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On the Surjectivity Properties of Perturbations of Maximal Monotone Operators in Non-Reflexive Banach Spaces

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We are concerned with surjectivity of perturbations of maximal monotone operators in non-reflexive Banach spaces. While in a reflexive setting, a classical surjectivity result due to Rockafellar gives a necessary and sufficient condition to maximal monotonicity, in a non-reflexive space we characterize maximality using an "enlarged" version of the duality mapping, introduced previously by Gossez.

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1. Introduction

Let X be a real Banach space and X^* its topological dual. We use the notation π and π_* for the duality product in $X \times X^*$ and in $X^* \times X^{**}$, respectively:

$$\begin{aligned} &\pi : X \times X^* \to \mathbb{R}, && \pi_* : X^* \times X^{**} \to \mathbb{R} \\ &\pi(x, x^*) = \langle x, x^* \rangle, && \pi_*(x^*, x^{**}) = \langle x^*, x^{**} \rangle. \end{aligned} \tag{1}$$

The norms on X, X^* and X^{**} will be denoted by $\| \cdot \|$. We also use the notation $\bar{\mathbb{R}}$ for the extended real numbers:

$$\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

Whenever necessary, we will identify X with its image under the canonical injection of X into X^{**}.

A point to set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

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and $T(x) = \{x^* \in X^* \mid (x, x^*) \in T\}$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X into X^*. The conjugate of f is $f^* : X^* \to \bar{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

Note that f^* is always convex and lower semicontinuous.

The *subdifferential* of f is the point to set operator $\partial f : X \rightrightarrows X^*$ defined at $x \in X$ by

$$\partial f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall y \in X\}.$$

For each $x \in X$, the elements $x^* \in \partial f(x)$ are called *subgradients* of f. The concept of *ε-subdifferential* of a convex function f was introduced by Brøndsted and Rockafellar [4]. It is a point to set operator $\partial_\varepsilon f : X \rightrightarrows X^*$ defined at each $x \in X$ as

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \ \forall y \in X\},$$

where $\varepsilon \geq 0$. Note that $\partial f = \partial_0 f$ and $\partial f(x) \subset \partial_\varepsilon f(x)$, for all $\varepsilon \geq 0$.

A convex function $f : X \to \bar{\mathbb{R}}$ is said to be proper if $f > -\infty$ and there exists a point $\hat{x} \in X$ for which $f(\hat{x}) < \infty$. Rockafellar proved that if f is proper, convex and lower semicontinuous, then ∂f is maximal monotone on X [18]. If $f : X \to \bar{\mathbb{R}}$ is proper, convex and lower semicontinuous, then f^* is proper and f satisfies the *Fenchel-Young inequality*: for all $x \in X$, $x^* \in X^*$,

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \quad f(x) + f^*(x^*) = \langle x, x^* \rangle \iff x^* \in \partial f(x). \tag{2}$$

Moreover, in this case, $\partial_\varepsilon f$ (and $\partial f = \partial_0 f$) may be characterized using f^*:

$$\begin{aligned} \partial f(x) &= \{x^* \in X^* \mid f(x) + f^*(x^*) = \langle x, x^* \rangle\}, \\ \partial_\varepsilon f(x) &= \{x^* \in X^* \mid f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon\}. \end{aligned} \tag{3}$$

The subdifferential and the ε-subdifferential of the function $\frac{1}{2}\|\cdot\|^2$ will be of special interest in this paper, and will be denoted by $J : X \rightrightarrows X^*$ and $J_\varepsilon : X \rightrightarrows X^*$ respectively

$$J(x) = \partial \, \frac{1}{2}\|x\|^2, \qquad J_\varepsilon(x) = \partial_\varepsilon \, \frac{1}{2}\|x\|^2.$$

Using $f(x) = (1/2)\|x\|^2$ in (3), it is trivial to verify that

$$\begin{aligned} J(x) &= \left\{ x^* \in X^* \ \Big| \ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle \right\} \\ &= \left\{ x^* \in X^* \mid \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle \right\} \end{aligned}$$

and

$$J_\varepsilon(x) = \left\{ x^* \in X^* \mid \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \varepsilon \right\}.$$

The operator J is widely used in Convex Analysis in Banach spaces and it is called the *duality mapping* of X. The operator J_ε was introduced by Gossez [11] to generalize some results concerning maximal monotonicity in reflexive Banach spaces to non-reflexive Banach spaces. It was also used in [10] to the study of locally maximal monotone operators in non-reflexive Banach spaces.

If X is a real *reflexive* Banach space and $T : X \rightrightarrows X^*$ is monotone, then T is maximal monotone if and only if

$$R(T(\cdot + z_0) + J) = X^*, \quad \forall z_0 \in X.$$

We shall prove a similar result for a class of maximal monotone operators in non-reflexive Banach spaces.

2. Basic definitions and theory

In this section we present the tools and results which will be used to prove the main results of this paper.

For $f : X \to \bar{\mathbb{R}}$, $\operatorname{conv} f : X \to \bar{\mathbb{R}}$ is the largest convex function majorized by f, and $\operatorname{cl} f : X \to \bar{\mathbb{R}}$ is the largest lower semicontinuous function majorized by f. It is trivial to verify that

$$\operatorname{cl} f(x) = \liminf_{y \to x} f(y), \qquad f^* = (\operatorname{conv} f)^* = (\operatorname{cl}\operatorname{conv} f)^*.$$

The functions $\operatorname{cl} f$ and $\operatorname{cl}\operatorname{conv} f$ are usually called the (lower semicontinuous) closure of f and the convex lower semicontinuous closure of f, respectively.

Fitzpatrick proved constructively that maximal monotone operators are representable by convex functions. Let $T : X \rightrightarrows X^*$ be maximal monotone. The *Fitzpatrick function of* T [9] is $\varphi_T : X \times X^* \to \bar{\mathbb{R}}$

$$\varphi_T(x, x^*) = \sup_{(y,y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \tag{4}$$

and *Fitzpatrick family* associated with T is

$$\mathfrak{F}_T = \left\{ h \in \bar{\mathbb{R}}^{X \times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and lower semicontinuous} \\ \langle x, x^* \rangle \le h(x, x^*), \quad \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}. \tag{5}$$

Theorem 2.1 ([9, Theorem 3.10]). *Let X be a real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Then for any $h \in \mathfrak{F}_T$ (5)*

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*$$

and φ_T (4) is the smallest element of the family $\mathfrak{F}_T$.

Fitzpatrick's results described above were rediscovered by Martínez-Legaz and Théra [15], and Burachik and Svaiter [7]. Since then, this area has been the subject of intense research.

The *indicator function* of $A \subset X$ is $\delta_A : X \to \bar{\mathbb{R}}$,

$$\delta_A(x) := \begin{cases} 0, & x \in A \\ \infty, & \text{otherwise.} \end{cases}$$

Using the indicator function we have another expression for Fitzpatrick function:

$$\varphi_T(x, x^*) = (\pi + \delta_T)^* (x^*, x).$$

The supremum of Fitzpatrick family is the $\mathcal{S}$-function, defined and studied by Burachik and Svaiter in [7], $\mathcal{S}_T : X \times X^* \to \bar{\mathbb{R}}$

$$\mathcal{S}_T(x, x^*) = \sup \left\{ h(x, x^*) \;\middle|\; \begin{array}{l} h : X \times X^* \to \bar{\mathbb{R}} \text{ convex lower semicontinuous} \\ h(x, x^*) \leq \langle x, x^* \rangle, \quad \forall (x, x^*) \in T \end{array} \right\}$$

or, equivalently (see [7, Eq. (35)], [6, Eq. 29])

$$\mathcal{S}_T = \operatorname{cl} \operatorname{conv}(\pi + \delta_T). \tag{6}$$

Some authors [2, 21, 3] attribute the $\mathcal{S}$-function to [16] although this work was *submitted* after the publication of [7]. Moreover, the content of [7], and specifically the $\mathcal{S}_T$ function, was presented on Erice workshop on July 2001, by R. S. Burachik [5]. A list of the talks of this congress, which includes [17], is available on the www[1]. It will also be noted that [6], the preprint of [7], was published (and available on www) at IMPA preprint server in August 2001.

Burachik and Svaiter also proved that the family $\mathcal{F}_T$ is invariant under the mapping

$$\mathcal{J} : \bar{\mathbb{R}}^{X \times X^*} \to \bar{\mathbb{R}}^{X \times X^*}, \qquad \mathcal{J}\, h(x, x^*) = h^*(x^*, x). \tag{7}$$

If $T : X \rightrightarrows X^*$ is maximal monotone, then [7]

$$\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T, \qquad \mathcal{J}\, \mathcal{S}_T = \varphi_T.$$

In particular, for any $h \in \mathcal{F}_T$,

$$h(x, x^*) \geq \langle x, x^* \rangle, \qquad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \tag{8}$$

A partial converse of this fact was proved in [8]: in a *reflexive* Banach space, if h is convex, lower semicontinuous and satisfy (8) then

$$T := \{(x, x^*) \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and $h \in \mathcal{F}_T$ [8]. In order to extend this result to non-reflexive Banach spaces, Marques Alves and Svaiter considered an extension of condition (8) to non-reflexive Banach spaces:

$$\begin{gathered} h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{gathered} \tag{9}$$

We shall prefer the synthetic notation $h \geq \pi$, $h^* \geq \pi_*$ for the above condition. The following result will be fundamental in our analysis

[1]http://www.polyu.edu.hk/~ama/events/conference/EriceItaly-OCA2001/Abstract.html

Theorem 2.2 ([12, Theorem 3.4]). *Let $h : X \times X^* \to \bar{\mathbb{R}}$ be a convex and lower semicontinuous function. If*

$$h \geq \pi, \qquad h^* \geq \pi_*$$

and $h(x, x^*) < \langle x, x^* \rangle + \varepsilon$, then for any $\lambda > 0$ there exists x_λ, x^*_λ such that

$$h(x_\lambda, x^*_\lambda) = \langle x_\lambda, x^*_\lambda \rangle, \qquad \|x_\lambda - x\| < \lambda, \quad \|x^*_\lambda - x^*\| < \varepsilon/\lambda.$$

Using Theorem 2.2, the authors proved [12] that condition (9) ensures that h represents a maximal monotone operator. Here we will be interested also in the case where the lower semicontinuity assumption is removed.

Theorem 2.3 ([12, Theorem 4.2, Corollary 4.4]). *Let $h : X \times X^* \to \bar{\mathbb{R}}$ be a convex function. If*

$$h \geq \pi, \qquad h^* \geq \pi_*$$

then

$$T = \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone and satisfy the restricted Brøndsted-Rockafellar property. Additionally, if h is also lower semicontinuous, then

$$T = \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}.$$

We will need the following immediate consequence of the above theorem:

Corollary 2.4. *Let $h : X \times X^* \to \bar{\mathbb{R}}$. If*

$$\operatorname{conv} h \geq \pi, \qquad h^* \geq \pi_*$$

then

$$\begin{aligned} T &= \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\} \\ &= \{(x, x^*) \in X \times X^* \mid \mathcal{J}h(x, x^*) = \langle x, x^* \rangle\} \end{aligned}$$

is maximal monotone,

$$T = \{(x, x^*) \in X \times X^* \mid \operatorname{cl}\operatorname{conv} h(x, x^*) = \langle x, x^* \rangle\}$$

$\operatorname{cl}\operatorname{conv} h \in \mathfrak{F}_T$ *and* $\mathcal{J}h \in \mathfrak{F}_T$*, where* $\mathcal{J}h(x, x^*) = h^*(x^*, x)$.

Proof. As the duality product is continuous in $X \times X^*$, $\operatorname{cl}\operatorname{conv} h \geq \pi$. As conjugation is invariant under the conv operation and the (lower semicontinuous) closure, $(\operatorname{cl}\operatorname{conv} h)^* = h^* \geq \pi_*$. To end the proof, apply Theorem 2.3 to $\operatorname{cl}\operatorname{conv} h$, observe that $\mathcal{J}h$ is convex, lower semicontinuous, $\mathcal{J}h \geq \pi$ and use definition (5). $\square$

In a non-reflexive Banach Space X, if $T : X \rightrightarrows X^*$ is maximal monotone and for some $h \in \mathfrak{F}_T$ it holds that $h \geq \pi$, $h^* \geq \pi_*$, then T behaves similarly to a maximal monotone operator in a *reflexive* Banach space. A natural question is: what is the class of maximal monotone operators (in non-reflexive Banach spaces) which have some function in Fitzpatrick family satisfying (9)? To answer this question, first let us recall the definition of maximal monotone operators of type NI [20].

Definition 2.5. A maximal monotone operator $T : X \rightrightarrows X^*$ is of type NI if

$$\inf_{(y,y^*)\in T} \langle y^* - x^*, x^{**} - y\rangle \le 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

In [22] it was observed that if T is a maximal monotone operators of type NI, then $\mathcal{S}_T$ satisfies condition (9).

Martínez-Legaz and Svaiter defined [14] (with a different notation), for $h : X \times X^* \to \bar{\mathbb{R}}$ and $(x_0, x_0^*) \in X \times X^*$

$$\begin{aligned} &h_{(x_0,x_0^*)} : X \times X^* \to \bar{\mathbb{R}}, \\ &\qquad h_{(x_0,x_0^*)}(x, x^*) := h(x + x_0, x^* + x_0^*) - [\langle x, x_0^*\rangle + \langle x_0, x^*\rangle + \langle x_0, x_0^*\rangle]. \end{aligned} \tag{10}$$

The operation $h \mapsto h_{(x_0,x_0^*)}$ preserves many properties of h, as convexity, lower semicontinuity and can be seen as the action of the group $(X \times X^*, +)$ on $\bar{\mathbb{R}}^{X\times X^*}$, because

$$\left(h_{(x_0,x_0^*)}\right)_{(x_1,x_1^*)} = h_{(x_0+x_1,x_0^*+x_1^*)}.$$

Moreover

$$\left(h_{(x_0,x_0^*)}\right)^* = (h^*)_{(x_0^*,x_0)},$$

where the rightmost x_0 is identified with its image under the canonical injection of X into X^{**}. Therefore,

1. $h \ge \pi \iff h_{(x_0,x_0)} \ge \pi$,
2. $\left(h_{(x_0,x_0^*)}\right)^* \ge \pi_* \iff (h^*)_{(x_0^*,x_0)} \ge \pi_*$,

We shall need the following theorem. Its proof is heavily based on these nice properties of the map $h \mapsto h_{(x_0,x_0^*)}$ and it is presented on the Appendix A.

Theorem 2.6 ([13, Theorem 1.2]). *Let $T : X \rightrightarrows X^*$ be maximal monotone. The following conditions are equivalent*

1. *T is of type NI,*
2. *there exists $h \in \mathcal{F}_T$ such that $h \ge \pi$ and $h^* \ge \pi_*$,*
3. *for all $h \in \mathcal{F}_T$, $h \ge \pi$ and $h^* \ge \pi_*$,*
4. *there exists $h \in \mathcal{F}_T$ such that*

$$\inf h_{(x_0,x_0^*)} + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*,$$

5. *for all $h \in \mathcal{F}_T$,*

$$\inf h_{(x_0,x_0^*)} + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

3. Surjectivity and maximal monotonicity in non-reflexive Banach spaces

We begin with two elementary technical results which will be useful.

Proposition 3.1. *The following statements holds:*

1. *For any $\varepsilon \geq 0$, if $y^* \in J_\varepsilon(x)$, then $|\,\|x\| - \|y^*\|\,| \leq \sqrt{2\varepsilon}$.*
2. *Let $T : X \rightrightarrows X^*$ be a monotone operator and $\varepsilon, M > 0$. Then,*

$$(T + J_\varepsilon)^{-1}\left(B_{X^*}[0, M]\right)$$

is bounded.

Proof. To prove item *1.*, let $\varepsilon \geq 0$ and $y^* \in J_\varepsilon(x)$. The desired result follows from the following inequalities:

$$\frac{1}{2}(\|x\| - \|y^*\|)^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \langle x, y^* \rangle \leq \varepsilon.$$

To prove item *2.*, take $(z, z^*) \in T$. If $x \in (T + J_\varepsilon)^{-1}(B[0, M])$ then there exists x^*, y^* such that

$$x^* \in T(x), \quad y^* \in J_\varepsilon(x), \qquad \|x^* + y^*\| \leq M.$$

Therefore, using Fenchel Young inequality (2), the monotonicity of T and the definition of J_ε we obtain

$$\begin{aligned}
\frac{1}{2}\|x - z\|^2 + \frac{1}{2}\|x^* + y^* - z^*\|^2 &\geq \langle x - z, x^* + y^* - z^* \rangle \\
&\geq \langle x - z, y^* \rangle \\
&\geq \left[\frac{1}{2}\|x\|^2 + \frac{1}{2}\|y^*\|^2 - \varepsilon\right] - \|z\|\|y^*\|.
\end{aligned}$$

Note also that

$$\|x - z\|^2 \leq \|x\|^2 + 2\|x\|\|z\| + \|z\|^2, \qquad \|x^* + y^* - z^*\|^2 \leq (M + \|z^*\|)^2.$$

Combining the above equations we obtain

$$\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + \|z^*\|)^2 \geq \frac{1}{2}\|y^*\|^2 - \|x\|\|z\| - \|z\|\|y^*\| - \varepsilon.$$

As $y^* \in J_\varepsilon(x)$, by item *1.*, we have $\|x\| \leq \|y^*\| + \sqrt{2\varepsilon}$. Therefore

$$\frac{1}{2}\|z\|^2 + \frac{1}{2}(M + \|z^*\|)^2 \geq \frac{1}{2}\|y^*\|^2 - 2\|y^*\|\|z\| - \|z\|\sqrt{2\varepsilon} - \varepsilon.$$

Hence, y^* is bounded. In fact,

$$\|y^*\| \leq 2\|z\| + \sqrt{4\|z\|^2 + 2\left[\|z\|\sqrt{2\varepsilon} + \varepsilon\right] + \|z\|^2 + (M + \|z^*\|)^2}.$$

As we already observed, $\|x\| \leq \|y^*\| + \sqrt{2\varepsilon}$ and so, x is also bounded. $\square$

Now we will prove that under monotonicity, dense range of some perturbation of a monotone operator is equivalent to surjectivity of that perturbation.

Lemma 3.2. *Let $T : X \rightrightarrows X^*$ be monotone and $\mu > 0$. Then the conditions below are equivalent*

1. $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$, *for any* $\varepsilon > 0$ *and* $z_0 \in X$,
2. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$, *for any* $\varepsilon > 0$ *and* $z_0 \in X$.

Proof. It suffices to prove the lemma for $\mu = 1$ and then, for the general case, consider $T' = \mu^{-1}T$. Now note that for any $z_0 \in X$ and $z_0^* \in X^*$, $T - \{(z_0, z_0^*)\}$ is also monotone. Therefore, it suffices to prove that $0 \in \overline{R(T + J_\varepsilon)}$, for any $\varepsilon > 0$ if and only if $0 \in R(T + J_\varepsilon)$, for any $\varepsilon > 0$. The "if" is easy to check. To prove the "only if", suppose that

$$0 \in \overline{R(T + J_\varepsilon)}, \quad \forall \varepsilon > 0.$$

First use item *2.* of Proposition 3.1 with $M = 1/2$ to conclude that there exists $\rho > 0$ such that

$$(T + J_{1/2})^{-1}\left(B_{X^*}[0, 1/2]\right) \subset B_X[0, \rho].$$

By assumption, for any $0 < \eta < \frac{1}{2}$ there exists $x_\eta \in X$, $x_\eta^*, y_\eta^* \in X^*$ such that

$$x_\eta^* \in T(x_\eta), \quad y_\eta^* \in J_\eta(x_\eta) \quad \text{and} \quad \|x_\eta^* + y_\eta^*\| < \eta < \frac{1}{2}. \tag{11}$$

As $J_\eta(x_\eta) \subset J_{1/2}(x_\eta)$, $x_\eta \in (T + J_{1/2})^{-1}(x_\eta^* + y_\eta^*)$ and so,

$$\|x_\eta\| \le \rho, \qquad \|y_\eta^*\| \le \rho + 1,$$

where the second inequality follows from the first one and item *1.* of Proposition 3.1. Therefore

$$\frac{1}{2}\|x_\eta^*\|^2 \le \frac{1}{2}\left(\|x_\eta^* + y_\eta^*\| + \|y_\eta^*\|\right)^2 \le \frac{1}{2}\eta^2 + \eta(\rho + 1) + \frac{1}{2}\|y_\eta^*\|^2,$$

$$\langle x_\eta, x_\eta^* \rangle = \langle x_\eta, x_\eta^* + y_\eta^* \rangle - \langle x_\eta, y_\eta^* \rangle \le \rho\eta - \langle x_\eta, y_\eta^* \rangle.$$

Combining the above inequalities we obtain

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \le \frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle + \eta(2\rho + 1) + \frac{1}{2}\eta^2.$$

The inclusion $y_\eta^* \in J_\eta(x_\eta)$ means that,

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|y_\eta^*\|^2 - \langle x_\eta, y_\eta^* \rangle \le \eta. \tag{12}$$

Hence, using the two above inequalities we conclude that

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^* \rangle \le 2\eta(\rho + 1) + \frac{1}{2}\eta^2.$$

To end the prove, take an arbitrary $\varepsilon > 0$. Choosing $0 < \eta < 1/2$ such that,

$$2\eta(\rho + 1) + \frac{1}{2}\eta^2 < \varepsilon,$$

we have

$$\frac{1}{2}\|x_\eta\|^2 + \frac{1}{2}\|x_\eta^*\|^2 + \langle x_\eta, x_\eta^*\rangle < \varepsilon, \quad x_\eta^* \in T(x_\eta).$$

According tho the above inequality, $-x_\eta^* \in J_\varepsilon(x_\eta)$. Hence $0 \in (T + J_\varepsilon)(x_\eta)$. □

In a reflexive Banach space, surjectivity of a monotone operator plus the duality mapping is equivalent to maximal monotonicity. This is a classical result of Rockafellar [19]. To obtain a partial extension of this result to non-reflexive Banach spaces, we must consider the "enlarged" duality mapping.

Lemma 3.3. *Let $T : X \rightrightarrows X^*$ be monotone and $\mu > 0$. If*

$$\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*, \quad \forall \varepsilon > 0, z_0 \in X$$

then $\overline{T}$, the closure of T in the norm-topology of $X \times X^*$, is maximal monotone and of type NI.

Proof. Note that $T + \mu J_\varepsilon = \mu(\mu^{-1}T + J\varepsilon)$. Therefore, it suffices to prove the lemma for $\mu = 1$ and then, for the general case, consider $T' = \mu^{-1}T$. The monotonicity of $\bar{T}$ follows from the continuity of the duality product.

Using the assumptions on T and Lemma 3.2 we conclude that $T(\cdot + z_0) + J_\varepsilon$ is onto, for any $\varepsilon > 0$ and $z_0 \in X$. Therefore, for any $(z_0, z_0^*) \in X \times X^*$ and $\varepsilon > 0$, there exists x_ε, x_ε^* such that

$$x_\varepsilon^* + z_0^* \in T(x_\varepsilon + z_0) \quad \text{and} \quad -x_\varepsilon^* \in J_\varepsilon(x_\varepsilon). \tag{13}$$

Note that the second inclusion in the above equation is equivalent to

$$\frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \leq \langle x_\varepsilon, -x_\varepsilon^*\rangle + \varepsilon. \tag{14}$$

To prove maximal monotonicity of $\bar{T}$, suppose that $(z_0, z_0^*) \in X \times X^*$ is monotonically related to $\bar{T}$. As $T \subset \bar{T}$

$$\langle z - z_0, z^* - z_0^*\rangle \geq 0, \quad \forall\ (z, z^*) \in T.$$

So, taking $\varepsilon > 0$ and $x_\varepsilon \in X$, $x_\varepsilon^* \in X^*$ as in (13) we conclude that

$$\langle x_\varepsilon, x_\varepsilon^*\rangle = \langle x_\varepsilon + z_0 - z_0, x_\varepsilon^* + z_0^* - z_0^*\rangle \geq 0,$$

which, combined with (14) yields

$$\frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \leq \varepsilon.$$

As $(x_\varepsilon + z_0, x_\varepsilon^* + z_0^*) \in T$, and ε is an arbitrary strictly positive number, we conclude that $(z_0, z_0^*) \in \bar{T}$, and $\bar{T}$ is maximal monotone.

It remains to prove that $\bar{T}$ is of type NI. Consider an arbitrary $(z_0, z_0^*) \in X \times X^*$ and $h \in \mathfrak{F}_{\bar{T}}$. Then, using (13), (14) we conclude that for any $\varepsilon > 0$, there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$h(x_\varepsilon + z_0, x_\varepsilon^* + z_0^*) = \langle x_\varepsilon + z_0, x_\varepsilon^* + z_0^*\rangle, \qquad \frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 \le \langle x_\varepsilon, -x_\varepsilon^*\rangle + \varepsilon.$$

The first equality above is equivalent to $h_{(z_0,z_0^*)}(x_\varepsilon, x_\varepsilon^*) = \langle x_\varepsilon, x_\varepsilon^*\rangle$. Therefore,

$$h_{(z_0,z_0^*)}(x_\varepsilon, x_\varepsilon^*) + \frac{1}{2}\|x_\varepsilon\|^2 + \frac{1}{2}\|x_\varepsilon^*\|^2 < \varepsilon,$$

that is,

$$\inf h_{(z_0,z_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0.$$

Now, use item *5.* of Theorem 2.6 to conclude that $\bar{T}$ is of type NI. □

Direct application of Lemma 3.3 gives the next corollary.

Corollary 3.4. *If $T : X \rightrightarrows X^*$ is monotone, closed, $\mu > 0$ and*

$$\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*, \quad \forall \varepsilon > 0, z_0 \in X$$

then T, is maximal monotone and of type NI.

Proof. Use Lemma 3.3 and the assumption $T = \bar{T}$. □

Lemma 3.5. *Let $T_1, T_2 : X \rightrightarrows X^*$ be maximal monotone and of type NI. Take*

$$h_1 \in \mathfrak{F}_{T_1}, \qquad h_2 \in \mathfrak{F}_{T_2}$$

and define

$$\begin{aligned} &h : X \times X^* \to \bar{\mathbb{R}} \\ &\qquad h(x, x^*) = (h_1(x, \cdot)\Box h_2(x, \cdot))(x^*) = \inf_{y^* \in X^*} h_1(x, y^*) + h_2(x, x^* - y^*), \end{aligned}$$

$$D_X(h_i) = \{x \in X \mid \exists\, x^*,\ h_i(x, x^*) < \infty\}, \quad i = 1, 2.$$

If

$$\bigcup_{\lambda > 0} \lambda(D_X(h_1) - D_X(h_2)) \tag{15}$$

is a closed subspace then

$$h \ge \pi, h^* \ge \pi_*, \qquad \mathcal{J}h \ge \pi, (\mathcal{J}h)^* \ge \pi_*,$$

$$\begin{aligned} T_1 + T_2 &= \{(x, x^*) \mid \mathcal{J}h(x, x^*) = \langle x, x^*\rangle\} \\ &= \{(x, x^*) \mid h(x, x^*) = \langle x, x^*\rangle\} \end{aligned}$$

and $T_1 + T_2$ is maximal monotone of type NI and

$$\mathcal{J}h, \operatorname{cl} h \in \mathfrak{F}_{T_1 + T_2}.$$

Proof. Since $h_1 \in \mathfrak{F}_{T_1}$ and $h_2 \in \mathfrak{F}_{T_2}$, we have $h_1 \geq \pi$ and $h_2 \geq \pi$. So

$$h_1(x, y^*) + h_2(x, x^* - y^*) \geq \langle x, y^* \rangle + \langle x, x^* - y^* \rangle = \langle x, x^* \rangle.$$

Taking the inf in y^* at the left-hand side of the above inequality we conclude that $h \geq \pi$.

Let $(x^*, x^{**}) \in X^* \times X^{**}$. Using the definition of h we have

$$\begin{aligned} & h^*(x^*, x^{**}) \\ = & \sup_{(z,z^*) \in X \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h(z, z^*) & (16) \\ = & \sup_{(z,z^*,y^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle z^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, z^* - y^*) & (17) \\ = & \sup_{(z,y^*,w^*) \in X \times X^* \times X^*} \langle z, x^* \rangle + \langle y^*, x^{**} \rangle + \langle w^*, x^{**} \rangle - h_1(z, y^*) - h_2(z, w^*) & (18) \end{aligned}$$

where we used the substitution $z^* = w^* + y^*$ in the last term. So, defining $H_1, H_2 : X \times X^* \times X^* \to \bar{\mathbb{R}}$

$$H_1(x, y^*, z^*) = h_1(x, y^*), \qquad H_2(x, y^*, z^*) = h_2(x, z^*), \tag{19}$$

we have

$$h^*(x^*, x^{**}) = (H_1 + H_2)^*(x^*, x^{**}, x^{**}).$$

Using (15), the Attouch-Brezis extension [1, Theorem 1.1] of Fenchel-Rockafellar duality theorem and (19) we conclude that the conjugate of the sum at the right hand side of the above equation is the *exact* inf-convolution of the conjugates. Therefore,

$$h^*(x^*, x^{**}) = \min_{(u^*,y^{**},z^{**})} H_1^*(u^*, y^{**}, z^{**}) + H_2^*(x^* - u^*, x^{**} - y^{**}, x^{**} - z^{**}).$$

Direct use of definition (19) yields

$$H_1^*(u^*, y^{**}, z^{**}) = h_1^*(u^*, y^{**}) + \delta_0(z^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}, \tag{20}$$

$$H_2^*(u^*, y^{**}, z^{**}) = h_2^*(u^*, z^{**}) + \delta_0(y^{**}), \quad \forall (u^*, y^{**}, z^{**}) \in X^* \times X^{**} \times X^{**}. \tag{21}$$

Hence,

$$h^*(x^*, x^{**}) = \min_{u^* \in X^*} h_1^*(u^*, x^{**}) + h_2^*(x^* - u^*, x^{**}). \tag{22}$$

Therefore, using that $h_1^* \geq \pi_*$, $h_2^* \geq \pi_*$, (22) and the same reasoning used to show that $h \geq \pi$ we have

$$h^* \geq \pi^*.$$

Up to now, we proved that $h \geq \pi$ and $h^* \geq \pi_*$(and $\mathcal{J}h \geq \pi$). So, using Theorem 2.3 we conclude that $S : X \rightrightarrows X^*$, defined as

$$S = \{(x, x^*) \in X \times X^* \mid \mathcal{J}h(x, x^*) = \langle x, x^* \rangle\},$$

is maximal monotone. As $\mathcal{J}h$ is convex and lower semicontinuous, $\mathcal{J}h \in \mathfrak{F}_S$.

We will prove that $T_1 + T_2 = S$. Take $(x, x^*) \in S$, that is, $\mathcal{J}h(x, x^*) = \langle x, x^*\rangle$. Using (22) we conclude that there exists $u^* \in X^*$ such that

$$h_1^*(u^*, x) + h_2^*(x^* - u^*, x) = \langle x, x^*\rangle.$$

We know that

$$h_1^*(u^*, x) \geq \langle x, u^*\rangle, \qquad h_2^*(x^* - u^*, x) \geq \langle x, x^* - u^*\rangle.$$

Combining these inequalities with the previous equation we conclude that these inequalities hold as equalities, and so

$$\begin{array}{ccc} u^* \in T_1(x), & x^* - u^* \in T_2(x), & x^* \in (T_1 + T_2)(x). \\ h_1(x, u^*) = \langle x, u^*\rangle, & h_2(x, x^* - u^*) = \langle x, x^* - u^*\rangle, & h(x, x^*) \leq \langle x, x^*\rangle. \end{array}$$

We proved that $S \subset T_1 + T_2$. Since $T_1 + T_2$ is monotone and S is maximal monotone, we have $T_1 + T_2 = S$ (and $\mathcal{J}h \in \mathcal{F}_{T_1+T_2}$). Note also that $h(x, x^*) \leq \langle x, x^*\rangle$ for any $(x, x^*) \in T_1 + T_2 = S$. As $h \geq \pi$, we have equality in $T_1 + T_2$. Therefore,

$$T_1 + T_2 \subset \{(x, x^*) \mid h(x, x^*) = \langle x, x^*\rangle\} \subset \{(x, x^*) \mid \operatorname{cl} h(x, x^*) \leq \langle x, x^*\rangle\}.$$

Since $h \geq \pi$ and the duality product π is *continuous* in $X \times X^*$, we also have $\operatorname{cl} h \geq \pi$. Hence, using the above inclusion we conclude that $\operatorname{cl} h$ coincides with π in $T_1 + T_2$. Therefore, $\operatorname{cl} h \in \mathcal{F}_{T_1+T_2}$ and the rightmost set in the above inclusions is $T_1 + T_2$. Hence

$$T_1 + T_2 = \{(x, x^*) \mid h(x, x^*) = \langle x, x^*\rangle\}.$$

Conjugation is invariant under the (lower semicontinuous) closure operation. Therefore,

$$(\operatorname{cl} h)^* = h^* \geq \pi_*$$

and so $T_1 + T_2$ is NI. We proved already that $\mathcal{J}h \in \mathcal{F}_{T_1+T_2}$. Using item *3.* of Theorem 2.6 we conclude that $(\mathcal{J}h)^* \geq \pi_*$. □

Theorem 3.6. *If $T : X \rightrightarrows X^*$ is a closed monotone operator, then the conditions below are equivalent*

1. $\overline{R(T(\cdot + z_0) + J)} = X^*$ *for all* $z_0 \in X$,
2. $\overline{R(T(\cdot + z_0) + J_\varepsilon)} = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$,
3. $R(T(\cdot + z_0) + J_\varepsilon) = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$,
4. *T is maximal monotone and of type NI.*

Proof. Item *1.* trivially implies item *2.*. Using Lemma 3.2 we conclude that, in particular, item *2.* implies item *3.*. Now use Corollary 3.4 to conclude that item *3.* implies item *4.*. Up to now we have *1.* ⇒ *2.* ⇒ *3.* ⇒ *4.*.

To complete the proof we will show that item *4.* implies item *1.*. So, assume that item *4.* holds, that is, T is of type NI. Take $z_0^* \in X^*$ and $z_0 \in X$. Define $T_0 = T - \{(z_0, z_0^*)\}$. Trivially

$$z_0^* \in \overline{R(T(\cdot + z_0) + J)} \iff 0 \in \overline{R(T_0 + J)}.$$

As the class NI is invariant under translations, in order to prove item *1.*, it is sufficient to prove that if T is of type NI, then $0 \in \overline{R(T+J)}$. Let $h \in \mathcal{F}_T$ and $\varepsilon > 0$. Define $p : X \times X^* \to \mathbb{R}$,

$$p(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2. \tag{23}$$

Item *5.* of Theorem 2.6 ensure us that there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2. \tag{24}$$

Direct calculations yields $p \geq \pi$ and $p^* \geq \pi_*$. We also know that $p \in \mathcal{F}_J$ and so J is of type NI. Define $H : X \times X^* \to \bar{\mathbb{R}}$,

$$H(x, x^*) = \inf_{y^* \in X^*} h(x, y^*) + p(x, x^* - y^*).$$

As $D(p) = X \times X^*$, we may apply Lemma 3.5 to conclude that $T + J$ is NI and $\operatorname{cl} H \in \mathcal{F}_{T+J}$. Using (24) we have

$$H(x_\varepsilon, 0) \leq h(x_\varepsilon, x_\varepsilon^*) + p(x_\varepsilon, -x_\varepsilon^*) < \varepsilon^2.$$

So, $\operatorname{cl} H(x_\varepsilon, 0) \leq H(x_\varepsilon, 0) < \langle x_\varepsilon, 0\rangle + \varepsilon^2$. Now use Theorem 2.2 to conclude that there exists $\bar{x}$, $\bar{x}^*$ such that

$$(\bar{x}, \bar{x}^*) \in T + J, \qquad \|\bar{x} - x_\varepsilon\| < \varepsilon, \qquad \|\bar{x}^* - 0\| < \varepsilon.$$

So, $\bar{x}^* \in R(T + J)$ and $\|\bar{x}^*\| < \varepsilon$. As $\varepsilon > 0$ is arbitrary, 0 is in the closure of $R(T + J)$. □

Corollary 3.7. *If $T : X \rightrightarrows X^*$ is a closed monotone operator then the conditions bellow are equivalent*

a. $\overline{R(T(\cdot + z_0) + \mu J)} = X^*$ *for all* $z_0 \in X$ *and some* $\mu > 0$,
b. $\overline{R(T(\cdot + z_0) + \mu J)} = X^*$ *for all* $z_0 \in X$, $\mu > 0$,
c. $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$ *and some* $\mu > 0$,
d. $\overline{R(T(\cdot + z_0) + \mu J_\varepsilon)} = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$, $\mu > 0$,
e. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$, *and some* $\mu > 0$,
f. $R(T(\cdot + z_0) + \mu J_\varepsilon) = X^*$ *for all* $\varepsilon > 0$, $z_0 \in X$, $\mu > 0$,
g. *T is maximal monotone and of type NI.*

Proof. Suppose that item *a.* holds. Define $T' = \mu^{-1}T$ and use Theorem 3.6 to conclude that T' is maximal monotone and of type NI. Therefore, $T = \mu T'$ is maximal monotone and of type NI, which means that *g.* holds.

Now assume that item *g.* holds, that is, T is maximal monotone and of type NI. Then, for all $\mu > 0$, $\mu^{-1}T$ is maximal monotone and of type NI, which implies item *b.*.

As the implication *b.* $\Rightarrow$ *a.* is trivial, we conclude that items *a.*, *b.*, *g.* are equivalent.

The same reasoning shows that items *c.*, *d.*, *g.* are equivalent and so on. □

A. Proof of Theorem 2.6

Proof. First let us prove that item *2.* and item *4.* are equivalent. So, suppose item *2.* holds and let $(x_0, x_0^*) \in X \times X^*$. Direct calculations yield

$$h_{(x_0,x_0^*)} \geq \pi, \qquad (h_{(x_0,x_0^*)})^* \geq \pi_*.$$

Using [12, Theorem 3.1, Eq. (12)] we conclude that condition item *4.* holds. For proving that item *4.* $\Rightarrow$ item *2.*, first note that, for any $(z, z^*) \in X \times X^*$,

$$h_{(z,z^*)}(0,0) \geq \inf_{(x,x^*)} h_{(z,z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2.$$

Therefore, using item *4.* we obtain

$$h(z, z^*) - \langle z, z^* \rangle = h_{(z,z^*)}(0,0) \geq 0.$$

Since (z, z^*) is an arbitrary element of $X \times X^*$ we conclude that $h \geq \pi$.

For proving that, $h^* \geq \pi_*$, take some $(y^*, y^{**}) \in X^* \times X^{**}$. First, use Fenchel-Young inequality to conclude that for any $(x, x^*), (z, z^*) \in X \times X^*$,

$$h_{(z,z^*)}(x, x^*) \geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle - \left(h_{(z,z^*)}\right)^* (y^* - z^*, y^{**} - z).$$

As $\left(h_{(z,z^*)}\right)^* = (h^*)_{(z^*,z)}$,

$$\begin{aligned}\left(h_{(z,z^*)}\right)^* (y^* - z^*, y^{**} - z) &= h^*(y^*, y^{**}) - \langle z, y^* - z^* \rangle - \langle z^*, y^{**} - z \rangle - \langle z, z^* \rangle \\ &= h^*(y^*, y^{**}) - \langle y^*, y^{**} \rangle + \langle y^* - z^*, y^{**} - z \rangle.\end{aligned}$$

Combining the two above equations we obtain

$$\begin{aligned}&h_{(z,z^*)}(x, x^*) \\ &\geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}).\end{aligned}$$

Adding $(1/2)\|x\|^2 + (1/2)\|x^*\|^2$ in both sides of the above inequality we have

$$\begin{aligned}&h_{(z,z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\geq \langle x, y^* - z^* \rangle + \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\qquad - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}).\end{aligned}$$

Note that

$$\langle x, y^* - z^* \rangle + \frac{1}{2}\|x\|^2 \geq -\frac{1}{2}\|y^* - z^*\|^2, \qquad \langle x^*, y^{**} - z \rangle + \frac{1}{2}\|x^*\|^2 \geq -\frac{1}{2}\|y^{**} - z\|^2.$$

Therefore, for any $(x, x^*), (z, z^*) \in X \times X^*$,

$$\begin{aligned}&h_{(z,z^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \\ &\geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 \\ &\qquad - \langle y^* - z^*, y^{**} - z \rangle + \langle y^*, y^{**} \rangle - h^*(y^*, y^{**}).\end{aligned}$$

Using now the assumption we conclude that the infimum, for $(x, x^*) \in X \times X^*$, at the left hand side of the above inequality is 0. Therefore, taking the infimum on $(x, x^*) \in X \times X^*$ at the left hand side of the above inequality and rearranging the resulting inequality we have

$$h^*(y^*, y^{**}) - \langle y^*, y^{**}\rangle \geq -\frac{1}{2}\|y^* - z^*\|^2 - \frac{1}{2}\|y^{**} - z\|^2 - \langle y^* - z^*, y^{**} - z\rangle.$$

Note that

$$\sup_{z^* \in X^*} -\langle y^* - z^*, y^{**} - z\rangle - \frac{1}{2}\|y^* - z^*\|^2 = \frac{1}{2}\|y^{**} - z\|^2.$$

Hence, taking the sup in $z^* \in X^*$ at the right hand side of the previous inequality we obtain

$$h^*(y^*, y^{**}) - \langle y^*, y^{**}\rangle \geq 0$$

and item *4.* holds. Now, using that item *2.* and item *4.* are equivalent it is trivial to verify that item *3.* and item *5.* are equivalent.

The second step is to prove that item *4.* and item *5.* are equivalent. So, assume that item *4.* holds, that is, for some $h \in \mathfrak{F}_T$,

$$\inf_{(x,x^*) \in X \times X^*} h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0, \quad \forall (x_0, x_0^*) \in X \times X^*.$$

Take $g \in \mathfrak{F}_T$, and $(x_0, x_0^*) \in X \times X^*$. First observe that, for any $(x, x^*) \in X \times X^*$, $g_{(x_0,x_0^*)}(x, x^*) \geq \langle x, x^*\rangle$ and

$$g_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq \langle x, x^*\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0.$$

Therefore,

$$\inf_{(x,x^*) \in X \times X^*} g_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \geq 0. \tag{25}$$

As the square of the norm is coercive, there exist $M > 0$ such that

$$\left\{(x, x^*) \in X \times X^* \mid h_{(x_0,x_0^*)}(x, x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 < 1\right\} \subset B_{X \times X^*}(0, M),$$

where

$$B_{X \times X^*}(0, M) = \left\{(x, x^*) \in X \times X^* \mid \sqrt{\|x\|^2 + \|x^*\|^2} < M\right\}.$$

For any $\varepsilon > 0$, there exists $(\tilde{x}, \tilde{x}^*)$ such that

$$\min\left\{1, \varepsilon^2\right\} > h_{(x_0,x_0^*)}(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2.$$

Therefore

$$\begin{gathered} \varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x}, \tilde{x}^*) + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 \geq h_{(x_0,x_0^*)}(\tilde{x}, \tilde{x}^*) - \langle \tilde{x}, \tilde{x}^*\rangle \geq 0, \\ M^2 \geq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2. \end{gathered} \tag{26}$$

In particular,

$$\varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) - \langle \tilde{x},\tilde{x}^*\rangle.$$

Now using Theorem 2.2 we conclude that there exists $(\bar{x},\bar{x}^*)$ such that

$$h_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) = \langle \bar{x},\bar{x}^*\rangle, \qquad \|\tilde{x}-\bar{x}\| < \varepsilon, \qquad \|\tilde{x}^*-\bar{x}^*\| < \varepsilon. \tag{27}$$

Therefore,

$$h(\bar{x}+x_0,\bar{x}^*+x_0^*) - \langle \bar{x}+x_0,\bar{x}^*+x_0^*\rangle = h_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) - \langle \bar{x},\bar{x}^*\rangle = 0,$$

and $(\bar{x}+x_0,\bar{x}^*+x_0^*) \in T$. As $g \in \mathfrak{F}_T$,

$$g(\bar{x}+x_0,\bar{x}^*+x_0^*) = \langle \bar{x}+x_0,\bar{x}^*+x_0^*\rangle,$$

and

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) = \langle \bar{x},\bar{x}^*\rangle. \tag{28}$$

Using the first line of (26) we have

$$\varepsilon^2 > h_{(x_0,x_0^*)}(\tilde{x},\tilde{x}^*) + \left[\frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle\right] - \langle \tilde{x},\tilde{x}^*\rangle \geq \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle.$$

Therefore,

$$\varepsilon^2 > \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + \langle \tilde{x},\tilde{x}^*\rangle. \tag{29}$$

Direct use of (27) gives

$$\begin{aligned}
\langle \bar{x},\bar{x}^*\rangle &= \langle \tilde{x},\tilde{x}^*\rangle + \langle \bar{x}-\tilde{x},\tilde{x}^*\rangle + \langle \tilde{x},\bar{x}^*-\tilde{x}^*\rangle + \langle \bar{x}-\tilde{x},\bar{x}^*-\tilde{x}^*\rangle \\
&\leq \langle \tilde{x},\tilde{x}^*\rangle + \|\bar{x}-\tilde{x}\|\,\|\tilde{x}^*\| + \|\tilde{x}\|\,\|\bar{x}^*-\tilde{x}^*\| + \|\bar{x}-\tilde{x}\|\,\|\bar{x}^*-\tilde{x}^*\| \\
&\leq \langle \tilde{x},\tilde{x}^*\rangle + \varepsilon[\|\tilde{x}^*\| + \|\tilde{x}\|] + \varepsilon^2
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{x}\|^2 + \|\bar{x}^*\|^2 \leq\ & (\|\tilde{x}\| + \|\bar{x}-\tilde{x}\|)^2 + (\|\tilde{x}^*\| + \|\bar{x}^*-\tilde{x}^*\|)^2 \\
&\leq \|\tilde{x}\|^2 + \|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2.
\end{aligned}$$

Combining the two above equations with (28) we obtain

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq \langle \tilde{x},\tilde{x}^*\rangle + \frac{1}{2}\|\tilde{x}\|^2 + \frac{1}{2}\|\tilde{x}^*\|^2 + 2\varepsilon[\|\tilde{x}\| + \|\tilde{x}^*\|] + 2\varepsilon^2$$

Using now (29) and the second line of (26) we conclude that

$$g_{(x_0,x_0^*)}(\bar{x},\bar{x}^*) + \frac{1}{2}\|\bar{x}\|^2 + \frac{1}{2}\|\bar{x}^*\|^2 \leq 2\varepsilon\, M\sqrt{2} + 3\varepsilon^2.$$

As ε is an arbitrary strictly positive number, using also (25) we conclude that

$$\inf_{(x,x^*)\in X\times X^*} g_{(x_0,x_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = 0.$$

Altogether, we conclude that if item *4.* holds then item *5.* holds. The converse item *5.* $\Rightarrow$ item *4.* is trivial to verify. Hence item *4.* and item *5.* are equivalent. As item *2.* is equivalent to item *4.* and item *3.* is equivalent to *5.*, we conclude that items *2.*, *3.*, *4.* and *5.* are equivalent.

Now we will prove that item *1.* is equivalent to item *3.* and conclude the proof of the theorem. First suppose that item *3.* holds. Since $\mathcal{S}_T \in \mathcal{F}_T$

$$(\mathcal{S}_T)^* \geq \pi_*.$$

As has already been observed, for any proper function h it holds that $(\operatorname{cl}\operatorname{conv} h)^* = h^*$. Therefore

$$(\mathcal{S}_T)^* = (\pi + \delta_T)^* \geq \pi_*,$$

that is,

$$\sup_{(y,y^*)\in T} \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**} \tag{30}$$

After some algebraic manipulations we conclude that (30) is equivalent to

$$\inf_{(y,y^*)\in T} \langle x^{**} - y, x^* - y^* \rangle \leq 0, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$

that is, T is of type (NI) and so item *1.* holds. If item *1.* holds, by the same reasoning we conclude that (30) holds and therefore $(\mathcal{S}_T)^* \geq \pi_*$. As $\mathcal{S}_T \in \mathcal{F}_T$, we conclude that item *2.* holds. As has been proved previously item *2.* $\Rightarrow$ item *3.*. $\square$

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