} denotes the inverse to $\eta : [0, 2\pi) \to \mathbb{R}^2$. Clearly, η^{-1} maps $\partial\Omega$ onto $[0, 2\pi)$ and $\xi \circ \eta^{-1}$ maps $\partial\Omega$ onto Ξ.

In the case of $\varphi_y(0) \ne a$ the first aim of the defender $\mathbf{D}$ is to catch the moving point $\varphi_y(t)$. Next he will move along the set Ξ keeping a constant distance to the point $(\Pi_\Omega \circ \psi_y)(t)$ up to the moment $\tau_3(y)$, if $\tau_3(y) < \infty$. In this connection we denote by x_y the solution of the Cauchy problem

$$x'(t) = \frac{\varphi_y(t) - x(t)}{\|\varphi_y(t) - x(t)\|}, \quad x(0) = a, \tag{5}$$

defined in a maximal interval $[0, \tau_4(y))$. The following four cases are possible:

(a) $\varphi_y(0) = a$ and $\tau_3(y) = \infty$, (b) $\varphi_y(0) = a$ and $\tau_3(y) < \infty$,
(c) $\varphi_y(0) \ne a$ and $\tau_3(y) = \infty$, (d) $\varphi_y(0) \ne a$ and $\tau_3(y) < \infty$.

We are now ready to define the trajectory $\sigma_{\mathbf{D}}(y)$.

Definition of $\sigma_{\mathbf{D}}(y)$ in the case of (a):

$$\sigma_{\mathbf{D}}(y) = \varphi_y.$$

Definition of $\sigma_{\mathbf{D}}(y)$ in the case of (b):

$$\sigma_{\mathbf{D}}(y)(t) = \begin{cases} \varphi_y(t), & \text{when } 0 \le t \le \tau_3(y), \\ \varphi_y(\tau_3(y)), & \text{when } \tau_3(y) < t. \end{cases}$$

Definition of $\sigma_{\mathbf{D}}(y)$ in the case of (c):

$$\sigma_{\mathbf{D}}(y)(t) = \begin{cases} x_y(t), & \text{when } 0 \le t < \tau_4(y), \\ \varphi_y(t), & \text{when } \tau_4(y) \le t. \end{cases}$$

Definition of $\sigma_{\mathbf{D}}(y)$ in the case of (d):

$$\sigma_{\mathbf{D}}(y)(t) = \begin{cases} x_y(t), & \text{when } 0 \le t < \tau_4(y), \\ \varphi_y(t), & \text{when } \tau_4(y) \le t \le \tau_3(y), \\ \varphi_y(\tau_3(y)), & \text{when } \tau_3(y) < t. \end{cases}$$

Obviously, in case (a) and (c), the invader never reaches the set Ω. If the invader never reaches Ω, any strategy of the defender is successfull. In case (b) and (d) the invader is captured; so that $\mathbf{D}$ guards Ω successfully provided the definition of $\sigma_{\mathbf{D}}(y)$ is correct (see the definition of $\sigma_{\mathbf{D}}(y)$ in case (c) and (d)) and $\sigma_{\mathbf{D}} \in \Sigma_{\mathbf{D}}(a,b)$.

Claim 1. We have

$$\tau_1(y) \ge 2d \quad \text{and} \quad \left\|\psi_y'(t)\right\| \le \frac{1}{2}, \quad \text{a.e. in } [0, \tau_1(y)).$$

Proof of Claim 1. Since $\operatorname{Lip}(y) \le \theta$ we have

$$\begin{aligned}\|y(t) - c\| \ge\ & \|y(0) - c\| - \theta t = \|b - c\| - \theta t \ge 2\theta(d + r_0) - \theta t \\ >\ & 2\theta(d + r_0) - 2\theta d = r_1,\end{aligned}$$

for $t \in [0, 2d)$, which implies the inequality $\tau_1(y) \ge 2d$. It is easy to verify that

$$\psi_y'(t) = \frac{r_0 y'(t)}{\|y(t) - c\|} - r_0 \langle y'(t), y(t) - c\rangle \frac{y(t) - c}{\|y(t) - c\|^3},$$

almost everywhere in $[0, \tau_1(y))$. Thus, for almost all $t \in [0, \tau_1(y))$,

$$\begin{aligned}\left\|\psi_y'(t)\right\|^2 &= \frac{r_0^2\|y'(t)\|^2}{\|y(t) - c\|^2} - \frac{r_0^2\langle y'(t), y(t) - c\rangle^2}{\|y(t) - c\|^4} \\ &\le \frac{r_0^2\|y'(t)\|^2}{\|y(t) - c\|^2} \le \frac{r_0^2\theta^2}{(2\theta r_0)^2} = \frac{1}{4},\end{aligned}$$

as claimed.

Claim 2. If $\varphi_y(0) \ne a$ then the Cauchy problem (5) has a unique solution x_y defined in an interval $[0, \tau_4(y))$ such that

$$\tau_4(y) \le \tau_1(y) \quad \text{and} \quad \lim_{t \uparrow \tau_4(y)} x_y(t) = \varphi_y(\tau_4(y)).$$

Proof of Claim 2. By Lemma 3.3, the right hand side

$$f(t,x) \stackrel{\text{def}}{=} \frac{\varphi_y(t) - x}{\|\varphi_y(t) - x\|}$$

of the differential equation considered is continuous in t and locally Lipschitz in x in the set

$$\left\{(t,x) \in [0,\infty) \times \mathbb{R}^2 :\ x \ne \varphi_y(t)\right\}.$$

Therefore, the Cauchy problem (5) has a unique solution x_y defined in a maximal domain $[0, \tau_4(y))$. Let us observe now that

$$\begin{aligned}
\frac{d}{dt}\|\varphi_y(t) - x(t)\| &= \left\langle \frac{\varphi_y(t) - x(t)}{\|\varphi_y(t) - x(t)\|}, \varphi_y'(t) - x'(t) \right\rangle \\
&= \left\langle \frac{\varphi_y(t) - x(t)}{\|\varphi_y(t) - x(t)\|}, \varphi_y'(t) - \frac{\varphi_y(t) - x(t)}{\|\varphi_y(t) - x(t)\|} \right\rangle \\
&= \left\langle \frac{\varphi_y(t) - x(t)}{\|\varphi_y(t) - x(t)\|}, \varphi_y'(t) \right\rangle - 1 \\
&\leq \left\|\varphi_y'(t)\right\| - 1,
\end{aligned}$$

for almost all $t \geq 0$ such that

$$t < \min\{\tau_4(y), \tau_1(y)\}.$$

Thus, it follows from Lemma 3.3 and *Claim 1* that

$$\frac{d}{dt}\|\varphi_y(t) - x(t)\| \leq \left\|\varphi_y'(t)\right\| - 1 \leq \operatorname{Lip}\varphi_y - 1 \leq \frac{1}{\theta}\operatorname{Lip}\psi_y - 1 \leq \frac{1}{2\theta} - 1 \leq -\frac{1}{2},$$

almost everywhere in $[0, \min\{\tau_4(y), \tau_1(y)\})$. Consequently,

$$0 \leq \|\varphi_y(t) - x(t)\| \leq \|\varphi_y(0) - x(0)\| - \frac{t}{2} \leq d - \frac{t}{2},$$

in the interval $[0, \min\{\tau_4(y), \tau_1(y)\})$, which yields the inequality

$$\min\{\tau_4(y), \tau_1(y)\} \leq 2d.$$

We have $\tau_1(y) \geq 2d$, by *Claim 1*, thus $\tau_4(y) \leq \tau_1(y)$. The condition

$$\lim_{t\uparrow\tau_4(y)} x_y(t) = \varphi_y(\tau_4(y))$$

is obvious, so the proof of *Claim 2* is complete.

The definition of $\sigma_{\mathbf{D}}(y)$ is correct, by *Claim 2*. Moreover, $\operatorname{Lip}(\varphi_y) \leq 1$, by Lemma 3.3, and $\operatorname{Lip}(x_y) \leq 1$, by the definition. Consequently $\sigma_{\mathbf{D}}(y) \in X(a)$. Since $\sigma_{\mathbf{D}} : Y(b) \to X(a)$ is a non-anticipating mapping the proof of part (a) of Theorem 2.7 is complete.

4. Attack

We shall prove here that the invader **I** wins the game when the line segment guarded is too long. Roughly speaking the invader will turn around the moving defender keeping distance ϱ to him and, simultaneously, he will be looking for a chance to enter the line segment guarded. In order to define a suitable strategy $\sigma_{\mathbf{I}}$ of the player **I** we need a lemma.

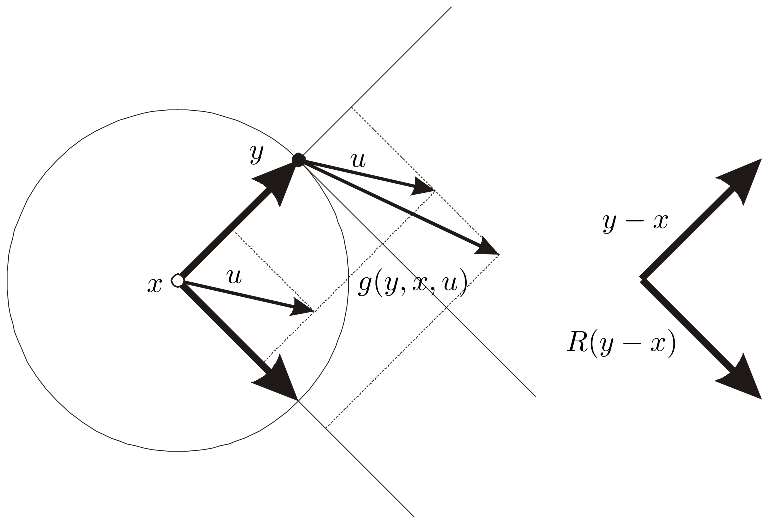

Figure 4.1: Velocity $g(y,x,u)$.

Define

$$G = \left\{(y,x) \in \mathbb{R}^2 \times \mathbb{R}^2 : \ y \neq x\right\}$$

and

$$g(y,x,u) = \left\langle \frac{y-x}{\|y-x\|}, u \right\rangle \frac{y-x}{\|y-x\|} + \sqrt{\theta^2 - \left\langle \frac{y-x}{\|y-x\|}, u \right\rangle^2} R \frac{y-x}{\|y-x\|}, \qquad (6)$$

for all $(y,x) \in G$ and all $u \in \mathbb{R}^2$, with $\|u\| \leq 1$, see Fig. 4.1.

Lemma 4.1. *Given arbitrary $\tau \geq 0$, $(b,a) \in G$ and $x \in X(a)$. Then the following statements hold true.*

(i) *The Cauchy problem*

$$y'(t) = g(y(t), x(t), x'(t)), \quad a.e. \ in \ [\tau, \infty),$$
$$y(\tau) = b,$$

has a unique solution $y : [\tau, \infty) \to \mathbb{R}^2$.

(ii)

$$y'(t) - x'(t) = \left(\sqrt{\theta^2 - \langle z(t), x'(t)\rangle^2} - \langle Rz(t), x'(t)\rangle\right) Rz(t),$$

almost everywhere in $[\tau, \infty)$, where

$$z(t) = \frac{y(t) - x(t)}{\|y(t) - x(t)\|}, \quad t \in [\tau, \infty).$$

(iii)

$$\|y(t) - x(t)\| = \|y(\tau) - x(\tau)\| = \|b - a\|, \quad t \in [\tau, \infty).$$

(iv) $y \in Y(b)$.

Proof. The function g is locally Lipschitz in y and continuous in the couple (x, u), so that the problem considered has a unique solution y defined in a maximal domain $[\tau, \Theta(x))$. Clearly

$$y'(t) = g(y(t), x(t), x'(t)) = \langle z(t), x'(t)\rangle z(t) + \sqrt{\theta^2 - \langle z(t), x'(t)\rangle^2} Rz(t),$$

for almost all $t \in [\tau, \Theta(x))$. Since $\{z(t), Rz(t)\}$ is an orthonormal basis in $\mathbb{R}^2$ we have also

$$x'(t) = \langle z(t), x'(t)\rangle z(t) + \langle Rz(t), x'(t)\rangle Rz(t),$$

for almost all $t \in [\tau, \Theta(x))$. Thus

$$\begin{aligned}\frac{d}{dt}\|y(t) - x(t)\| &= \left\langle \frac{y(t) - x(t)}{\|y(t) - x(t)\|}, y'(t) - x'(t)\right\rangle \\ &= \left\langle z(t), \left(\sqrt{\theta^2 - \langle z(t), x'(t)\rangle^2} - \langle Rz(t), x'(t)\rangle\right) Rz(t)\right\rangle \\ &= 0, \quad \text{a.e. in } [\tau, \Theta(x)).\end{aligned}$$

Consequently

$$\|y(t) - x(t)\| = \|y(\tau) - x(\tau)\| = \|b - a\| > 0, \quad t \in [\tau, \Theta(x)).$$

Since $[\tau, \Theta(x))$ is a maximal domain of the solution y it must be $\Theta(x) = \infty$. Note now that

$$\|g(y, x, u)\|^2 = \left\langle \frac{y - x}{\|y - x\|}, u\right\rangle^2 + \theta^2 - \left\langle \frac{y - x}{\|y - x\|}, u\right\rangle^2 = \theta^2,$$

for all $(y, x) \in G$ and all $u \in \mathbb{R}^2$, with $\|u\| \leq 1$. Hence

$$\|y'(t)\| = \theta, \quad \text{a.e. in } [\tau, \infty),$$

which implies that $y \in Y(b)$. The proof of Lemma 4.1 is complete.

We have stated at the beginning of this Section that the invader **I** will turn around **D**. In order to justify this statement we need next lemma. The next lemma will also be used in the proof of another property of the strategy $\sigma_{\mathbf{I}}$ mentioned above. With the aid of Theorem 1 of [2] one can prove the following

Lemma 4.2. *Given a Lipschitz function $\zeta : [\tau, \infty) \to \mathbb{R}^2$ satisfying the condition $\|\zeta(t)\| > 0$, $t \in [\tau, \infty)$. Then there exists an absolutely continuous function $\gamma : [\tau, \infty) \to \mathbb{R}$ such that*

$$\zeta(t) = \|\zeta(t)\| e(\gamma(t)), \quad t \in [\tau, \infty), \tag{7}$$

and

$$\gamma'(t) = \left\langle Re(\gamma(t)), \frac{\zeta'(t)}{\|\zeta(t)\|}\right\rangle, \quad a.e. \text{ in } [\tau, \infty). \tag{8}$$

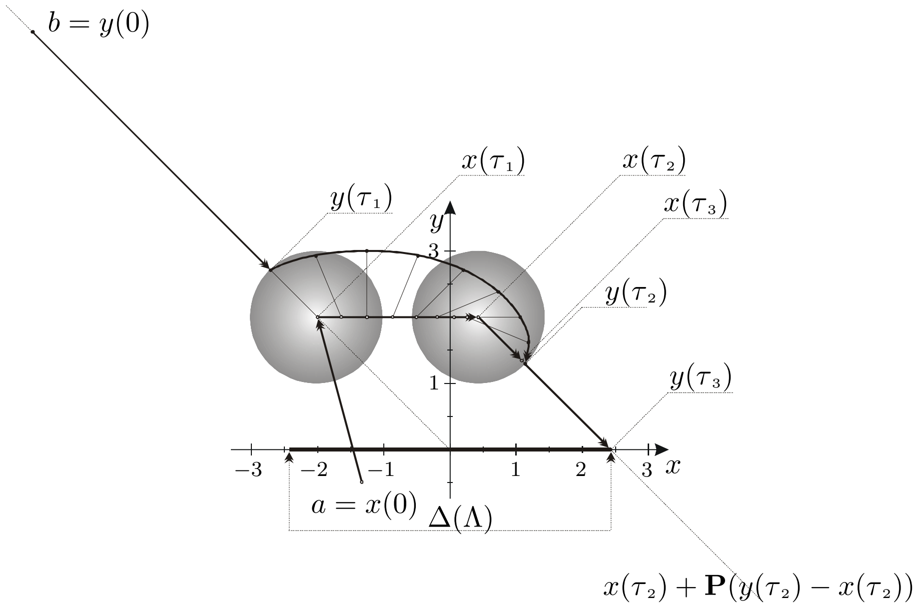

Figure 4.2: Three stages of the game.

We are now ready to define a winning strategy for the invader. Suppose $\Lambda > \Lambda_\varrho$, where Λ_ϱ is defined by formula (1). Let initial positions $a, b \in \mathbb{R}^2$, with

$$b \notin \Delta(\Lambda) \quad \text{and} \quad \|b - a\| \geq \varrho,$$

be fixed arbitrarily. We shall construct, for each $x \in X(a)$, a trajectory $\sigma_{\mathbf{I}}(x) \in Y(b)$. The construction will be divided into three steps, see Fig. 4.2.

In the first step we consider a situation when, at the beginning of the game, the invader is far away from the defender, i.e. the distance between them is greater than ρ. Then he will try to reach the origin (lying in $\Delta(\Lambda)$) moving along a corresponding straight line with his maximal speed. Clearly, the defender will try to protect the origin, so we have to consider a situation when the distance between **I** and **D** becomes equal to ρ, say at a moment τ_1. From this moment we pass to the second step of the construction when "the invader will turn around the moving defender keeping distance ϱ to him and, simultaneously, he will be looking for a chance to enter the line segment guarded". For $t \geq \tau_1$, the invader's trajectory $y(t)$ will be generated by vector field g, defined by (6), up to an eventual moment τ_2 at which **I** could enter the set $\Delta(\Lambda)$ moving along a straight line (the last step of the construction). Let us take an $x \in X(a)$.

Step 1. Define

$$y_b(t) = \left(1 - \frac{\theta t}{\|b\|}\right) b, \quad t \geq 0,$$

$$T_1(x) = \{t \geq 0 : \ \|y_b(t) - x(t)\| = \varrho\},$$

$$\tau_1(x) = \begin{cases} \min T_1(x), & \text{when } T_1(x) \neq \varnothing, \\ \infty, & \text{when } T_1(x) = \varnothing. \end{cases}$$

We put

$$\sigma_{\mathbf{I}}(x)(t) = y_b(t), \quad \text{for all } t \in [0,\infty), \text{ when } \tau_1(x) = \infty,$$
$$\sigma_{\mathbf{I}}(x)(t) = y_b(t), \quad \text{for all } t \in [0,\tau_1(x)], \text{ when } \tau_1(x) < \infty.$$

In the case of $\tau_1(x) = \infty$ the trajectory $\sigma_{\mathbf{I}}(x)$ has been constructed. Otherwise, we pass to

Step 2. Denote by ψ_x the solution to the problem (see Lemma 4.1)

$$y'(t) = g(y(t), x(t), x'(t)), \quad \text{a.e. in } [\tau_1(x), \infty),$$
$$y(\tau_1(x)) = y_b(\tau_1(x)),$$

and define

$$T_2(x) = \{t \geq \tau_1(x) : (\psi_x(t) + \mathbf{P}(\psi_x(t) - x(t))) \cap \Delta(\Lambda) \neq \varnothing\},$$

and

$$\tau_2(x) = \begin{cases} \inf T_2(x), & \text{when } T_2(x) \neq \varnothing, \\ \infty, & \text{when } T_2(x) = \varnothing. \end{cases}$$

We put

$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(t), \quad \text{for all } t \in [\tau_1(x), \infty), \text{ when } \tau_2(x) = \infty,$$
$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(t), \quad \text{for all } t \in [\tau_1(x), \tau_2(x)], \text{ when } \tau_2(x) < \infty.$$

In the case of $\tau_2(x) = \infty$ the trajectory $\sigma_{\mathbf{I}}(x)$ has been constructed. Otherwise, we pass to

Step 3. For all $t \in [\tau_2(x), \infty)$, we put

$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(\tau_2(x)) + \theta(t - \tau_2(x)) \frac{\psi_x(\tau_2(x)) - x(\tau_2(x))}{\|\psi_x(\tau_2(x)) - x(\tau_2(x))\|},$$

which completes the construction of the trajectory $\sigma_{\mathbf{I}}(x)$.

Note that
$$\left\|\sigma_{\mathbf{I}}(x)'(t)\right\| = \theta, \quad \text{a.e. in } [0,\infty).$$

Since $x \in X(a)$ was chosen arbitrarily, we have defined a mapping $\sigma_{\mathbf{I}} : X(a) \to Y(b)$. It is easy to check that $\sigma_{\mathbf{I}} : X(a) \to Y(b)$ is a non-anticipating mapping, so $\sigma_{\mathbf{I}} \in \Sigma_{\mathbf{I}}(a,b)$.

We are going to prove that the strategy $\sigma_{\mathbf{I}}$ rushes for the line segment $\Delta(\Lambda)$. Let us fix an arbitrary $x \in X(a)$. We have to show that

$$\|\sigma_{\mathbf{I}}(x)(t) - x(t)\| \geq \varrho, \quad \text{for all } t \geq 0, \tag{9}$$

and there exists an $s \geq 0$ such that $\sigma_{\mathbf{I}}(x)(s) \in \Delta(\Lambda)$.

Claim 1. Condition (9) is true.

Proof of Claim 1. Condition (9) holds true in the case of $\tau_1(x) = \infty$, by the definition. If $\tau_1(x) < \infty$ and $\tau_2(x) = \infty$ then, by part *(iii)* of Lemma 4.1, condition (9) holds true as well. It remains to check that

$$\|\sigma_{\mathbf{I}}(x)(t) - x(t)\| \geq \varrho, \quad \text{for all } t \geq \tau_2(x),$$

in the case of $\tau_2(x) < \infty$. In this case we have

$$\|\sigma_{\mathbf{I}}(x)(t) - x(t)\| \geq \varrho,$$

for $t \in [0, \tau_2(x)]$, and (see *Step 3*)

$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(\tau_2(x)) + \theta(t - \tau_2(x)) z^*, \quad t > \tau_2(x),$$

where

$$z^* = \frac{\psi_x(\tau_2(x)) - x(\tau_2(x))}{\|\psi_x(\tau_2(x)) - x(\tau_2(x))\|}.$$

Since

$$\begin{aligned}\frac{d}{dt}\langle \sigma_{\mathbf{I}}(x)(t) - x(t), z^* \rangle &= \left\langle \sigma_{\mathbf{I}}(x)'(t) - x'(t), z^* \right\rangle = \langle \theta z^* - x'(t), z^* \rangle \\ &= \theta - \langle x'(t), z^* \rangle \geq \theta - \|x'(t)\| \geq \theta - 1 > 0,\end{aligned}$$

almost everywhere in $[\tau_2(x), \infty)$, we have

$$\begin{aligned}\|\sigma_{\mathbf{I}}(x)(t) - x(t)\| &\geq \langle \sigma_{\mathbf{I}}(x)(t) - x(t), z^* \rangle \\ &\geq \langle \sigma_{\mathbf{I}}(x)(\tau_2(x)) - x(\tau_2(x)), z^* \rangle + (\theta - 1)(t - \tau_2(x)) \\ &= \varrho + (\theta - 1)(t - \tau_2(x)),\end{aligned}$$

for all $t > \tau_2(x)$. This completes the proof of *Claim 1.*

Claim 2. If $\tau_1(x) = \infty$ or $\tau_2(x) < \infty$ then there exists an $s \geq 0$ such that $\sigma_{\mathbf{I}}(x)(s) \in \Delta(\Lambda)$.

Proof of Claim 2. Taking $s = \frac{\|b\|}{\theta}$, in the case of $\tau_1(x) = \infty$, we obtain

$$\sigma_{\mathbf{I}}(x)(s) = y_b(s) = \left(1 - \frac{\theta s}{\|b\|}\right) b = 0 \in \Delta(\Lambda).$$

If $\tau_2(x) < \infty$ then, by the definition,

$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(\tau_2(x)) + \theta(t - \tau_2(x)) \frac{\psi_x(\tau_2(x)) - x(\tau_2(x))}{\|\psi_x(\tau_2(x)) - x(\tau_2(x))\|},$$

for all $t \geq \tau_2(x)$. Moreover,

$$\psi_x(\tau_2(x)) + \lambda_0(\psi_x(\tau_2(x)) - x(\tau_2(x))) \in \Delta(\Lambda),$$

for a $\lambda_0 \geq 0$. Indeed, there exists a decreasing sequence $t_k \geq \tau_2(x)$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} t_k = \tau_2(x)$$

and

$$\psi_x(t_k) + \mathbf{P}(\psi_x(t_k) - x(t_k)) \cap \Delta(\Lambda) \neq \varnothing, \quad k \in \mathbb{N}.$$

Thus, there exists a sequence $\lambda_k \geq 0$, $k \in \mathbb{N}$, such that

$$\psi_x(t_k) + \lambda_k(\psi_x(t_k) - x(t_k)) \stackrel{\text{def}}{=} c_k \in \Delta(\Lambda), \quad k \in \mathbb{N}.$$

It follows from

$$\|\psi_x(t_k) - \psi_x(0)\| \leq t_k\theta \leq t_1\theta, \quad k \in \mathbb{N},$$

that the sequence $\{\lambda_k\}$ is bounded since, by Lemma 4.1,

$$\begin{aligned} \lambda_k &= \frac{\langle c_k - \psi_x(t_k), \psi_x(t_k) - x(t_k)\rangle}{\|\psi_x(t_k) - x(t_k)\|^2} \leq \frac{\|c_k - \psi_x(t_k)\|}{\|\psi_x(t_k) - x(t_k)\|} \\ &= \frac{\|c_k - \psi_x(t_k)\|}{\varrho}, \end{aligned}$$

for all $k \in \mathbb{N}$. Now, taking a subsequence $\{\lambda_{k_j}\}$ converging to a $\lambda_0 \geq 0$, we obtain

$$\begin{aligned} &\psi_x(\tau_2(x)) + \lambda_0(\psi_x(\tau_2(x)) - x(\tau_2(x))) \\ &= \lim_{j\to\infty}\left(\psi_x(t_{k_j}) + \lambda_{k_j}(\psi_x(t_{k_j}) - x(t_{k_j}))\right) \in \Delta(\Lambda). \end{aligned}$$

Thus, for

$$s = \tau_2(x) + \frac{\lambda_0}{\theta}\|\psi_x(\tau_2(x)) - x(\tau_2(x))\|,$$

we obtain

$$\begin{aligned} \sigma_{\mathbf{I}}(x)(s) &= \psi_x(\tau_2(x)) + \theta(s - \tau_2(x))\frac{\psi_x(\tau_2(x)) - x(\tau_2(x))}{\|\psi_x(\tau_2(x)) - x(\tau_2(x))\|} \\ &= \psi_x(\tau_2(x)) + \lambda_0(\psi_x(\tau_2(x)) - x(\tau_2(x))) \in \Delta(\Lambda), \end{aligned}$$

as claimed.

In order to finish the proof of part (b) of Theorem 2.7 it is enough to show that $\tau_2(x) < \infty$. Assuming the contrary, i.e. $\tau_2(x) = \infty$, we will arrive at a contradiction.

Let us observe first that

$$\sigma_{\mathbf{I}}(x)(t) = \psi_x(t) \neq 0 \in \Delta(\Lambda),$$

by the definition and the assumption $\tau_2(x) = \infty$, and

$$\|\psi_x(t) - x(t)\| = \varrho,$$

for all $t \geq \tau_1(x)$, by part *(iii)* of Lemma 4.1. It thus follows from Lemma 4.2 that there exist absolutely continuous functions $\alpha, \beta : [\tau_1(x), \infty) \to \mathbb{R}$ such that

$$\psi_x(t) - x(t) = \|\psi_x(t) - x(t)\| \, e(\alpha(t)) = \varrho e(\alpha(t)), \tag{10}$$

$$\psi_x(t) = \|\psi_x(t)\| \, e(\beta(t)), \tag{11}$$

for all $t \geq \tau_1(x)$. We have

$$0 \notin \psi_x(t) + \mathbf{P}(\psi_x(t) - x(t)), \quad t \geq \tau_1(x),$$

by the assumption $\tau_2(x) = \infty$. It follows from this (see (10) and (11)) that the relation

$$|\alpha(t) - \beta(t)| = \pi \pmod{2\pi}$$

is impossible in the interval $[\tau_1(x), \infty)$. Without loss of generality we can assume that

$$|\alpha(\tau_1(x)) - \beta(\tau_1(x))| < \pi.$$

Thus, by the continuity of both α and β,

$$|\alpha(t) - \beta(t)| < \pi, \quad t \in [\tau_1(x), \infty). \tag{12}$$

Claim 3.

$$\alpha'(t) \geq \frac{\theta - 1}{\varrho}, \quad \text{a.e. in } [\tau_1(x), \infty).$$

Proof of Claim 3. Set

$$z(t) = \frac{\psi_x(t) - x(t)}{\|\psi_x(t) - x(t)\|}, \quad t \geq \tau_1(x).$$

Employing (10), Lemma 4.2 and part *(ii)* of Lemma 4.1 we obtain

$$\begin{aligned}
\alpha'(t) &= \langle Re(\alpha(t)), \psi_x'(t) - x'(t) \rangle = \frac{1}{\varrho} \langle Rz(t), \psi_x'(t) - x'(t) \rangle \\
&= \frac{1}{\varrho} \left(\sqrt{\theta^2 - \langle z(t), x'(t) \rangle^2} - \langle Rz(t), x'(t) \rangle \right) \\
&= \frac{1}{\varrho} \frac{\theta^2 - \langle z(t), x'(t) \rangle^2 - \langle Rz(t), x'(t) \rangle^2}{\sqrt{\theta^2 - \langle z(t), x'(t) \rangle^2} + \langle Rz(t), x'(t) \rangle} \\
&= \frac{1}{\varrho} \frac{\theta^2 - \|x'(t)\|^2}{\sqrt{\theta^2 - \langle z(t), x'(t) \rangle^2} + \langle Rz(t), x'(t) \rangle} \geq \frac{1}{\varrho} \frac{\theta^2 - \|x'(t)\|^2}{\sqrt{\theta^2} + \|x'(t)\|} \\
&= \frac{\theta - \|x'(t)\|}{\varrho} \geq \frac{\theta - 1}{\varrho}, \quad \text{a.e. in } [\tau_1(x), \infty),
\end{aligned}$$

which finishes the proof of *Claim 3.*

In view of *Claim 3*, by (12), we obtain

$$\lim_{t\to\infty}\beta(t)=\lim_{t\to\infty}\alpha(t)=\infty.$$

It means (see (12) once more) that vectors $\psi_x(t)-x(t)$ and $\psi_x(t)$ rotate with more or less the same speed. In order to obtain a contradiction we are now going to show that the rotation of $\psi_x(t)-x(t)$ is faster. Take

$$t_0=\min\{t\in[\tau_1(x),\infty):\ \beta(t)=0\pmod{2\pi}\}$$

and, for each $n\in\mathbb{N}$, define

$$t_n=\min\{t\in[t_{n-1},\infty):\ \beta(t)\geq\pi+\beta(t_{n-1})\}.$$

Clearly,

$$\beta(t_n)=\beta(t_0)+n\pi,\quad n=0,1,...,\tag{13}$$

and, by the condition $\psi_x(t)\notin\Delta(\Lambda),\ t\geq0,$

$$\psi_{x,2}(t_n)=0,\qquad\psi_{x,1}(t_{2n})<-\Lambda,\qquad\psi_{x,1}(t_{2n+1})>\Lambda,\quad n=0,1,...,\tag{14}$$

where

$$\psi_x(t)=(\psi_{x,1}(t),\psi_{x,2}(t)),\quad t\geq\tau_1(x).$$

Claim 4. We have:

(a)

$$\begin{aligned}&\psi_{x,1}(t_{2n+1})-\psi_{x,1}(t_{2n})\\&\leq\frac{\theta^2\varrho}{\theta^2-1}(\cos\alpha(t_{2n+1})-\cos\alpha(t_{2n}))+\frac{\theta\varrho}{\theta^2-1}(\alpha(t_{2n+1})-\alpha(t_{2n})),\end{aligned}$$

for $n=0,1,...,$ and

(b)

$$\begin{aligned}&\psi_{x,1}(t_{2n})-\psi_{x,1}(t_{2n-1})\\&\geq\frac{\theta^2\varrho}{\theta^2-1}(\cos\alpha(t_{2n})-\cos\alpha(t_{2n-1}))-\frac{\theta\varrho}{\theta^2-1}(\alpha(t_{2n})-\alpha(t_{2n-1})),\end{aligned}$$

for $n=1,2,...$

Proof of Claim 4. Set

$$z(t)=\frac{\psi_x(t)-x(t)}{\|\psi_x(t)-x(t)\|},\quad t\geq\tau_1(x),$$

and

$$(u_1(t),u_2(t))=(\langle z(t),x'(t)\rangle,\langle Rz(t),x'(t)\rangle),\quad\text{a.e. in }[\tau_1(x),\infty).$$

Now, let us fix an $n \geq 0$ and define

$$\mu(t) = \psi_{x,1}(t) - \psi_{x,1}(t_{2n}),$$
$$\nu(t) = \frac{\theta^2 \varrho}{\theta^2 - 1}(\cos\alpha(t) - \cos\alpha(t_{2n})) + \frac{\theta\varrho}{\theta^2 - 1}(\alpha(t) - \alpha(t_{2n})),$$

for $t \in [t_{2n}, t_{2n+1}]$. Since $\mu(t_{2n}) = \nu(t_{2n})$ it is enough to show that $\mu'(t) \leq \nu'(t)$ a.e. in $[t_{2n}, t_{2n+1}]$. We have (see (10))

$$\begin{aligned}\psi_x'(t) &= \langle z(t), x'(t)\rangle z(t) + \sqrt{\theta^2 - \langle z(t), x'(t)\rangle^2} Rz(t) \\ &= u_1(t) e(\alpha(t)) + \sqrt{\theta^2 - u_1^2(t)} Re(\alpha(t)),\end{aligned}$$

almost everywhere in $[\tau_1(x), \infty)$,which implies that

$$\mu'(t) = \psi_{x,1}'(t) = -u_1(t)\cos\alpha(t) + \sqrt{\theta^2 - u_1^2(t)}\sin\alpha(t), \quad \text{a.e. in } [\tau_1(x), \infty).$$

Moreover,

$$\nu'(t) = \frac{\theta\varrho}{\theta^2 - 1}(\theta\sin\alpha(t) + 1)\alpha'(t),$$

and (see the proof of *Claim 3*)

$$\alpha'(t) = \frac{1}{\varrho}\left(\sqrt{\theta^2 - u_1^2(t)} - u_2(t)\right),$$

almost everywhere in $[t_{2n}, t_{2n+1}]$. Thus it suffices to prove that

$$-u_1\cos\alpha + \sqrt{\theta^2 - u_1^2}\sin\alpha \leq \frac{\theta}{\theta^2 - 1}(\theta\sin\alpha + 1)\left(\sqrt{\theta^2 - u_1^2} - u_2\right),$$

for all $\alpha \in \mathbb{R}$ and all $u = (u_1, u_2) \in \mathbb{R}^2$, with $u_1^2 + u_2^2 \leq 1$. Since $\theta > 1$ the above inequality is equivalent to the following one:

$$-\left(\theta^2 - 1\right)u_1\cos\alpha + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)\sin\alpha \leq \theta\left(\sqrt{\theta^2 - u_1^2} - u_2\right).$$

We have

$$\begin{aligned}&-\left(\theta^2 - 1\right)u_1\cos\alpha + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)\sin\alpha \\ &\leq \sqrt{\left(\theta^2 - 1\right)^2 u_1^2 + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)^2} \\ &= \theta\sqrt{\left(\sqrt{\theta^2 - u_1^2} - u_2\right)^2 - \left(\theta^2 - 1\right)\left(1 - u_1^2 - u_2^2\right)} \\ &\leq \theta\left(\sqrt{\theta^2 - u_1^2} - u_2\right),\end{aligned}$$

which completes the proof of the assertion (a).

Let us fix an $n \geq 1$ and define

$$\mu(t) = \psi_{x,1}(t) - \psi_{x,1}(t_{2n-1}),$$
$$\nu(t) = \frac{\theta^2 \varrho}{\theta^2 - 1}(\cos\alpha(t) - \cos\alpha(t_{2n-1})) - \frac{\theta\varrho}{\theta^2 - 1}(\alpha(t) - \alpha(t_{2n})),$$

for $t \in [t_{2n-1}, t_{2n}]$. It is enough to show this time that $\mu'(t) \geq \nu'(t)$, a.e. in $[t_{2n-1}, t_{2n}]$. Since

$$\mu'(t) = \psi'_{x,1}(t) = -u_1(t)\cos\alpha(t) + \sqrt{\theta^2 - u_1^2(t)}\sin\alpha(t),$$
$$\nu'(t) = \frac{\theta\varrho}{\theta^2 - 1}(\theta\sin\alpha(t) - 1)\alpha'(t),$$
$$\alpha'(t) = \frac{1}{\varrho}\left(\sqrt{\theta^2 - u_1^2(t)} - u_2(t)\right),$$

almost everywhere in $[t_{2n-1}, t_{2n}]$, it suffices to prove that

$$-u_1\cos\alpha + \sqrt{\theta^2 - u_1^2}\sin\alpha \geq \frac{\theta}{\theta^2 - 1}(\theta\sin\alpha - 1)\left(\sqrt{\theta^2 - u_1^2} - u_2\right)$$

for all $\alpha \in \mathbb{R}$ and all $u = (u_1, u_2) \in \mathbb{R}^2$, with $u_1^2 + u_2^2 \leq 1$. The above inequality is equivalent to the following one:

$$-\left(\theta^2 - 1\right)u_1\cos\alpha + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)\sin\alpha \geq -\theta\left(\sqrt{\theta^2 - u_1^2} - u_2\right)$$

and we have

$$\begin{aligned}
& -\left(\theta^2 - 1\right)u_1\cos\alpha + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)\sin\alpha \\
\geq & -\sqrt{\left(\theta^2 - 1\right)^2 u_1^2 + \left(\theta^2 u_2 - \sqrt{\theta^2 - u_1^2}\right)^2} \\
= & -\theta\sqrt{\left(\sqrt{\theta^2 - u_1^2} - u_2\right)^2 - (\theta^2 - 1)(1 - u_1^2 - u_2^2)} \\
\geq & -\theta\left(\sqrt{\theta^2 - u_1^2} - u_2\right).
\end{aligned}$$

This completes the proof of the assertion (b) and the proof of *Claim 4*.

We are now ready to finish the proof of part (b) of Theorem 2.7. Take

$$\epsilon = 2\Lambda - 2\Lambda_\varrho = 2\Lambda - \left(\frac{2\theta^2\varrho}{\theta^2 - 1} + \frac{\theta\varrho}{\theta^2 - 1}\pi\right).$$

It follows from *Claim 4* and formula (14) that

$$\begin{aligned}
&\frac{2\theta^2\varrho}{\theta^2-1}+\frac{\theta\varrho}{\theta^2-1}\pi+\epsilon=2\Lambda\leq\psi_{x,1}\left(t_{2n+1}\right)-\psi_{x,1}\left(t_{2n}\right)\\
&\leq\frac{\theta^2\varrho}{\theta^2-1}\left(\cos\alpha\left(t_{2n+1}\right)-\cos\alpha\left(t_{2n}\right)\right)+\frac{\theta\varrho}{\theta^2-1}\left(\alpha\left(t_{2n+1}\right)-\alpha\left(t_{2n}\right)\right)\\
&\leq\frac{2\theta^2\varrho}{\theta^2-1}+\frac{\theta\varrho}{\theta^2-1}\left(\alpha\left(t_{2n+1}\right)-\alpha\left(t_{2n}\right)\right),
\end{aligned}$$

for $n=0,1,...$, and

$$\begin{aligned}
&-\frac{2\theta^2\varrho}{\theta^2-1}-\frac{\theta\varrho}{\theta^2-1}\pi-\epsilon=-2\Lambda\geq\psi_{x,1}\left(t_{2n}\right)-\psi_{x,1}\left(t_{2n-1}\right)\\
&\geq\frac{\theta^2\varrho}{\theta^2-1}\left(\cos\alpha\left(t_{2n}\right)-\cos\alpha\left(t_{2n-1}\right)\right)-\frac{\theta\varrho}{\theta^2-1}\left(\alpha\left(t_{2n}\right)-\alpha\left(t_{2n-1}\right)\right)\\
&\geq-\frac{2\theta^2\varrho}{\theta^2-1}-\frac{\theta\varrho}{\theta^2-1}\left(\alpha\left(t_{2n}\right)-\alpha\left(t_{2n-1}\right)\right),
\end{aligned}$$

for $n=1,2,...$ Thus,

$$\alpha\left(t_{n+1}\right)-\alpha\left(t_n\right)>\pi+\frac{\theta^2-1}{\theta\varrho}\epsilon,\quad n=0,1,...$$

Now, involving formula (13) we obtain

$$\begin{aligned}
&\alpha\left(t_{n+1}\right)-\beta\left(t_{n+1}\right)-\left(\alpha\left(t_n\right)-\beta\left(t_n\right)\right)\\
&=\alpha\left(t_{n+1}\right)-\alpha\left(t_n\right)-\left(\beta\left(t_{n+1}\right)-\beta\left(t_n\right)\right)\\
&=\alpha\left(t_{n+1}\right)-\alpha\left(t_n\right)-\pi\\
&>\frac{\theta^2-1}{\theta\varrho}\epsilon,\quad n=0,1,...,
\end{aligned}$$

which implies that (compare with (12))

$$\lim_{n\to\infty}\left(\alpha\left(t_n\right)-\beta\left(t_n\right)\right)=\infty,$$

a contradiction. The part (b) of Theorem 2.7 is proved.

5. Patrolling a channel

Let $\beta\in\left(0,\frac{\pi}{2}\right)$ be such that

$$\sin\beta=\frac{1}{\theta}.$$

Define

$$\Lambda_r^*=\frac{\theta^2 r}{\theta^2-1}\cos\beta+\frac{\theta r}{\theta^2-1}\left(\frac{\pi}{2}+\beta\right)$$

and

$$C_r=\left[-\Lambda_r,\Lambda_r\right]\times\mathbb{R}\subset\mathbb{R}^2,$$

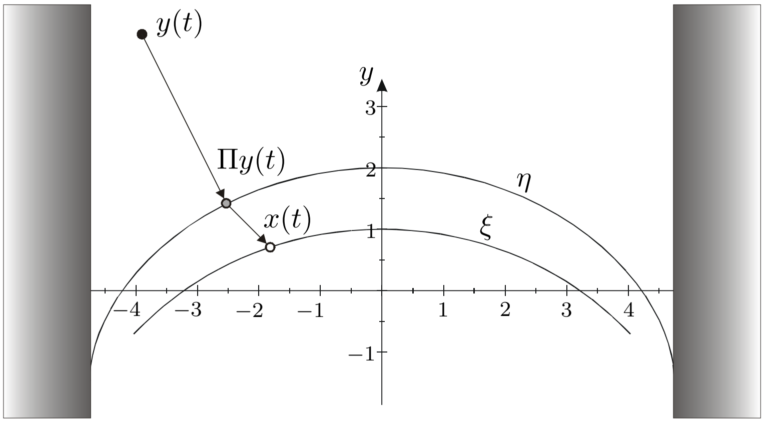

Figure 5.1: Curves η and ξ in channel, for $\theta=\sqrt{2}$, $r=1$.

for $r>0$. It is easy to check that

$$\Lambda_r^* > \frac{\theta^2 r}{\theta^2-1} + \frac{\theta r}{\theta^2-1}\frac{\pi}{2} > r,$$

for all $r>0$. Suppose now that both players **D** and **I** are located in the channel (corridor) C_r at the points a, b respectively. The aim of the invader **I** is to get past **D** keeping the distance to him no less than ϱ. The following two statements hold true.

A. There exists a $\delta_{\mathbf{D}} > \varrho$ such that if $r<\varrho$ then **D** can prevent the passage for all $a, b \in C_r$, with $\|a-b\| \geq \delta_{\mathbf{D}}$.

B. There exists a $\delta_{\mathbf{I}} > \varrho$ such that if $r>\varrho$ then, for all $a, b \in C_r$, with $\|a-b\| \geq \delta_{\mathbf{I}}$, **D** cannot prevent the passage.

One can prove the statement **A** making use of two curves:

$$\eta(\alpha) = \begin{bmatrix} -\Lambda_r^* \\ 0 \end{bmatrix} + \frac{\theta^2 r}{\theta^2-1}\begin{bmatrix} 1-\cos\alpha \\ \sin\alpha \end{bmatrix} + \frac{\theta r}{\theta^2-1}\begin{bmatrix} \alpha \\ 0 \end{bmatrix},$$
$$\xi(\alpha) = \eta(\alpha) - re(\alpha),$$

where $\alpha \in [-\beta, \pi+\beta]$, see Fig. 5.1, similarly as it was done in Section 3.

In order to prove statement **B** one can follow Section 4 using *Claim 4* with a suitable modification.

The number

$$L_\varrho \overset{\text{def}}{=} 2\Lambda_\varrho^* = \frac{2\theta^2\varrho}{\theta^2-1}\cos\beta + \frac{2\theta\varrho}{\theta^2-1}\left(\frac{\pi}{2}+\beta\right)$$

can be called the critical width of the channel (corridor) patrolled. Let us look now at an analogous formula (9.6.1), p. 269, of [5]:

$$L_c = \frac{2lw}{w^2-1}\left(\sqrt{w^2-1} - \cos^{-1}\frac{1}{w}\right),$$

where $l=\varrho$ and $w=\theta$ in our notation. It was stated there that **D** cannot guard the corridor if $L>L_c$, see statement (9.6.2) on the same page. Such a statement is evidently false since

$$\frac{2lw}{w^2-1}\left(\sqrt{w^2-1}-\cos^{-1}\frac{1}{w}\right)<2l,$$

for all $w>1$, and $\mathbf{D}$ can guard each corridor of width $L<2l$ being centered in the corridor at a fixed point. The solution of the problem of patrolling a channel given in Example 9.6.4 of [5] is thus incorrect.

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