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On the Characterization of a Class of Laminates for 2×2 Symmetric Gradients

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We report on our attempts to disprove the implication from rank-one convexity to quasiconvexity for 2×2 symmetric matrices. As a by-product, we have reached a characterization of some laminates, belonging to a special class which we call 3-edge-laminates.

Keywords: Vector calculus of variations, quasiconvexity, polyconvexity, rank-one convexity, homogeneous gradient Young measure, laminate

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1. Introduction

The question addressed in this work is the following: in the 2×2 symmetric case, is every homogeneous gradient Young measure a laminate? More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$, $f : \mathbb{M}^{m\times N} \to \mathbb{R}$ a continuous function defined on matrices and consider functionals of the type

$$I(u) = \int_\Omega f(\nabla u(x))\, dx.$$

In order to apply the so-called direct method of the Calculus of Variations to the above functional, one needs to ensure the (sequential) weak-* lower semicontinuity of I; and while in the scalar case ($N = 1$ or $m = 1$) the convexity of f is both a necessary and sufficient condition, in the vectorial case ($N, m > 1$) this condition is still sufficient, but is far from being necessary. More than 50 years ago Morrey showed in [12] that the adequate notion of convexity for the vector case is quasiconvexity of

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the integrand f, which can be defined by

$$f(A) \leq \int_{(0,1)^N} f(A + \nabla\xi(x))\, dx$$

for every $A \in \mathbb{M}^{m\times N}$ and every $\xi \in W_0^{1,\infty}((0,1)^N, \mathbb{R}^m)$. Here we prefer, however, to use the following equivalent definition:

$$f(A) \leq \int_{(0,1)^N} f(A + \nabla\xi(x))\, dx$$

for every $A \in \mathbb{M}^{m\times N}$ and every $(0,1)^N$-periodic $\xi \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$.

The quasiconvexity condition is in general very hard to verify, namely due to its nonlocal character. Thus two other related convexity conditions were introduced, which are easier to check: a sufficient condition, called polyconvexity, introduced by Ball in [1] (f is polyconvex if there is a convex function g such that

$$f(A) = g(M(A))$$

for every $A \in \mathbb{M}^{m\times N}$, where $M(A)$ is the vector of all minors of the matrix A); and, on the other hand, a necessary condition called rank-one convexity (f is rank-one convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

for every $t \in [0,1]$ and every $A, B \in \mathbb{M}^{m\times N}$ with $\operatorname{rank}\{A - B\} \leq 1$). While in the scalar case all these concepts are equivalent, in the vector case the polyconvexity condition is strictly stronger than quasiconvexity and weaker than convexity.

Consider now the implication from rank-one convexity to quasiconvexity. For the general case $N \geq 2$, $m \geq 3$, Šverák ([19]) disproved such implication with a remarkable counterexample, whose crucial point was the fact that when the dimension of the target space is $m \geq 3$, it is possible to find periodic deformations whose gradients take values in a subspace of $\mathbb{M}^{m\times N}$ having very few rank-one directions. Still today there does not exist any other such example. (Using Šverák's counterexample, Kristensen in [11] proved that when $N \geq 2, m \geq 3$ there is no local condition equivalent to quasiconvexity.) As to special cases, as has been known for a long time, when f is quadratic equivalence holds for every $N, m > 1$ (see [20], [21]; and for a more general proof, see [2]); while for 2×2 diagonal matrices, Müller ([13]) showed that equivalence holds.

What remains to be cleared up is the general $m = 2$ case: does rank-one convexity imply quasiconvexity in this case, or not? In this work we report on our attempts to find a negative answer, through a counterexample, to such question, even in the special case of 2×2 symmetric matrices.

But we prefer to use an alternative approach, as follows. The question whether rank-one convexity implies quasiconvexity can be restated in terms of laminates and homogeneous gradient Young measures: is every homogeneous gradient Young measure a laminate?

Laminates are a special subclass of probability measures (associated to rank-one convexity), which can be understood, at least conceptually, in a constructive way ([17]). The basic idea comes from the (H_k) conditions ([7]): a set of pairs $\{(\lambda_i, A_i)\}_{1\le i\le k}$ where $\lambda_i > 0$, $\sum_i \lambda_i = 1$, $A_i \in \mathbb{M}^{m\times N}$ is said to satisfy the (H_k) condition if:

1. for $k = 2$, $\operatorname{rank}\{A_1 - A_2\} \le 1$;
2. for $k \ge 2$, then, up to a permutation, $\operatorname{rank}\{A_1 - A_2\} \le 1$; and defining, for $2 \le i \le k-1$,

$$\begin{cases} \theta_1 = \lambda_1 + \lambda_2 & B_1 = \frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1+\lambda_2} \\ \theta_i = \lambda_{i+1} & B_i = A_{i+1} \end{cases}$$

then the pairs $(\theta_i, B_i)_{1\le i\le k-1}$ satisfy (H_{k-1}) (an example appears below, see (10) and Figure 3.1). A finite-order laminate μ_k is then defined by

$$\mu_k := \sum_{i=1}^{k} \lambda_i \delta_{A_i}.$$

In particular, fixing l, we call $(l-1)$-th order laminate to μ_l if it is generated by a set of pairs $\{(\lambda_i, A_i)\}_{1\le i\le l}$ satisfying the (H_l) condition. Then a laminate μ is any weak-* limit, in the sense of measures, of sequences of finite-order laminates

$$\mu_k \overset{*}{\rightharpoonup} \mu.$$

Laminates can also be characterized as the probability measures μ (with support on a compact set $K \subset \mathbb{M}^{m\times N}$) for which Jensen's inequality

$$f\left(\int_K A\, d\mu(A)\right) \le \int_K f(A)\, d\mu(A)$$

holds for rank-one convex functions f (see [17]).

On the other hand, concerning homogeneous gradient Young measures, they can be characterized, similarly, as the probability measures satisfying Jensen's inequality for quasiconvex functions f ([10]). Alternatively, homogeneous gradient Young measures can be defined as the probability measures μ for which there is a sequence $(u_j) \subset W^{1,\infty}(\Omega, \mathbb{R}^m)$ satisfying

$$u_j \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(\Omega, \mathbb{R}^m),$$

for which the sequence (∇u_j) generates the Young measure μ (in the sense that

$$\varphi(\nabla u_j) \overset{*}{\rightharpoonup} \int_K \varphi(A)\, d\mu(A) \quad \text{in } L^\infty(\Omega)$$

for each continuous φ). For simplicity, we will omit the term "homogeneous".

Similarly, a polyconvex measure is a probability measure for which Jensen's inequality holds for polyconvex functions ([17]). It turns out that polyconvex measures can be also characterized as the probability measures μ commuting with the minors of the matrices:

$$M\left(\int_K A\, d\mu(A)\right) = \int_K M(A)\, d\mu(A).$$

For each fixed barycenter

$$P_0 = \int_K A\, d\mu(A),$$

the set of all laminates is a convex subset of the set of all gradient Young measures, which in turn is a convex subset of the set of all polyconvex measures, itself convex.

In the $m = 2$ case, several authors have tried to answer the question of equivalence between quasiconvexity and rank-one convexity (see e.g. [14], [15], [18]), without success. Several explicit computations about this problem can be found in [9]. Other numerical computations on related subjects can be found in [3], [4], [5], [6]. The interested reader may find general reviews on this subjects in [16]. A general reference on the Calculus of Variations is [8].

Here we follow the attempt [15] of Pedregal to adapt the approach [19] of Šverák to the space of 2×2 symmetric matrices, using measures supported on the 8 vertices of the cube $[-1, 1]^3$. Pedregal has attempted the following strategy: to generate a point Q^- in the set of gradient Young measures, as extreme in this set as possible, with the aim of showing the impossibility of generating such Q^- as a laminate. Gradient Young measures with barycenter $(0, 0, 0)$ were used in his attempt.

Our contribution, in this work, aims at analyzing what we believe to be one of the best choices to try and find a counterexample for the 2×2 symmetric case, following the same strategy of [15], with other specially chosen barycenters. We have thus considered, besides $(0, 0, 0)$, for several reasons (in particular to keep the symmetry between the x and y coordinates, which translates into symmetry of the different sets of measures), the barycenters $(\frac{1}{3}, \frac{1}{3}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. But unfortunately also we have finally succeeded (see Theorem 2.5) to generate their corresponding laminates.

This is why we have then changed our focus, towards the problem of characterizing exactly the corresponding members of a precise class of laminates. Indeed, the characterization we have reached in Theorem 2.6 below concerns only the laminates of a certain type, which we call 3-edge-laminates (see Definition 2.2). They seem to generate (through convexity) all the laminates, but we were unable to prove this because, as is known and we have rediscovered, it is amazingly difficult, in general, to prove rigorously (in concrete examples) that a given gradient Young measure or polyconvex measure is not a laminate. Similarly, if one fixes an arbitrary polyconvex measure then it looks equally difficult to prove that it is not a gradient Young measure. All the computations done do not seem to relieve our doubts: they just reinforce our feeling that the relationship between rank-one convexity and quasiconvexity is not at all trivial or superficial; and (beyond the question of being able to find or not a counterexample) that both concepts, of laminates and gradient Young measures, are not yet well understood.

The organization of this work is as follows. In Section 2 we explain in detail the results obtained in this work, culminating in Theorems 2.5 and 2.6. This is complemented by Section 3, which proves Theorem 2.5, namely by exhibiting sets of points generating the laminates stated in it. As to Section 4, it deals with sets of polyconvex measures; while Section 5 concerns sets of gradient Young measures. In Section 6, after some preliminaries, we prove Theorem 2.6, characterizing the extreme points of the three sets of 3-edge-laminates (i.e. those corresponding to the barycenter $(a, a, 0)$ with $a = 0$, $a = \frac{1}{3}$, $a = \frac{1}{2}$), which is our main result. Finally, Section 7 concerns the conjecture (which seems to us quite reasonable) that the extreme points of the general set of laminates are (in the 2×2 symmetric case) themselves 3-edge-laminates. We report here on our computational experiments aimed at confirming (or invalidating) such conjecture; and after exploiting the whole power of a high capacity personal computer (namely: by expressing each laminate point exactly through rational coordinates, as a quotient of two integer numbers with 16 decimal digits), we have found nothing to contradict such conjecture.

As a concluding remark, we feel that to prove nonequivalence between laminates and gradient Young measures (for 2×2 symmetric matrices) one would need to find a means of enlarging the set of gradient Young measures which one is able to generate directly. Indeed, given the fact that the set of gradient Young measures is a 3-dimensional convex set with nonempty interior, it is quite unfortunate that one is now able to generate directly just a 1-dimensional subset (whose extreme points lie, moreover, generally in the interior of the convex set of laminates, instead of reaching, at least, its boundary)!

2. Definitions and description of our results

Since we start by generalizing, in a sense, the work of [15], namely by extending its study to two more barycenters, we use its same notations (namely those of [15, Section 4]).

Thus we consider barycenters which are 2×2 symmetric matrices of the form

$$P_0' = \begin{pmatrix} \alpha_1 + \alpha_3 & \alpha_3 \\ \alpha_3 & \alpha_2 + \alpha_3 \end{pmatrix},$$

represented by points

$$P_0 = (\alpha_1, \alpha_2, \alpha_3) \in [-1, 1]^3;$$

and using $[0,1]^2$-periodic functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we consider Lipschitz deformations $u : \mathbb{R}^2 \to \mathbb{R}^2$ of the type

$$u(x) = \nabla\varphi(x) + P_0' x \tag{1}$$

with u the superposition of 3 sawtooth waves with oscillations along directions $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively, meaning that $\nabla u(x)$ is, pointwise a.e., a symmetric matrix

$$\begin{pmatrix} x + z & z \\ z & y + z \end{pmatrix},$$

represented by a vector (x, y, z) assuming only 8 different values, namely the 8 vertices $\{-1, 1\}^3$ of the cube. We thus have gradient Young measures supported on the above 8 vertices; and likewise for polyconvex measures and laminates. Such measures are characterized by their barycenter P_0 and by the weights a, b, c on 3 vertices of the cube, hence we represent them as (compact convex) sets in (a, b, c)-space, as follows.

To simplify the presentation, instead of probability measures we use measures with total mass $:= 576$ $(= 24^2)$, so that our relevant vector measures become triples (a, b, c)

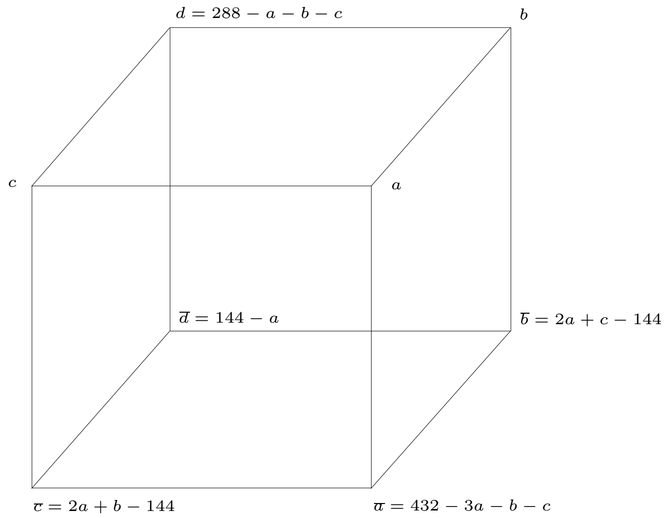

Figure 2.1: $P_0 = (0, 0, 0)$

of integer numbers, with few exceptions (which involve only the simple rational numbers 64.8, $74.(6) = 74 + \frac{2}{3}$, 76.5, $106.(6) = 106 + \frac{2}{3}$, $157.(09) = 157 + \frac{1}{11}$, 158.4); we thus avoid writing lots of cumbersome fractions.

Let us start by recalling the result of [15], concerning the barycenter $P_0 = (0, 0, 0)$, so that the reader of that paper becomes smoothly acquainted with our own geometric representations, before proceeding to the other two barycenters which are considered here.

With $P_0 = (0, 0, 0)$, as shown below (in Section 4), the weights on the vertices $\{-1, 1\}^3$ of the cube, for each polyconvex measure, can be represented as follows (see Figure 2.1):

$$a \mapsto (1, 1, 1), \qquad b \mapsto (-1, 1, 1), \qquad c \mapsto (1, -1, 1),$$
$$d = 288 - a - b - c \mapsto (-1, -1, 1),$$
$$\overline{a} = 432 - 3a - b - c \mapsto (1, 1, -1), \qquad \overline{b} = 2a + c - 144 \mapsto (-1, 1, -1),$$
$$\overline{c} = 2a + b - 144 \mapsto (1, -1, -1), \qquad \overline{d} = 144 - a \mapsto (-1, -1, -1).$$

Using this notation, the polyconvex measures for the barycenter $P_0 = (0, 0, 0)$ constitute the polyhedron in (a, b, c)-space which is the convex hull of its vertices:

$$\begin{gathered} A = (0, 144, 144), \qquad B_0 = (72, 0, 0), \qquad B_1 = (72, 216, 0), \\ B_2 = (72, 0, 216), \qquad C = (144, 0, 0). \end{gathered} \tag{2}$$

(This 3-dimensional solid is easily visualized: B_0, B_1, B_2 are the vertices of a vertical triangle which is the common basis of two opposite pyramids having vertex at A, C respectively, see Figure 2.2.)

Inside this set of polyconvex measures lies the corresponding set of gradient Young measures, obtained in [15] from the Riemann-Lebesgue lemma (see [17]) for periodic

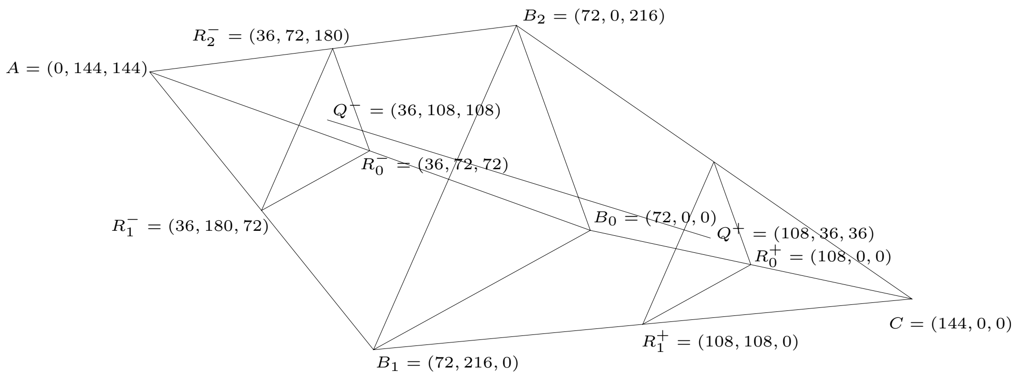

Figure 2.2: $P_0 = (0, 0, 0)$

gradients, which is a straight-line segment as shown below (see Section 5). For the barycenter zero, the extremities of this segment are

$$Q^- = (36, 108, 108), \qquad Q^+ = (108, 36, 36).$$

We prefer, however, in order to simplify further the geometric picture of the relationship between gradient Young measures and laminates, to present each one of these sets of polyconvex measures in (a, b, c)-space through its intersection with the bisector plane $b = c$. For example, the edge B_1B_2 – with extremities $(a, b, c) = (72, 216, 0)$ and (a, c, b) – is thus represented by its point of intersection with the bisector plane: $B = (72, 108)$. In this way the above polyhedron (which is the set of polyconvex measures) becomes represented by a polygon, the convex hull of its 4 vertices:

$$A = (0, 144), \qquad B_0 = (72, 0), \qquad B = (72, 108), \qquad C = (144, 0),$$

see Figure 2.3. (Notice: in this figure, and also in the next ones, Q_+^- and Q_0^- represent the points of intersection of the vertical line through Q^- with the boundary of the above polygon; similarly for Q_+^+ and Q_0^+. The reader should not pay attention, for the moment, to the points in these figures which are denoted using the letter R, namely R^-, R_0^-, ...; indeed, these points will be the subject of Theorem 2.6.)

Concerning the generated segment of gradient Young measures, mentioned above, it is contained in the bisector plane, with its extremities being now represented by

$$Q^- = (36, 108), \qquad Q^+ = (108, 36). \tag{3}$$

As to the set of laminates, its intersection with the bisector plane – as happens with the set of polyconvex measures – coincides with its orthogonal projection into this plane.

We present now some definitions, aimed at simplifying the notation.

Definition 2.1. $\mathcal{D}$ will denote the class of Lipschitz deformations $u : \mathbb{R}^2 \to \mathbb{R}^2$ of the type (1), while $\mathcal{D}_\chi$ denotes its subset consisting of deformations $u(\cdot) \in \mathcal{D}$ which can be expressed as

$$u_1(x, y) = \int_0^x \chi_1(t - \delta_1)\, dt + \int_0^{x+y} \chi_3(t - \delta_3)\, dt,$$

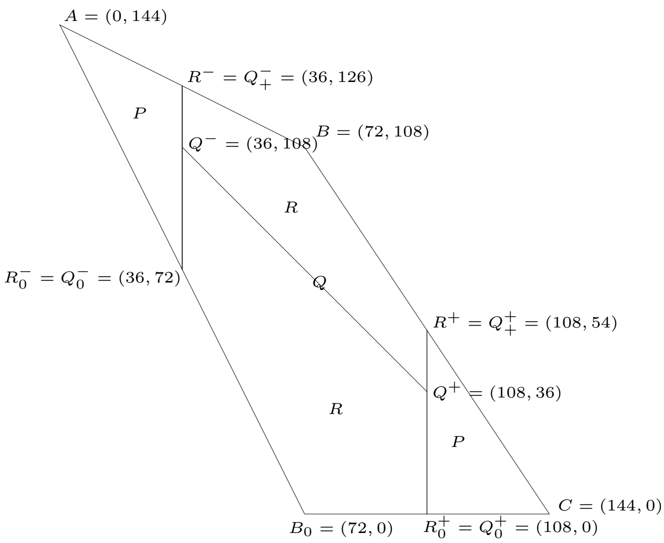

Figure 2.3: $P_0 = (0, 0, 0)$

$$u_2(x, y) = \int_0^y \chi_2(t - \delta_2)\, dt + \int_0^{x+y} \chi_3(t - \delta_3)\, dt,$$

with $\delta_i \in (0, 1)$ and

$$\chi_i(s) := \begin{cases} 1, & s \in (0, s_i), \\ -1, & s \in (s_i, 1), \end{cases}$$

extended periodically to $\mathbb{R}$, where

$$s_i := \frac{1}{2}\ (1 + \alpha_i)\,.$$

In the search for a counterexample, one important question is how to obtain all the gradient Young measures which can be generated directly from the Riemann-Lebesgue lemma (i.e. not indirectly through laminates); and the class $\mathcal{D}_\chi$ is just the natural generalization and parametrization, for a general barycenter $P_0 = (\alpha_1, \alpha_2, \alpha_3)$, of the special deformation appearing in [15]. We were unable to write down a more general expression for the deformations in the class $\mathcal{D}$, capable of yielding more extreme gradient Young measures, namely outside of the Q-segment (defined in Definition 2.3(a) below).

Definition 2.2. We call 3-edge-laminate to any third order laminate, supported on edges of $[-1, 1]^3$, which lies on an edge of the set of all laminates.

Definition 2.3. For each fixed barycenter, the intersection of the bisector plane $b = c$:

(a) with the set of gradient Young mesures obtained through the Riemann-Lebesgue lemma by using deformations $u \in \mathcal{D}_\chi$ is denoted by Q-segment;
(b) with the convex hull of the 3-edge-laminates is denoted by R-polygon;
(c) with the set of polyconvex measures is denoted by P-polygon.

Remark 2.4. Notice that we are able (see Section 4) to generate directly all the polyconvex measures (which is not true for gradient Young measures or laminates).

Using such notations, in trying to reach the answer "no" (to the question starting the introduction), the aim would be: to show that the extremities Q^-, Q^+ of the Q-segment could not be reached by laminates. However, for the barycenter $(0,0,0)$ such aim was frustrated in [15, Proposition 4.1] (see also below, Section 3), showing that the measures

$$Q_0^- = (36, 72), \qquad Q_+^- = (36, 126)$$

are indeed laminates, so that Q^- belongs to the set of laminates. The same happens with Q^+: just apply symmetry.

We proceed now to present our own work concerning the two other barycenters P_0. Starting with

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$$

one obtains for polyconvex measures the following weights on the vertices of $[-1,1]^3$:

$$\begin{gathered} a \mapsto (1,1,1), \qquad b \mapsto (-1,1,1), \qquad c \mapsto (1,-1,1), \\ d = 288 - a - b - c \mapsto (-1,-1,1), \qquad \overline{a} = 640 - 3a - b - c \mapsto (1,1,-1), \\ \overline{b} = 2a + c - 256 \mapsto (-1,1,-1), \qquad \overline{c} = 2a + b - 256 \mapsto (1,-1,-1), \\ \overline{d} = 160 - a \mapsto (-1,-1,-1). \end{gathered}$$

Using this notation, as shown below (see Section 4), the corresponding set of polyconvex measures has extreme points:

$$\begin{gathered} A = (74.(6), 106.(6), 106.(6))\,, \\ B_0 = (128,0,0)\,, \qquad B_1 = (128,160,0)\,, \qquad B_2 = (128,0,160)\,, \\ C_0 = (160,0,0)\,, \qquad C_1 = (160,128,0)\,, \qquad C_2 = (160,0,128) \end{gathered} \tag{4}$$

(yielding again two opposite pyramids, but now a vertical plane cuts a triangular face, in the second pyramid, with vertices C_0, C_1, C_2) ; so that its P-polygon is the convex hull of its vertices

$$\begin{gathered} A = (74.(6), 106.(6))\,, \qquad B_0 = (128,0)\,, \qquad B = (128,80)\,, \\ C_0 = (160,0)\,, \qquad C = (160,64)\,, \end{gathered}$$

see Figure 2.4.

On the other hand (directly) by the Riemann-Lebesgue lemma, we were able to obtain (see Section 5) no more than the Q-segment having extremities

$$Q^- = (100, 92), \qquad Q^+ = (156, 36). \tag{5}$$

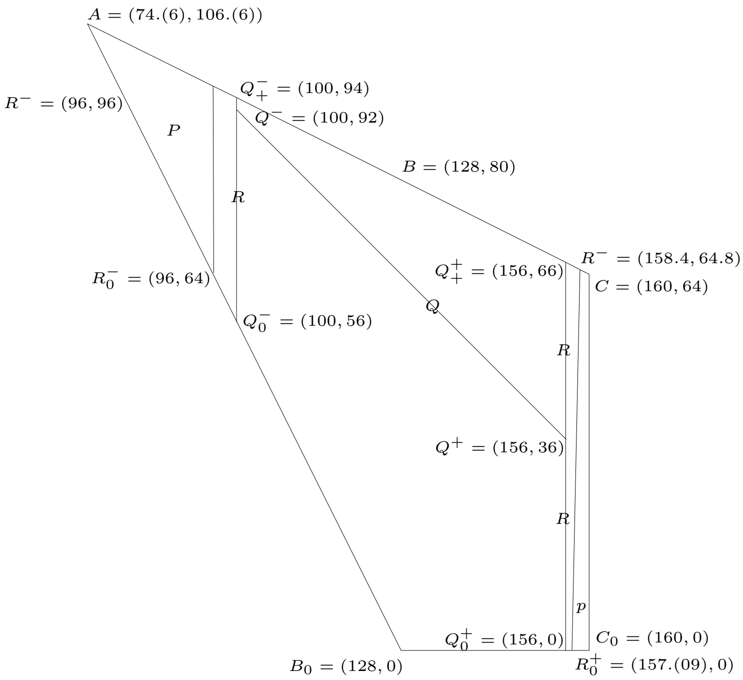

Figure 2.4: $P_0 = (\frac{1}{3}, \frac{1}{3}, 0)$

Thus, concerning the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$, our aim was to show these Q^-, Q^+ to be out of reach of laminates; but it got frustrated, when we came to the conclusion (as proved in Section 3) that one may indeed obtain, as laminates,

$$Q_0^- = (100, 56), \qquad Q_+^- = (100, 94), \qquad Q_0^+ = (156, 0), \qquad Q_+^+ = (156, 66), \tag{6}$$

showing that Q^-, Q^+ and the whole corresponding Q-segment are inside the set of laminates.

Finally, for the barycenter

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right),$$

we denote the weights of polyconvex measures by:

$$\begin{gathered} a \mapsto (1,1,1), \qquad b \mapsto (-1,1,1), \qquad c \mapsto (1,-1,1), \\ d = 288 - a - b - c \mapsto (-1,-1,1), \\ \overline{a} = 756 - 3a - b - c \mapsto (1,1,-1), \qquad \overline{b} = 2a + c - 324 \mapsto (-1,1,-1), \\ \overline{c} = 2a + b - 324 \mapsto (1,-1,-1), \qquad \overline{d} = 180 - a \mapsto (-1,-1,-1). \end{gathered}$$

Then, as shown below (see Section 4), the extreme points of the corresponding set of polyconvex measures are:

$$\begin{gathered} A = (120, 84, 84), \\ B_0 = (162, 0, 0), \qquad B_1 = (162, 126, 0), \qquad B_2 = (162, 0, 126), \\ C_0 = (180, 0, 0), \qquad C_1 = (180, 108, 0), \qquad C_2 = (180, 0, 108) \end{gathered} \tag{7}$$

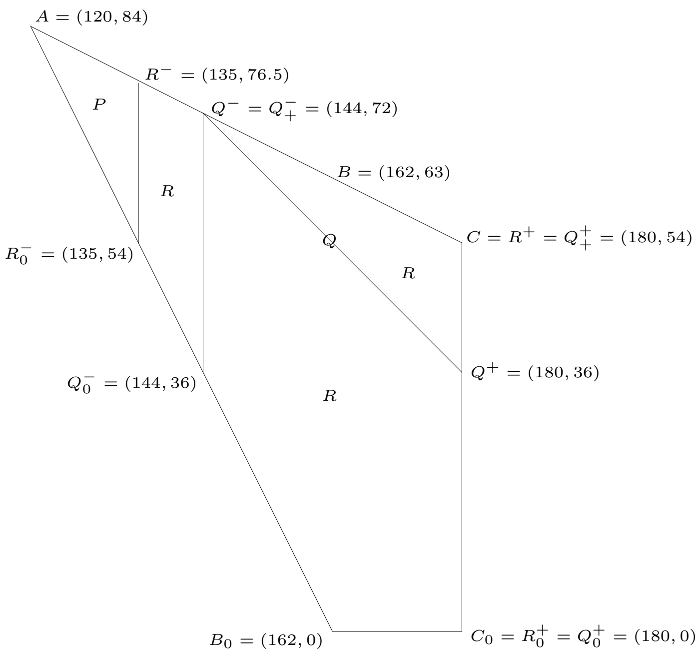

Figure 2.5: $P_0 = (\frac{1}{2}, \frac{1}{2}, 0)$

(yielding again: two opposite pyramids with the second one cut by a vertical plane); so that its corresponding P-polygon is the convex hull of its vertices

$$A = (120, 84), \qquad B_0 = (162, 0)\,, \qquad B = (162, 63)\,,$$
$$C_0 = (180, 0)\,, \qquad C = (180, 54)\,,$$

see Figure 2.5. As to the Q-segment, it has now (see Section 5) extremities

$$Q^- = (144, 72), \qquad Q^+ = (180, 36); \tag{8}$$

which, again, are convex combinations of the following laminates (see Section 3):

$$Q_0^- = (144, 36), \qquad Q_+^- = (144, 72), \qquad Q_0^+ = (180, 0), \qquad Q_+^+ = (180, 54). \tag{9}$$

We may summarize the preceding discussion, on the possibility of obtaining as laminates the points

$$Q_0^-, \qquad Q_+^-, \qquad Q_0^+, \qquad Q_+^+,$$

described in (11), (6), (9), in the next

Theorem 2.5. *The following points belong to the R-polygon (defined in Definitions* 2.3 *and* 2.2*) generated by starting with the weight* 576 *from each barycenter* P_0*:*

$Q_0^- = (36, 72)$, $Q_+^- = (36, 126)$, $Q_0^+ = (108, 0)$, $Q_+^+ = (108, 54)$ *for* $P_0 = (0, 0, 0)$;

$Q_0^- = (100, 56)$, $Q_+^- = (100, 94)$, $Q_0^+ = (156, 0)$, $Q_+^+ = (156, 66)$ *for* $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$;

$Q_0^- = (144, 36)$, $Q_+^- = (144, 72)$, $Q_0^+ = (180, 0)$, $Q_+^+ = (180, 54)$ *for* $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Theorem 2.5 is our first result (proved in Section 3), while our second (and main) result (proved in Section 6) is

Theorem 2.6. *The extreme points of the R-polygon of Theorem* 2.5 *are, besides the points* B_0, B *listed above (see Figures* 2.3, 2.4, 2.5*):*

for the barycenter $P_0 = (0,0,0)$,

$$R_0^- = (36,72), \qquad R^- = (36,126),$$
$$R_0^+ = (108,0), \qquad R^+ = (108,54)\,;$$

for the barycenter $P_0 = \left(\frac{1}{3},\frac{1}{3},0\right)$,

$$R_0^- = (96,64), \qquad R^- = (96,96),$$
$$R_0^+ = (157.(09),0), \qquad R^+ = (158.4,64.8);$$

and, for the barycenter $P_0 = \left(\frac{1}{2},\frac{1}{2},0\right)$,

$$R_0^- = (135,54), \qquad R^- = (135,76.5),$$
$$R_0^+ = C_0 = (180,0), \qquad R^+ = C = (180,54).$$

3. Presenting the sets of points which generate Q_0^-, Q_+^-, Q_0^+ and Q_+^+

As in [15], instead of providing the required sets of pairs (satisfying some (H_k) condition) which generate a specific laminate, we provide a set of points from which one can obtain univocally the mentioned set of pairs. Notice that these sets of points are not unique in general.

Proof of Theorem 2.5. A set of points which gives Q_0^- for the barycenter zero is ([15])

$$\begin{gathered} P_0 = (0,0,0), \qquad P_1 = \left(-\frac{1}{2},1,1\right), \qquad P_2 = \left(\frac{1}{10},-\frac{1}{5},-\frac{1}{5}\right), \\ P_3 = \left(1,-\frac{5}{7},1\right), \qquad P_4 = \left(-\frac{1}{11},-\frac{1}{11},-\frac{5}{11}\right), \\ P_5 = (1,1,-1), \qquad P_6 = (-1,-1,0), \end{gathered} \tag{10}$$

see Figure 3.1. Indeed, starting with the weight 576 from the barycenter $P_0 = (0,0,0)$, the above set of points generates the measure $(a,b,c) = (36,72,72)$, hence $(a,c,b) = (36,72,72)$ and $(a,\frac{b+c}{2}) = (36,72)$. Similarly one reaches the measures

$$(36,180,72), \qquad (36,72,180), \qquad (108,0,0), \qquad (108,108,0), \qquad (108,0,108).$$

This shows that, for this barycenter, the following points indeed belong (as stated in Theorem 2.5) to the R-polygon:

$$Q_0^- = (36,72), \qquad Q_+^- = (36,126), \qquad Q_0^+ = (108,0), \qquad Q_+^+ = (108,54). \tag{11}$$

This is a result of [15], which we have included here for convenience of the reader.

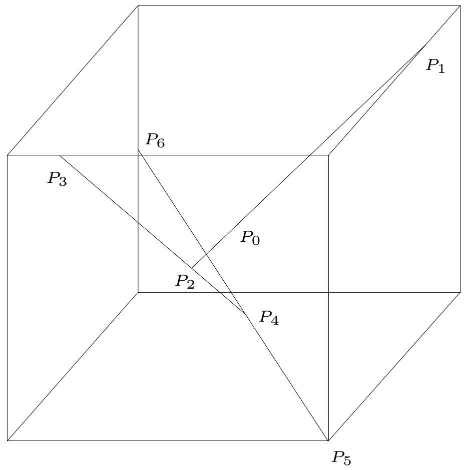

Figure 3.1: $P_0 = (0, 0, 0)$

For the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$, we have: the measure associated with the set of points

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(1, 1, -\frac{1}{3}\right), \qquad P_2 = \left(-\frac{3}{29}, -\frac{3}{29}, \frac{19}{87}\right),$$
$$P_3 = \left(-\frac{23}{41}, 1, 1\right), \qquad P_4 = \left(\frac{1}{65}, -\frac{431}{1105}, \frac{1}{65}\right),$$
$$P_5 = \left(1, -\frac{15}{17}, 1\right), \qquad P_6 = \left(-1, \frac{2}{17}, -1\right)$$

is $(100, 132, 56)$, thus yielding the point $(100, 94)$ of the R-polygon.

Similarly for the other measures associated to this barycenter, as follows: the measure $(100, 56, 56)$, hence the point $(100, 56)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(1, -\frac{1}{15}, 1\right), \qquad P_2 = \left(\frac{29}{157}, \frac{199}{471}, -\frac{35}{157}\right),$$
$$P_3 = \left(-\frac{11}{53}, 1, 1\right), \qquad P_4 = \left(\frac{25}{89}, \frac{25}{89}, -\frac{791}{1513}\right),$$
$$P_5 = \left(1, 1, \frac{15}{17}\right), \qquad P_6 = \left(-1, -1, \frac{2}{17}\right);$$

the measure $(156, 0, 0)$, hence the point $(156, 0)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(\frac{19}{21}, -1, -1\right), \qquad P_2 = \left(\frac{347}{1293}, \frac{209}{431}, \frac{49}{431}\right),$$
$$P_3 = \left(-\frac{89}{129}, 1, -1\right), \qquad P_4 = \left(\frac{151}{369}, \frac{151}{369}, \frac{3787}{13653}\right),$$
$$P_5 = \left(1, 1, -\frac{2}{111}\right), \qquad P_6 = \left(-1, -1, \frac{109}{111}\right);$$

and the measure $(156, 132, 0)$, hence the point $(156, 66)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(-1, \frac{19}{21}, -1\right), \qquad P_2 = \left(\frac{583}{1497}, \frac{463}{1497}, \frac{21}{499}\right),$$
$$P_3 = \left(\frac{32}{33}, -1, -1\right), \qquad P_4 = \left(\frac{1}{12}, 1, \frac{29}{49}\right),$$
$$P_5 = \left(1, 1, \frac{29}{49}\right), \qquad P_6 = \left(-1, 1, \frac{29}{49}\right).$$

Similarly for the measures associated to the barycenter $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$: the measure $(144, 36, 36)$, hence the point $(144, 36)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(\frac{1}{6}, 1, 1\right), \qquad P_2 = \left(\frac{19}{34}, \frac{7}{17}, -\frac{3}{17}\right),$$
$$P_3 = \left(1, \frac{1}{11}, 1\right), \qquad P_4 = \left(\frac{9}{19}, \frac{9}{19}, -\frac{23}{57}\right),$$
$$P_5 = \left(1, 1, -\frac{2}{3}\right), \qquad P_6 = \left(-1, -1, \frac{1}{3}\right);$$

the measure $(144, 36, 108)$, hence the point $(144, 72)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(1, 1, -\frac{1}{4}\right), \qquad P_2 = \left(0, 0, \frac{1}{4}\right),$$
$$P_3 = \left(-\frac{3}{7}, 1, 1\right), \qquad P_4 = \left(\frac{1}{11}, -\frac{7}{33}, \frac{1}{11}\right),$$
$$P_5 = \left(-1, \frac{1}{3}, -1\right), \qquad P_6 = \left(1, -\frac{2}{3}, 1\right);$$

the measure $C_0 = (180, 0, 0)$, hence the point $C_0 = (180, 0)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(1, -\frac{1}{2}, -1\right), \qquad P_2 = \left(\frac{5}{11}, \frac{13}{22}, \frac{1}{11}\right),$$
$$P_3 = \left(-\frac{1}{5}, 1, -1\right), \qquad P_4 = \left(\frac{7}{13}, \frac{7}{13}, \frac{3}{13}\right),$$
$$P_5 = (-1, -1, 1), \qquad P_6 = (1, 1, 0);$$

and the measure $C_1 = (180, 108, 0)$, hence the point $C = (180, 54)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(-\frac{1}{2}, 1, -1\right), \qquad P_2 = \left(1, \frac{1}{4}, \frac{1}{2}\right),$$
$$P_3 = \left(1, 1, \frac{1}{2}\right), \qquad P_4 = \left(1, -1, \frac{1}{2}\right).$$

The proof is complete.

The reader should be aware of the fact that what is difficult here is not to prove Theorem 2.5, but to find adequate points P_i to generate the stated measures.

Notice also that all the sets of points presented in this work have all their odd points (i.e. $P_1,\ P_3,\ P_5,\ P_6$) on edges of the $[-1,1]^3$ cube: this makes sense for someone searching extreme laminates.

4. Characterization of the sets of polyconvex measures

In this section and the next one we consider polyconvex measures and gradient Young measures not as probability measures but as measures having total mass= p^2; as pointed out in Section 2 with $p = 24$, this is convenient to avoid many cumbersome fractions when treating concrete examples.

Here we wish to determine the set of all the possible polyconvex measures supported on the vertices of $[-1,1]^3$, with barycenter $P_0 = (\alpha_1\,,\ \alpha_2\,,\ \alpha_3)$.

Denoting again by a, b, c, d the weights generated on the four upper vertices

$$(1,1,1), \qquad (-1,1,1), \qquad (1,-1,1), \qquad (-1,-1,1)$$

of $[-1,1]^3$; and by $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ the weights generated on its four lower vertices,

$$(1,1,-1), \qquad (-1,1,-1), \qquad (1,-1,-1), \qquad (-1,-1,-1);$$

and defining the parameters

$$s_i := \frac{1}{2}\ (1+\alpha_i)\,,$$

$$\gamma := p^2\ \left[\left(s_1 - \frac{1}{2}\right)\ \left(s_2 - \frac{1}{2}\right) +\ s_1 s_3\ +\ s_2 s_3\right],$$

one easily reaches the following characterization:

Proposition 4.1. *The set of possible weights of polyconvex measures can be represented as the set in (a,b,c)-space described by the restrictions*

$$\begin{gathered} a \geq 0, \qquad b \geq 0, \qquad c \geq 0, \\ d := p^2 s_3 - a - b - c \geq 0 \\ \overline{a} := \gamma + \frac{p^2}{2}\left(s_1 + s_2 - \frac{1}{2}\right) - 3a - b - c \geq 0 \\ \overline{b} := -\gamma + \frac{p^2}{2}\left(\frac{1}{2} - s_1 + s_2\right) + 2a + c \geq 0 \\ \overline{c} := -\gamma + \frac{p^2}{2}\left(\frac{1}{2} + s_1 - s_2\right) + 2a + b \geq 0 \\ \overline{d} := \gamma - \frac{p^2}{2}\left(s_1 + s_2 + 2s_3 - \frac{3}{2}\right) - a\ \geq 0. \end{gathered}$$

In particular, for $p = 24$ and for $\alpha_3 = 0$, $\alpha_1 = \alpha_2$, the P-polyhedron is the convex hull of its extreme points (2), (4), (7) *(for $\alpha_1 = 0,\ \frac{1}{3},\ \frac{1}{2}$, respectively).*

As was recalled in the introduction, (for each fixed barycenter) the set of polyconvex measures contains the set of gradient Young measures, which in turn contains the set of laminates. Consequently, the P-polygon contains both the Q-segment and the R-polygon.

5. Characterization of the gradient Young measures generated by $\mathcal{D}_\chi$ deformations

Each deformation $u(\cdot) \in \mathcal{D}_\chi$ (see Definition 2.1) generates (as described in the proof below) weights $a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d}$ as in Section 4. Or, in other words, each such deformation $u(\cdot)$ generates a gradient Young measure with barycenter P_0, represented by the triple (a, b, c), consisting of the weights generated on the first 3 of these vertices,

$$(1,1,1), \qquad (-1,1,1), \qquad (1,-1,1);$$

which may be compared with laminates (a, b, c) having weights a, b, c generated, on these vertices, by sets of points contained in $[-1,1]^3$ and having barycenter P_0. In particular, if one fixes $P_0 := (\alpha, \alpha, 0)$, with $\alpha = 0, \frac{1}{3}, \frac{1}{2}$, then examples of such sets of points appear in the proof of Theorem 2.5.

Proposition 5.1. *The gradient Young measures generated by deformations $u(\cdot) \in \mathcal{D}_\chi$ are all the points of a segment, namely the convex hull of its extremities Q^-, Q^+:*

$$Q^- := (a^-, d_2 - a^-, d_3 - a^-), \qquad Q^+ := (a^+, d_2 - a^+, d_3 - a^+),$$

with

$$d_2 := p^2 \ s_2 \ s_3 \,, \qquad d_3 := p^2 \ s_1 \ s_3 \,, \qquad s_i := \frac{1}{2} \ (1 + \alpha_i) \,,$$

$$a^- := p^2 \left(\frac{[s_1 + s_2 + s_3 - 1]^+}{2} \right)^2, \qquad a^+ := p^2 \left[s_1 \ s_2 - \left(\frac{[s_1 + s_2 - s_3]^+}{2} \right)^2 \right],$$

where

$$[x]^+ := \max\{0, x\}.$$

In particular, for $p = 24$ and for $\alpha_3 = 0$, $\alpha_1 = \alpha_2$, the Q-segment is the convex hull of its extreme points (3), (5), (8) *(for $\alpha_1 = 0,\ \frac{1}{3},\ \frac{1}{2}$, respectively).*

Proof. To compute the weights a, b, c, d and $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ generated by this general deformation $u(\cdot) \in \mathcal{D}_\chi$, one has to compute the areas of the corresponding regions (denoted by the same letters $a, b, c, \dots$) determined on the square $[0, p]^2$ by the lines

$$x = p\ \delta_1, \qquad x = p\ \delta_1 + p\ s_1, \qquad y = p\ \delta_2, \qquad y = p\ \delta_2 + p\ s_2,$$
$$x + y = p\ \delta_3, \qquad x + y = p\ \delta_3 + p\ s_3,$$
$$x + y = p + p\ \delta_3, \qquad x + y = p + p\ \delta_3 + p\ s_3.$$

One easily checks, geometrically, that

$$\overline{a} = p^2\ s_1\ s_2\ - a, \qquad \overline{b} = p^2\,(1 - s_1)\, s_2\ - b, \qquad \overline{c} = p^2\ s_1\,(1 - s_2) - c$$

and

$$d = p^2\ s_3 - a - b - c, \qquad \overline{d} = p^2\ [(1 - s_1)\ (1 - s_2) - s_3] + a + b + c.$$

On the other hand, we must have

$$b = p^2\ s_2\ s_3 - a, \qquad c = p^2\ s_1\ s_3 - a.$$

In this way one expresses the coordinates $b, c, d, \ldots$ as affine functions of a (dependent on the chosen P_0). Therefore the gradient Young measures generated by deformations $u(\cdot) \in \mathcal{D}_\chi$ form a segment; and to characterize it and thus end the proof, we only need to obtain its extreme values. But these are obtained by plugging in the extreme values a^-, a^+ of a, whose expressions are those stated above.

Remark 5.2. The results of Sections 4, 5 can be easily extended from the cube $[-1, 1]^3$ to a rectangular parallelepiped

$$[-A_1, A_1] \times [-A_2, A_2] \times [-A_3, A_3],$$

where $A_1, A_2, A_3 \in (0, +\infty)$.

6. Characterization of the 3-edge-laminates

One easily checks that the three P-polygons considered in Section 2 all have the same form, their only difference being that the vertices C_0, C collapse, in the case of the barycenter zero, into the unique vertex C. (One may also observe the following: for the other 2 barycenters, if one extends the edges B_0C_0, BC then they meet at the point $(288, 0)$ which is, however, out of reach for the polyconvex measures.) We leave the proof of the next proposition to the interested reader; it is similar to the proof of Theorem 2.5, but considerably easier, since it involves only the discovery of 3 first order laminates and 2 second order laminates:

Proposition 6.1. *For each one of the above* 3 *barycenters, the points* B_0, B *of the P-polygon always belong to the corresponding R-polygon (see Figures* 2.3, 2.4, 2.5*).*

But the main aim of this section is the determination of the extreme points R_0^-, R^- of the R-polygon along the edges E_0^-, E^- (i.e. those joining the vertices B_0A, BA); and also of the extreme points R_0^+, R^+ of the R-polygon along the edges E_0^+, E^+ (i.e. B_0C_0, BC assuming, in case $P_0 = (0, 0, 0)$, $C_0 := C$), see Figures 2.3, 2.4, 2.5.

In reality, more precisely, what we do below is the determination of the extreme points R_0^-, R_1^-, R_2^- (respectively R_0^+, R_1^+, R_2^+) along the edges E_0^-, E_1^-, E_2^- (respectively E_0^+, E_1^+, E_2^+) of the convex hull of the set of all 3-edge-laminates, see e.g. Figure 2.2. We believe these (together with B_0, B_1, B_2) to be also the extreme points of the set of all the laminates, but were unable to prove it.

One remarkable feature here is that, for some barycenters, the Q-segment is entirely contained in the interior of the corresponding R-polygon, hence does not reach at least its boundary, as one would expect. This is what happens for the barycenter $\left(\frac{1}{3}, \frac{1}{3}, 0\right)$; while for $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ the lower value of the coordinate a along the laminate is strictly smaller than its lower value along the Q-segment. This situation is unfortunate for the search of counterexamples, but we were unable to improve it, as remarked at the end of the introduction.

Proof of Theorem 2.6. (a) For the barycenter $P_0 = (0,0,0)$ the points R_0^-, R^-, R_0^+, R^+ are generated by (10).

For the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$ the measure $R_0^- = (96, 64, 64)$, hence the point $R_0^- = (96, 64)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(-\frac{1}{15}, 1, 1\right), \qquad P_2 = \left(\frac{25}{57}, \frac{3}{19}, -\frac{5}{19}\right),$$
$$P_3 = \left(1, -\frac{3}{13}, 1\right), \qquad P_4 = \left(\frac{3}{11}, \frac{3}{11}, -\frac{7}{11}\right),$$
$$P_5 = (-1,-1,0), \qquad P_6 = (1,1,-1);$$

while $R_1^- = (96, 128, 64)$, hence $R_2^- = (96, 64, 128)$ and the point $R^- = (96, 96)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(1, -\frac{1}{15}, 1\right), \qquad P_2 = \left(\frac{3}{19}, \frac{25}{57}, -\frac{5}{19}\right),$$
$$P_3 = \left(1, 1, -\frac{3}{5}\right), \qquad P_4 = \left(-\frac{1}{2}, 0, 0\right),$$
$$P_5 = (0,-1,-1), \qquad P_6 = (-1,1,1).$$

Still for the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$, the measure $R_0^+ = (157.(09), 0, 0)$, hence the point $R_0^+ = (157.(09), 0)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(\frac{19}{21}, -1, -1\right), \qquad P_2 = \left(\frac{47}{117}, \frac{29}{59}, \frac{7}{59}\right),$$
$$P_3 = \left(-\frac{2}{3}, 1, -1\right), \qquad P_4 = \left(\frac{7}{17}, \frac{7}{17}, \frac{5}{17}\right),$$
$$P_5 = (1,1,0), \qquad P_6 = (-1,-1,1);$$

while $R_1^+ = (158.4, 129.6, 0)$, hence $R_2^+ = (158.4, 0, 129.6)$ and the point $R^+ = (158.4, 64.8)$, is generated by

$$P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right), \qquad P_1 = \left(\frac{19}{21}, -1, -1\right), \qquad P_2 = \left(\frac{101}{339}, \frac{47}{113}, \frac{7}{113}\right),$$
$$P_3 = (-1,1,-1), \qquad P_4 = \left(\frac{139}{301}, \frac{103}{301}, \frac{59}{301}\right),$$
$$P_5 = \left(1, -\frac{7}{11}, -1\right), \qquad P_6 = \left(\frac{1}{10}, 1, 1\right).$$

For the barycenter $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ the measure $R_0^- = (135, 54, 54)$, hence the point

$R_0^- = (135, 54)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(\frac{1}{6}, 1, 1\right), \qquad P_2 = \left(\frac{37}{62}, \frac{11}{31}, -\frac{9}{31}\right),$$
$$P_3 = \left(1, \frac{1}{21}, 1\right), \qquad P_4 = \left(\frac{17}{37}, \frac{17}{37}, -\frac{27}{37}\right),$$
$$P_5 = (-1, -1, 0), \qquad P_6 = (1, 1, -1);$$

while $R_1^- = (135, 99, 54)$, hence $R_2^- = (135, 54, 99)$ and the point $R^- = (135, 76.5)$, is generated by

$$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \qquad P_1 = \left(1, 1, -\frac{1}{4}\right), \qquad P_2 = \left(-\frac{1}{9}, -\frac{1}{9}, \frac{11}{36}\right),$$
$$P_3 = \left(-\frac{7}{13}, 1, 1\right), \qquad P_4 = \left(\frac{1}{21}, -\frac{11}{21}, \frac{1}{21}\right),$$
$$P_5 = (-1, 0, -1), \qquad P_6 = (1, -1, 1).$$

Finally, for the barycenter $P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, the measures $R_0^+ = C_0$, $R^+ = C$ are generated as indicated in the proof of Theorem 2.5.

(b) It remains only to show that these measures are extreme, in the sense explained after Proposition 6.1.

Consider the barycenter $P_0 = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$. One wishes to show that the measure $(96, 128, 64)$ is R_1^-, namely the extreme point along the segment which is the convex hull of $(74 + \frac{2}{3}, 106 + \frac{2}{3}, 106 + \frac{2}{3})$ and $(128, 160, 0)$ (i.e. along the edge E_1^- of the corresponding set of polyconvex measures).

Parametrize the part of this edge E_1^- having $a < 96$:

$$(a, b, c) = (a, 32 + a, 256 - 2\,a), \qquad a \in \left[74 + \frac{2}{3}, 96\right).$$

For each a, the weights obtained on the remaining vertices of the cube are:

$$d = 0 \mapsto (-1, -1, 1), \qquad \overline{a} = 352 - 2a \mapsto (1, 1, -1), \qquad \overline{b} = 0 \mapsto (-1, 1, -1),$$
$$\overline{c} = 3a - 224 \mapsto (1, -1, -1), \qquad \overline{d} = 160 - a \mapsto (-1, -1, -1).$$

Our aim is to show that there exists no set of points generating weights $(a, b, c) \in E_1^-$ having $a \in \left[74 + \frac{2}{3}, 96\right)$. We begin by choosing the edges of the cube upon which one could place each one of the 4 points P_1, P_3, P_5, P_6. Since $d = 0 = \overline{b}$, only the edges $S_{ab}, S_{a\overline{a}}, S_{c\overline{c}}, S_{\overline{a}\ \overline{c}}, S_{\overline{c}\overline{d}}$ may be used. (Here S_{ab}, say, is the edge of the cube which joins the vertices holding weights a, b; i.e. $S_{ab} = co\ \{(1, 1, 1), (-1, 1, 1)\}$.)

We begin by choosing an edge to hold P_1, so that P_0P_1 is rank-one. We have two possibilities:

– either (b_1) $\quad (\,P_1 \in S_{ab}$ or $P_1 \in S_{\overline{c}\overline{d}}\,)$;

– or else (b_2) $\quad (\,P_1 \in S_{a\overline{a}}$ or $P_1 \in S_{ac}\,)$.

Then it suffices to convince oneself that none of these choices works, by exploring wisely all the available possibilities. Indeed, each one of them leads to a situation in which one of the restrictions to apply simply turns out to be impossible to satisfy.

For the other edges, one shows similarly that the extreme points on the edges are the ones shown in part (a) above.

The proof is complete.

7. A computational attempt to characterize the 3 sets of laminates

After having computed the extreme points stated in Theorem 2.6, the following question comes naturally to one's mind: are the vertical segments $S^- := [R_0^-, R^-]$, $S^+ := [R_0^+, R^+]$ (see Figures 2.3, 2.4, 2.5) extreme in the intersection of the bisector plane with the corresponding general set of laminates, in each case? (Or, more precisely, considering the 3-dimensional picture and using the same notation as in the proof of Theorem 2.5: are the vertical triangles $T^- := co\{R_0^-, R_1^-, R_2^-\}$ and $T^+ := co\{R_0^+, R_1^+, R_2^+\}$ extreme faces of the set of laminates?) If one could ensure this, then the intersection of the set of laminates with the bisector plane, in each case, would become completely characterized as the convex hull of the 3 vertical segments S^-, S^+ and $S := \{B_0, B\}$. (Or, in the 3-dimensional picture: then each set of laminates would be exactly the convex hull of the 3 vertical triangles T^-, T^+ and $T := co\{B_0, B_1, B_2\}$, see e.g. Figure 2.2.)

To show the plausibility of this conjecture, we have tried to characterize the extreme values of the first coordinate a (the weight on the vertex $(1,1,1)$ of $[-1,1]^3$), in each one of the above set of laminates (independently of the weights on the other vertices of the cube). Or, in other words, to find the extreme values of the coordinate a, regardless of restricting attention to edges of the corresponding set of laminates. To avoid any bias coming from wishful thinking, we have constructed (in a personal computer) *exact samples* of all the possible third order laminates. Since all the corresponding sets of points constructed here have all their odd points on edges of $[-1,1]^3$, in trying to construct a third order laminate starting from one of the chosen barycenters ($(a,a,0)$ with $a = 0$, $a = \frac{1}{3}$, $a = \frac{1}{2}$), the choices one has to make, concerning each odd point (namely P_1, P_3, P_5 or P_6) lead to less than a dozen possibilities. On the contrary, concerning each even point (i.e. P_2 or P_4) the possibilities are, instead, all the points of a straight-line segment, which we call an even segment; and our strategy has been to divide each such segment into $n = 100$ pieces, all with equal length. In this way we have generated blindly many hundreds of thousands of different third order laminates for each barycenter.

(Notice: the expression "exact samples" is used above – see middle of the preceding paragraph – in the following sense: our coordinates have been represented as quotients of integers with 16 decimal digits, so that the sets of points we have generated have exact coordinates and exact weight-distributions, hence yield exact – i.e. not approximate – laminate points.)

One might also wonder whether by using fourth order laminates it would be possible to obtain a more extreme value of a, namely a value not reachable by third order laminates only. In order to try and discard such possibility, we have also generated fourth order laminates on the computer. But since we have, in this case, 3 even segments instead of 2, we had to reduce the number n of divisions from 100 to just 30, due to computer memory limitations.

The computations thus performed tend to indicate that it is sufficient to consider third order laminates.

The conclusions we have reached from all these computations simply confirmed the validity of the conjecture stated at the beginning of this section, namely that any laminate is a convex combination of 3-edge-laminates (in the 2×2 symmetric case).

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