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Variational Principles in L^∞ with Applications to Antiplane Shear and Plane Stress Plasticity

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The yield set of a polycrystal is characterized by means of a variational principle in L^∞ obtained via Γ-convergence of a class of power-law functionals in the setting of $\mathcal{A}$-quasiconvexity. Our results apply, in particular, to the model cases of antiplane shear and plane stress plasticity.

Keywords: Antiplane shear, $\mathcal{A}$-quasiconvexity, Γ-convergence, lower semicontinuity, plane stress, polycrystal plasticity, yield set, yield surface

35F99, 35J70, 49K20, 49S05, 74C05 Mathematics Subject Classification:

1. Introduction

The problem of characterizing the effective yield set of a polycrystal has received considerable attention in recent years. In particular, the issue of the optimality of the classical Sachs and Bishop-Hill-Taylor bounds [1], [16], [19], which provide natural inner and outer estimates for the effective yield set, has been recently studied by many authors (see, e.g., Garroni & Kohn [10], Goldsztein [12], [13], Kohn & Little [14]). A new approach for this type of problems has been proposed by Garroni, Nesi, & Ponsiglione in [11], where an efficient mathematical derivation of the (first-failure) dielectric breakdown model as a limiting case of the power-law model via De Giorgi's Γ-convergence is provided, leading to a new variational principle for the effective yield set (in the context of dielectric breakdown) which is less degenerate than the traditional one. The Γ-convergence results in [11], which concern power-law type functionals acting on gradients, have been recently generalized by Bocea & Nesi [2] to more general linear PDE constraints on the underlying fields in the framework of $\mathcal{A}$-quasiconvexity. In particular, this leads to variational characterizations of the yield (strength) set in the setting of electrical resistivity (where the underlying fields are divergence-free). The aim of this paper is to extend the results in [2] to a framework

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which is relevant to treating models of Polycrystal Plasticity, where the underlying fields take values in stress space $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, are divergence free and, as described below, not just one, but several (depending on the number of slip systems present in the basic crystal) distinct pointwise constraints need to be simultaneously verified. In what follows we provide, for the convenience of the reader, a brief description of the physical context. More details can be found for example in [10], [11], [12], [14], and references therein.

A polycrystal is a collection of grains, or single crystals, which are bonded together in different orientations. The yield of a single crystal is determined by a closed convex subset K of the space of symmetric 3×3 real matrices $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$. The shapes and orientations of the grains in a polycrystal (this is called the texture of the polycrystal) are described by a piecewise constant rotation-valued function $R : \Omega \to \mathrm{SO}(3)$, where $R(x)$ is constant in each grain and indicates the orientation of the grain which contains the point $x \in \Omega$. If the yield set of the basic crystal is K, the stress in the polycrystal occupying the region $\Omega \subset \mathbb{R}^3$ must satisfy the constraint

$$\sigma(x) \in R(x) K R^T(x), \quad x \in \Omega. \tag{1}$$

The set of all average stresses $\overline{\sigma} := ⨍_\Omega \sigma(x)dx$, where σ satisfies the pointwise constraint (1) and the equilibrium equation

$$\mathrm{Div}\ \sigma = 0 \ \text{ in } \Omega, \tag{2}$$

is called the effective yield set of the polycrystal. It is given by

$$K_{\mathrm{eff}} := \left\{ \overline{\sigma} := ⨍_\Omega \sigma(x)dx \ : \ (1) \text{ and } (2) \text{ hold} \right\}.$$

Yield in a crystalline solid is associated with a finite number of slip systems which depend on the atomic lattice, each being determined by a pair (n_k, m_k) of orthogonal vectors, where n_k is the normal to the slip plane, and m_k is the direction of slip. In this case, we have

$$K = \left\{ A \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} \ : \ \langle A, \mu_k \rangle \le \tau_k^{\mathrm{critical}},\ k = 1, \cdots, s \right\},$$

where s stands for the number of slip systems, $\tau_k^{\mathrm{critical}}$ is the critical shear stress for the k-th slip system, and μ_k, defined by

$$\mu_k := \frac{1}{2}\left(m_k \otimes n_k + n_k \otimes m_k \right),$$

is the k-th slip tensor. Describing the yield set K_{eff}, given K and some information on the texture of the polycrystal, is the main goal of Polycrystal Plasticity.

The plan of the paper is as follows. In Section 2 we give the necessary background on $\mathcal{A}$-quasiconvexity and Γ-convergence needed in the sequel. In Section 3 we state and prove a Γ-convergence result for a general class of power-law functionals. Section 4 of the paper is devoted to the characterization of the effective yield set of a polycrystal in terms of a variational principle in L^∞. In addition, we show that our results apply in the model settings of antiplane shear and plane stress polycrystal plasticity.

2. $\mathcal{A}$-quasiconvexity and Γ-convergence

Let $N, d, l \in \mathbb{N}$ be given, $\Omega \subset \mathbb{R}^N$ open and bounded, $1 < p < \infty$, and let $p' := p/(p-1)$ be the Hölder conjugate exponent of p. Consider a family of linear operators $A^{(1)}, A^{(2)}, \cdots, A^{(N)} \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^l)$, and define the differential operator $\mathcal{A} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$ by

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}. \tag{3}$$

Precisely,

$$\langle \mathcal{A}v, u \rangle := \left\langle \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}, u \right\rangle = -\sum_{i=1}^{N} \int_\Omega A^{(i)} v \frac{\partial u}{\partial x_i} dx \quad \text{for all } u \in W_0^{1,p'}(\Omega; \mathbb{R}^l). \tag{4}$$

Here $W^{-1,p}(\Omega; \mathbb{R}^l)$ stands for the dual of $W_0^{1,p'}(\Omega; \mathbb{R}^l)$; it is well known that F belongs to $W^{-1,p}(\Omega; \mathbb{R}^l)$ if and only if there exist $f_1, f_2, \cdots, f_N \in L^p(\Omega; \mathbb{R}^l)$ such that

$$\langle F, u \rangle = \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial u}{\partial x_i} dx \quad \text{for all } u \in W_0^{1,p'}(\Omega; \mathbb{R}^l).$$

We assume that the operator $\mathcal{A}$ satisfies the following constant rank property:

$$\textit{there exists } r \in \mathbb{N} \textit{ such that } \mathrm{rank}(\mathbb{A}(w)) = r \textit{ for all } w = (w_1, \cdots, w_N) \in S^{N-1}, \tag{5}$$

where

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^l).$$

The constant rank property plays an important role in the theory of compensated compactness developed by Murat and Tartar (see, e.g., [15], [17], and [18]). Let $Q = (0,1)^N$ be the unit cube in $\mathbb{R}^N$.

Definition 2.1. A function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if

$$g(A) \le \int_Q g(A + w(x)) dx$$

for all $A \in \mathbb{R}^d$, and all Q-periodic $w \in C^\infty(Q; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(x)dx = 0$.

The notion of $\mathcal{A}$-quasiconvexity (without the periodicity assumption on the test functions) has been first investigated by Dacorogna [5]. Fonseca & Müller have shown in [9] that if $\mathcal{A}$ satisfies constant rank property (5), $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $(u, v) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d$ is measurable, and $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand then, under suitable growth assumptions, $\mathcal{A}$-quasiconvexity of $g(x,u,\cdot)$ is a necessary and sufficient condition for the sequential lower semicontinuity of integral functionals of the form

$$(u,v) \mapsto \int_\Omega g(x,u(x),v(x))dx$$

along sequences such that $u_n \to u$ in measure, $v_n \to v$ in L^p, and $\mathcal{A}v_n \to 0$ in $W^{-1,p}$. In particular, we have the following

Proposition 2.2 (see [9, Theorem 3.7]). *Let $1 \leq p \leq +\infty$ and suppose that $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,+\infty)$ is a normal integrand such that $z \mapsto g(x,u,z)$ is $\mathcal{A}$-quasiconvex and continuous for $\mathcal{L}^N$-a.e. $x \in \Omega$, and all $u \in \mathbb{R}^d$. If $1 \leq p < +\infty$, assume further that there exists a locally bounded function $a : \Omega \times \mathbb{R}^d \to [0,+\infty)$ such that*

$$0 \leq g(x,u,v) \leq a(x,u)(1+|v|^p),$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, and all $(u,v) \in \mathbb{R}^m \times \mathbb{R}^d$. If

$$u_n \to u \ \text{ in measure,}$$

$$v_n \rightharpoonup v \ \text{ in } L^p(\Omega;\mathbb{R}^d), \tag{6}$$

and

$$\mathcal{A}v_n \to 0 \ \text{ in } W^{-1,p}(\Omega;\mathbb{R}^l) \tag{7}$$

then

$$\int_\Omega g(x,u(x),v(x))dx \leq \liminf_{n\to\infty} \int_\Omega g(x,u_n(x),v_n(x))dx. \tag{8}$$

If $p = +\infty$, then (8) still holds provided that in (6) the weak convergence of v_n to v in $L^p(\Omega;\mathbb{R}^d)$ is replaced by the weak convergence in $L^\infty(\Omega;\mathbb{R}^d)$, and in (7) $\mathcal{A}v_n \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^l)$ is replaced by $\mathcal{A}v_n = 0$.*

Next we recall the definition of De Giorgi's Γ-convergence (see [7], [8]) in metric spaces. For a comprehensive introduction to the subject we refer to [6]. See also [3], and [4].

Definition 2.3. Let X be a metric space. A sequence $\{I_p\}$ of functionals $I_p : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to $\Gamma(X)$-converge to $I : X \to \overline{\mathbb{R}}$ (we write $\Gamma(X) - \lim_{p\to\infty} I_p = I$) if

(i) for every $u \in X$ and $\{u_p\} \subset X$ such that $u_p \to u$ in X, we have

$$I(u) \leq \liminf_{p\to\infty} I_p(u_p);$$

(ii) for every $u \in X$ there exists a sequence $\{u_p\} \subset X$ such that $u_p \to u$ in X, and

$$I(u) = \lim_{p\to\infty} I_p(u_p).$$

3. Γ-convergence of power-law functionals

Let $m \in \mathbb{N}$ be a positive integer, Ω an open, bounded domain in $\mathbb{R}^N (N \geq 1)$ and, for $i = 1, 2, \cdots, m$, consider Carathéodory integrands $f_i : \Omega \times \mathbb{R}^d \to [0, +\infty)$ such that

$$f_i(x, \cdot) \text{ is } \mathcal{A}\text{-quasiconvex for } \mathcal{L}^N\text{-a.e. } x \in \Omega, i \in \{1, 2, \cdots, m\}. \tag{9}$$

Assume that there exists a constant $C > 0$ such that for every $i \in \{1, 2, \cdots, m\}$ we have

$$f_i(x, v) \leq C\,(1 + |v|) \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \text{ and all } v \in \mathbb{R}^d. \tag{10}$$

Moreover, we assume that

$$\sum_{i=1}^{m} f_i(x, v) \geq c|v| \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \text{ and all } v \in \mathbb{R}^d, \tag{11}$$

where $c > 0$ is a positive constant.

Theorem 3.1. *Let Ω be an open, bounded domain in $\mathbb{R}^N (N \geq 1)$ and, for $i = 1, 2, \cdots, m$, let $f_i : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be Carathéodory integrands satisfying* (9), (10), *and* (11). *Define $J_{m,p}, J_{m,\infty} : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ by*

$$J_{m,p}(w) := \begin{cases} \left(\displaystyle\int_\Omega \left(\sum_{i=1}^{m} f_i(x, w(x))^p \right) dx \right)^{1/p} & \text{if } w \in L^p(\Omega; \mathbb{R}^d) \text{ and } \mathcal{A}w = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$J_{m,\infty}(w) := \begin{cases} \displaystyle\max_{i \in \{1, \cdots, m\}} \operatorname*{ess\,sup}_{x \in \Omega} f_i(x, w(x)) & \text{if } w \in L^\infty(\Omega; \mathbb{R}^d) \text{ and } \mathcal{A}w = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

respectively. Then

(*i*) *for every $w \in L^1(\Omega; \mathbb{R}^d)$, and $\{w_p\} \subset L^1(\Omega; \mathbb{R}^d)$ such that $w_p \rightharpoonup w$ weakly in $L^1(\Omega; \mathbb{R}^d)$, we have*

$$J_{m,\infty}(w) \leq \liminf_{p \to \infty} J_{m,p}(w_p). \tag{12}$$

(*ii*) *for every $w \in L^1(\Omega; \mathbb{R}^d)$, there exists a sequence $\{w_p\} \subset L^1(\Omega; \mathbb{R}^d)$ such that $w_p \to w$ strongly in $L^1(\Omega; \mathbb{R}^d)$, and*

$$\limsup_{p \to \infty} J_{m,p}(w_p) \leq J_{m,\infty}(w). \tag{13}$$

In particular,

$$\Gamma(L^1(\Omega; \mathbb{R}^d)) - \lim_{p \to \infty} J_{m,p} = J_{m,\infty}.$$

Proof. Let $\{w_p\} \subset L^1(\Omega;\mathbb{R}^d)$ be such that $w_p \rightharpoonup w$ weakly in $L^1(\Omega;\mathbb{R}^d)$. We need to show that (12) holds. After eventually passing to a subsequence we may assume, without loss of generality, that

$$w_p \in L^p(\Omega;\mathbb{R}^d), \quad \mathcal{A}w_p = 0, \tag{14}$$

and

$$\liminf_{p\to\infty} J_{m,p}(w_p) = \lim_{p\to\infty} J_{m,p}(w_p) < +\infty. \tag{15}$$

In view of (9) and Jensen's inequality, for any $i \in \{1,\cdots,m\}$ and $q \geq 1$, $f_i(x,\cdot)^q$ is $\mathcal{A}$-quasiconvex for $\mathcal{L}^N$-a.e. $x \in \Omega$. In addition, by (10),

$$f_i(x,v)^q \leq 2^{q-1}C^q\left(1+|v|^q\right), \tag{16}$$

for $i \in \{1,\cdots,m\}$, $\mathcal{L}^N$-a.e. $x \in \Omega$, and all $v \in \mathbb{R}^d$. For any $p > q > 1$, we have, by Hölder's inequality,

$$\|w_p\|_{L^q(\Omega;\mathbb{R}^d)} \leq \|w_p\|_{L^p(\Omega;\mathbb{R}^d)}\left(\mathcal{L}^N(\Omega)\right)^{\frac{p-q}{pq}}.$$

In addition, the coercivity condition (11) yields

$$\|w_p\|_{L^p(\Omega;\mathbb{R}^d)} \leq \frac{1}{c}\left(\int_\Omega\left(\sum_{i=1}^m f_i(x,w_p(x))\right)^p dx\right)^{\frac{1}{p}} \leq \frac{1}{c}J_{m,p}(w_p)m^{\frac{p-1}{p}}.$$

We deduce that

$$\|w_p\|_{L^q(\Omega;\mathbb{R}^d)} \leq \frac{\left(\mathcal{L}^N(\Omega)\right)^{\frac{p-q}{pq}}}{c}m^{\frac{p-1}{p}}J_{m,p}(w_p) \leq CJ_{m,p}(w_p),$$

where $C > 0$ is a constant which only depends on m and $\mathcal{L}^N(\Omega)$ (one may take, for example, $C := \frac{m}{c}\max\{\mathcal{L}^N(\Omega),1\}$). Thus, by (15), $\{w_p\}$ is bounded in $L^q(\Omega;\mathbb{R}^d)$. Since $q > 1$ we can extract a subsequence (not relabelled) such that $w_p \rightharpoonup w$ weakly in $L^q(\Omega;\mathbb{R}^d)$, as $p \to \infty$. Taking (14) and (16) into account, and in view of the $\mathcal{A}$-quasiconvexity and continuity of each $f_i(x,\cdot)^q$ for $\mathcal{L}^N-$ a.e. $x \in \Omega$, we are in the position to apply Proposition 2.2. We obtain that

$$\int_\Omega f_i(x,w(x))^q dx \leq \liminf_{p\to\infty}\int_\Omega f_i(x,w_p(x))^q dx, \tag{17}$$

for every $i \in \{1,\cdots,m\}$. On the other hand, for $i \in \{1,\cdots,m\}$, we have that

$$\left(\liminf_{p\to\infty}\int_\Omega\sum_{i=1}^m f_i(x,w_p(x))^q dx\right)^{1/q} \leq \limsup_{p\to\infty}\left(\int_\Omega\sum_{i=1}^m f_i(x,w_p(x))^q dx\right)^{1/q}. \tag{18}$$

Thus, in view of (17) and (18), we deduce that

$$J_{m,q}(w) \leq \left(\sum_{i=1}^{m} \liminf_{p\to\infty} \int_\Omega f_i(x, w_p(x))^q dx \right)^{\frac{1}{q}} \leq \limsup_{p\to\infty} J_{m,q}(w_p) \tag{19}$$

for all $q \geq 1$. Next, for $q < p$, we have

$$J_{m,q}(w_p) \leq \left(\left(\mathcal{L}^N(\Omega)\right)^{1-\frac{q}{p}} \sum_{i=1}^{m} \|f_i(\cdot, w_p(\cdot))\|^q_{L^p(\Omega)} \right)^{\frac{1}{q}}. \tag{20}$$

Putting $a_i := \|f_i(\cdot, w_p(\cdot))\|^p_{L^p(\Omega)}$, $i = 1, \cdots, m$, in the inequality

$$\left(\sum_{i=1}^{m} a_i^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq m^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{i=1}^{m} a_i \right)^{\frac{1}{p}},$$

which is valid for all $a_1, \cdots, a_m \geq 0$, we obtain that

$$\left(\sum_{i=1}^{m} \|f_i(\cdot, w_p(\cdot))\|^q_{L^p(\Omega)} \right)^{\frac{1}{q}} \leq m^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{i=1}^{m} \|f_i(\cdot, w_p(\cdot))\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} = m^{\frac{1}{q}-\frac{1}{p}} J_{m,p}(w_p). \tag{21}$$

Combining (20) and (21) we deduce that

$$J_{m,q}(w_p) \leq \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}-\frac{1}{p}} J_{m,p}(w_p).$$

Thus, passing to $\limsup$ as $p \to \infty$, we have

$$\limsup_{p\to\infty} J_{m,q}(w_p) \leq \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}} \lim_{p\to\infty} J_{m,p}(w_p)$$

which, together with (19), gives

$$J_{m,q}(w) \leq \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}} \liminf_{p\to\infty} J_{m,p}(w_p). \tag{22}$$

Thus,

$$\|f_i(\cdot, w(\cdot))\|_{L^q(\Omega)} \leq \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{q}} \liminf_{p\to\infty} J_{m,p}(w_p) \tag{23}$$

for every $i \in \{1, \cdots, m\}$. We now claim that $f_i(\cdot, w(\cdot)) \in L^\infty(\Omega)$ for every $i \in \{1, \cdots, m\}$. Indeed, let $x \in \Omega$ be a Lebesgue point for $f_i(\cdot, w(\cdot)) \in L^1(\Omega)$. For any ball $B(x, r) \subset \Omega$, and for $p > 1$ sufficiently large, we have

$$\begin{aligned} \int_{B(x,r)} f_i(y, w_p(y)) dy &\leq \left(\int_\Omega (f_i(y, w_p(y)))^p \, dy \right)^{1/p} \left(\mathcal{L}^N(B(x,r))\right)^{(p-1)/p} \\ &\leq J_{m,p}(w_p) \left(\mathcal{L}^N(B(x,r))\right)^{(p-1)/p}, \end{aligned} \tag{24}$$

where we have used Hölder's inequality. Letting $p \to \infty$, we obtain

$$\limsup_{p\to\infty} \int_{B(x,r)} f_i(y, w_p(y))dy \le \lim_{p\to\infty} J_{m,p}(w_p)\mathcal{L}^N(B(x,r)). \tag{25}$$

Applying Proposition 2.2 again, we deduce that

$$\int_{B(x,r)} f_i(y, w(y))dy \le \liminf_{p\to\infty} \int_{B(x,r)} f_i(y, w_p(y))dy.$$

Combining this with (25), we have that

$$\frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} f_i(y, w(y))dy \le \lim_{p\to\infty} J_{m,p}(w_p).$$

Since $\mathcal{L}^N$-almost every $x \in \Omega$ is a Lebesgue point for $f_i(\cdot, w(\cdot))$, passing to the limit $r \to 0^+$ in the above inequality yields

$$f_i(x, w(x)) \le \lim_{p\to\infty} J_{m,p}(w_p), \quad \mathcal{L}^N - \text{a.e. } x \in \Omega.$$

Since $i \in \{1, \cdots, m\}$ was arbitrary, and taking into account (15), it follows that $f_i(\cdot, w(\cdot)) \in L^\infty(\Omega)$ for every $i \in \{1, \cdots, m\}$, as claimed.

Letting $q \to \infty$ in (23) we obtain that

$$\|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)} \le \liminf_{p\to\infty} J_{m,p}(w_p),$$

for all $i \in \{1, \cdots, m\}$. Hence

$$\max_{i\in\{1,\cdots,m\}} \|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)} \le \liminf_{p\to\infty} J_{m,p}(w_p).$$

Finally, since $f_i(\cdot, w(\cdot)) \in L^\infty(\Omega)$ for all $i \in \{1, \cdots, m\}$, (11) gives $w \in L^\infty(\Omega; \mathbb{R}^d)$. Moreover, since $w_p \in L^1(\Omega; \mathbb{R}^d)$, $\mathcal{A}w_p = 0$, and $w_p \rightharpoonup w$ weakly in $L^1(\Omega; \mathbb{R}^d)$, it follows that $\mathcal{A}w = 0$. We conclude that (12) holds.

Let $w \in L^1(\Omega; \mathbb{R}^d)$, and consider the constant sequence $\{w_p\} \subset L^1(\Omega; \mathbb{R}^d)$, $w_p := w$ for all $p \in \mathbb{N}$. To verify that (13) holds, we assume without loss of generality that $J_{m,\infty}(w) < +\infty$. This implies that $w \in L^\infty(\Omega; \mathbb{R}^d)$ and $\mathcal{A}w = 0$. Using the fact that for each $i \in \{1, \cdots, m\}$ we have $f_i(x, w(x)) \le \|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, we obtain that

$$\left(\int_\Omega \sum_{i=1}^m f_i(x, w(x))^p dx \right)^{\frac{1}{p}} \le \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{p}} \left(\sum_{i=1}^m \|f_i(\cdot, w(\cdot))\|^p_{L^\infty(\Omega)} \right)^{\frac{1}{p}}.$$

Hence

$$\begin{aligned} J_{m,p}(w_p) = J_{m,p}(w) &\leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{p}} \left(\sum_{i=1}^m \|f_i(\cdot, w(\cdot))\|^p_{L^\infty(\Omega)}\right)^{\frac{1}{p}} \\ &\leq \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{p}} \left(m \cdot \max_{i\in\{1,\cdots,m\}} \|f_i(\cdot, w(\cdot))\|^p_{L^\infty(\Omega)}\right)^{\frac{1}{p}} \\ &= \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{p}} \max_{i\in\{1,\cdots,m\}} \|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)} = \left(m\mathcal{L}^N(\Omega)\right)^{\frac{1}{p}} J_{m,\infty}(w). \end{aligned}$$

Passing to the limit supremum as $p \to \infty$ we obtain

$$\limsup_{p\to\infty} J_{m,p}(w_p) \leq J_{m,\infty}(w),$$

and this concludes the proof. □

4. Variational characterization of the yield set of a polycrystal: two model cases

In this section we specialize to the case where given a function $B \in L^p\left(\Omega; \mathbb{M}^{N\times N}\right)$ the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}B := \mathrm{Div}B = \begin{pmatrix} \mathrm{div}B^{(1)} \\ \mathrm{div}B^{(2)} \\ \vdots \\ \mathrm{div}B^{(N)} \end{pmatrix},$$

where, for $i = 1, \cdots, N$, $B^{(i)}(x) := (B_{i1}(x), B_{i2}(x), \cdots, B_{iN}(x))$ stands for the i-th row of the matrix $B(x)$, $x \in \Omega$. Thus, taking $d = N^2$, and $l = N$, the differential constraint $\mathcal{A}B = 0$ can be written in the form

$$\sum_{k=1}^N A^{(k)} \frac{\partial B}{\partial x_k} = 0$$

provided that we define, for $i, k = 1, \cdots, N$ and $j = 1, \cdots, N^2$,

$$A^{(k)}_{ij} = \begin{cases} \delta_{i(j-(k-1)N)} & \text{if } (k-1)N + 1 \leq j \leq kN \\ 0 & \text{else,} \end{cases}$$

where the symbol δ_{ij} stands for the Kronecker's delta. We note that the constant rank condition (5) is satisfied since for every $w \in S^{N-1}$ we have

$$\ker(\mathbb{A}(w)) = \left\{V \in \mathbb{M}^{N\times N} \ : \ wV = 0\right\},$$

and thus $\dim(\ker \mathbb{A}(w)) = N^2 - N$.

We assume in what follows that the pointwise constraint (1) on the stress may be written in the form

$$\sigma(x) \in \left\{\eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \ : \ f_i(x, \eta) \leq 1 \text{ for all } i = 1, \cdots, m\right\}, \tag{26}$$

where $f_i : Q \times \mathbb{M}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}$ $(i = 1, \cdots, m)$ are Carathéodory integrands satisfying our hypotheses (9), (10), and (11). In this case the yield set of the polycrystal becomes

$$K_{\mathrm{eff}} = \Bigg\{ \eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \ : \ \exists\, \sigma \in L^\infty\left(Q; \mathbb{M}^{N\times N}_{\mathrm{sym}}\right) \text{ such that } \eta = \int_Q \sigma(x)dx, \ \mathrm{Div}\ \sigma = 0,$$
$$f_i(x, \sigma(x)) \le 1 \ \mathcal{L}^N\text{–a.e. } x \in Q, \ i = 1, \cdots, m \Bigg\},$$

or, equivalently,

$$K_{\mathrm{eff}} = \Bigg\{ \eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \ : \ \exists\, \sigma \in L^\infty\left(Q; \mathbb{M}^{N\times N}_{\mathrm{sym}}\right) \text{ such that } \int_Q \sigma(x)dx = 0, \ \mathrm{Div}\ \sigma = 0,$$
$$f_i(x, \sigma(x) + \eta) \le 1 \ \mathcal{L}^N\text{–a.e. } x \in Q, \ i = 1, \cdots, m \Bigg\}. \tag{27}$$

For $\eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$ we consider the variational principle

$$j^{\mathrm{eff}}_{m,p}(\eta) := \inf \left\{ \left(\int_Q \left(\sum_{i=1}^m f_i(x, B(x) + \eta)^p \right) dx \right)^{1/p} \ : \ B \in L^p\left(Q; \mathbb{M}^{N\times N}_{\mathrm{sym}}\right), \right.$$
$$\left. \int_Q B\, dx = 0, \ \mathrm{Div}\ B = 0 \right\}. \tag{28}$$

In view of our Theorem 3.1, well-known arguments in the theory of Γ-convergence (see, e.g., [4] and [6]) imply, in particular, that for any $\eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}}$, $j^{\mathrm{eff}}_{m,p}(\eta)$ converges, as $p \to \infty$, to $j^{\mathrm{eff}}_{m,\infty}(\eta)$ given by

$$j^{\mathrm{eff}}_{m,\infty}(\eta) := \inf \left\{ \max_{1\le i\le m} \operatorname*{ess\,sup}_{x\in Q} f_i(x, B(x) + \eta) \ : \ B \in L^\infty\left(Q; \mathbb{M}^{N\times N}_{\mathrm{sym}}\right), \right.$$
$$\left. \int_Q B(x)\, dx = 0, \ \mathrm{Div}\ B = 0 \right\}. \tag{29}$$

The effective yield set K_{eff} can now be characterized in terms of the limiting variational principle $j^{\mathrm{eff}}_{m,\infty}$.

Theorem 4.1.

$$K_{\mathrm{eff}} = \left\{ \eta \in \mathbb{M}^{N\times N}_{\mathrm{sym}} \ : \ j^{\mathrm{eff}}_{m,\infty}(\eta) \le 1 \right\}. \tag{30}$$

Proof. Let $\eta \in K_{\mathrm{eff}}$. By (27), there exists $\sigma \in L^\infty\left(Q; \mathbb{M}^{N\times N}_{\mathrm{sym}}\right)$ such that $\int_Q \sigma(x)dx = 0$, $\mathrm{Div}\ \sigma = 0$, and $f_i(x, \sigma(x) + \eta) \le 1$ for $\mathcal{L}^N$–a.e. $x \in Q$, $i = 1, \cdots, m$. We have

$j^{\text{eff}}_{m,\infty}(\eta) \le \max_{1\le i\le m} \operatorname{ess\,sup}_{x\in Q} f_i(x,\sigma(x)+\eta) \le 1$. Conversely, let $\eta \in \mathbb{M}^{N\times N}_{\text{sym}}$ be such that

$$j^{\text{eff}}_{m,\infty}(\eta) \le 1. \tag{31}$$

Consider a sequence $\{\sigma_n\} \subseteq L^\infty\left(Q;\mathbb{M}^{N\times N}_{\text{sym}}\right)$ such that $\operatorname{Div}\sigma_n = 0$, $\int_Q \sigma_n(x)dx = 0$ for any $n\in\mathbb{N}$, and

$$\lim_{n\to\infty}\left(\max_{1\le i\le m} \operatorname*{ess\,sup}_{x\in Q} f_i(x,\sigma_n(x)+\eta)\right) = j^{\text{eff}}_{m,\infty}(\eta). \tag{32}$$

The coercivity condition (11) implies that the sequence $\{\sigma_n\}$ is bounded in $L^\infty\big(Q;\mathbb{M}^{N\times N}_{\text{sym}}\big)$. Thus, we may extract a subsequence of $\{\sigma_n\}$ (not relabelled) such that $\sigma_n \rightharpoonup \sigma$ weakly* in $L^\infty\left(Q;\mathbb{M}^{N\times N}_{\text{sym}}\right)$, with $\operatorname{Div}\sigma = 0$, and $\int_Q \sigma(x)dx = 0$. Let $x\in Q$ be a Lebesgue point for each of the $f_i(\cdot,\sigma(\cdot)+\eta)$, $i=1,\cdots,m$. By Proposition 2.2 we deduce that

$$\int_{B(x,r)} f_i(y,\sigma(y)+\eta)dy \le \liminf_{n\to\infty} \int_{B(x,r)} f_i(y,\sigma_n(y)+\eta)dy, \quad i=1,\cdots,m,$$

for sufficiently small $r>0$. Thus, (32) yields

$$\frac{1}{\mathcal{L}^N(B(x,r))}\int_{B(x,r)} f_i(y,\sigma(y)+\eta)dy \le j^{\text{eff}}_{m,\infty}(\eta).$$

Letting $r\to 0^+$, since almost every point $x\in Q$ is a Lebesgue point for all $f_i(\cdot,\sigma(\cdot)+\eta)$, $i=1,\cdots,m$, we have that $f_i(x,\sigma(x)+\eta) \le j^{\text{eff}}_{m,\infty}(\eta)$ for $\mathcal{L}^N$-a.e. $x\in Q$, $i=1,\cdots,m$. Taking (31) into account, we deduce that $\eta\in K_{\text{eff}}$. Thus, (30) holds. □

We now consider two well-known models (see, e.g., Kohn & Little [14], whose presentation we follow) for which our results are relevant.

4.1. Plane stress

We consider stresses given by

$$\sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) & 0 \\ \sigma_{12}(x) & \sigma_{22}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x\in\Omega\subset\mathbb{R}^2. \tag{33}$$

In plane stress the slip tensor is given by $\mu = \frac{1}{2}(m\otimes n + n\otimes m)$, where $m\perp n, \mu_{13}=\mu_{23}=\mu_{33}=0$. The slip tensor belongs to the space spanned by the tensors $\mu^{(1)} = \frac{1}{2}\left(m^{(1)}\otimes n^{(1)} + n^{(1)}\otimes m^{(1)}\right)$ and $\mu^{(2)} = \frac{1}{2}\left(m^{(2)}\otimes n^{(2)} + n^{(2)}\otimes m^{(2)}\right)$, with $m^{(1)} = (1,0,0)$, $n^{(1)} = (0,1,0)$, $m^{(2)} = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, and $m^{(2)} = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)$. In what follows we identify the stress with the 2×2 upper-left corner of the 3×3 matrix in (33). The yield set of the basic crystal has the form

$$K = K_{M_1N_1} := \left\{\sigma = (\sigma_{ij})\in\mathbb{M}^{2\times 2}_{\text{sym}} \;:\; |\sigma_{12}|\le M_1, |\sigma_{11}-\sigma_{22}|\le 2,\; |\sigma_{11}+\sigma_{22}|\le N_1\right\}.$$

If the rotation describing the orientations of the grain containing the point $x \in \Omega$ is given by

$$R(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \tag{34}$$

then the pointwise constraint (1) becomes

$$R^T(x)\sigma(x)R(x) \in K_{M_1N_1}, \tag{35}$$

or, equivalently,

$$\begin{pmatrix} \sigma_{11}\cos^2\theta + \sigma_{12}\sin 2\theta + \sigma_{22}\sin^2\theta & \sigma_{12}\cos 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta \\ \sigma_{12}\cos 2\theta - \frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta & \sigma_{11}\sin^2\theta(x) - \sigma_{12}\sin 2\theta + \sigma_{22}\cos^2\theta \end{pmatrix}(x) \in K_{M_1N_1}.$$

Taking $N = 2$, $m = 3$, this can be written in the form (26), where the functions $f_i (i \in \{1,2,3\})$ are defined by

$$f_1(x,\eta) := \frac{1}{M_1}\left|\eta_{12}\cos(2\theta(x)) - \frac{1}{2}(\eta_{11}-\eta_{22})\sin(2\theta(x))\right|,$$

$$f_2(x,\eta) := \frac{1}{2}\left|(\eta_{11}-\eta_{22})\cos(2\theta(x)) + 2\eta_{12}\sin(2\theta(x))\right|,$$

and

$$f_3(x,\eta) := \frac{1}{N_1}\left|\eta_{11}+\eta_{22}\right|.$$

With these choices our results apply. Indeed, it is easy to check that the functions defined above satisfy our $\mathcal{A}$-quasiconvexity condition (9) ($f_i(x,\cdot)$ is, in fact, convex for all $i \in \{1,2,3\}$), and the growth condition (10). It remains to show that the coercivity condition (11) holds as well. To this aim first note that for all $x \in \Omega$ and $\eta = (\eta_{ij})_{i,j=1,2} \in \mathbb{M}^{2\times 2}_{\text{sym}}$ we have

$$|\eta_{12}| \le M_1 f_1(x,\eta) + f_2(x,\eta), \qquad |\eta_{11}+\eta_{22}| \le N_1 f_3(x,\eta),$$

and

$$|\eta_{11}-\eta_{22}| \le 2f_2(x,\eta) + 2M_1 f_1(x,\eta).$$

This gives

$$|\eta_{11}| \le \frac{N_1}{2} f_3(x,\eta) + f_2(x,\eta) + M_1 f_1(x,\eta),$$

and

$$|\eta_{22}| \le \frac{N_1}{2} f_3(x,\eta) + f_2(x,\eta) + M_1 f_1(x,\eta)$$

for all $x \in \Omega$ and $\eta \in \mathbb{M}^{2\times 2}_{\text{sym}}$. It follows that

$$|\eta| \le M_1(2+\sqrt{2}) f_1(x,\eta) + (2+\sqrt{2}) f_2(x,\eta) + N_1 f_3(x,\eta), \quad \text{for all } x \in \Omega \text{ and } \eta \in \mathbb{M}^{2\times 2}_{\text{sym}},$$

which yields (11), with $c = \left(\max\{N_1, 2+\sqrt{2}, (2+\sqrt{2})M_1\}\right)^{-1}$.

4.2. Antiplane shear

In antiplane shear the stress is given by

$$\sigma(x) = \begin{pmatrix} 0 & 0 & \sigma_{13}(x) \\ 0 & 0 & \sigma_{23}(x) \\ \sigma_{31}(x) & \sigma_{32}(x) & 0 \end{pmatrix}, \quad x \in \Omega \subset \mathbb{R}^2,$$

which corresponds to slip tensors $\mu = \frac{1}{2}(m \otimes n + n \otimes m)$, with $\mu_{11} = \mu_{22} = \mu_{33} = \mu_{12} = 0$, belonging to the two-dimensional space spanned by $\mu^{(1)} = \frac{1}{2}(e_1 \otimes e_3 + e_3 \otimes e_1)$ and $\mu^{(2)} = \frac{1}{2}(e_2 \otimes e_3 + e_3 \otimes e_2)$, where e_1, e_2, and e_3 are the vectors in the canonical basis of $\mathbb{R}^3$. Assume that we are dealing with a crystal with a deficient supply of slip systems, which in this case means that there are just four basic slip systems with slip tensors $\pm\mu^{(1)}, \pm\mu^{(2)}$ and critical stresses $\pm M, \pm 1$. After identifying the stress with a vector field in the plane, the yield set of our basic crystal is

$$K = K_M := \left\{ \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2 \ : \ |\sigma_1| \le M, \ |\sigma_2| \le 1 \right\}.$$

For consistency, the rotations in (1) are taken such that they keep the x_3-axis fixed, and thus we may identify them with rotations of the x_1x_2-plane. The constraint (1) then reads

$$R^T(x)\sigma(x) \in K_M, \quad x \in \Omega \subset \mathbb{R}^2.$$

With $R : \Omega \to \mathrm{SO}(2)$ given by (34) and $\sigma : \Omega \to \mathbb{R}^2, \sigma(x) = (\sigma_1(x), \sigma_2(x))^T$, this means that we must have

$$|\sigma_1(x)\cos\theta(x) + \sigma_2(x)\sin\theta(x)| \le M,$$

and

$$|\sigma_2(x)\cos\theta(x) - \sigma_1(x)\sin\theta(x)| \le 1,$$

for all $x \in \Omega$. These requirements can be written in the form (26) provided that we choose $N = m = 2$, and that $f_1, f_2 : \Omega \times \mathbb{R}^2 \to [0, +\infty)$ are given by

$$f_1(x, \eta) := \frac{1}{M} \left|\eta_1 \cos\theta(x) + \eta_2 \sin\theta(x)\right|,$$

and

$$f_2(x, \eta) := \left|\eta_2 \cos\theta(x) - \eta_1 \sin\theta(x)\right|.$$

Again, our results apply: (9) and (10) hold, and it is immediate to verify that the coercivity condition (11) is also satisfied, with $c = (2\max\{M, 1\})^{-1}$.

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References

[1] J. Bishop, R. Hill: A theory of the plastic distortion of a polycrystalline aggregate under combined stresses, Phil. Mag. A 42 (1951) 414–427.

[2] M. Bocea, V. Nesi: Γ-convergence of power-law functionals, variational principles in L^∞, and applications, SIAM J. Math. Anal. 39(5) (2008) 1550–1576.

[3] A. Braides: Γ-Convergence for Beginners, Oxford University Press, Oxford (2002).

[4] A. Braides, A. Defranceschi: Homogenization of Multiple Integrals, Oxford Lect. Ser. in Mathematics and its Applications 12, Oxford University Press, Oxford (1998).

[5] B. Dacorogna: Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals, Lecture Notes in Mathematics 922, Springer, Berlin (1982).

[6] G. Dal Maso: An Introduction to Γ-Convergence, Progress In Nonlinear Differential Equations and Their Applications 8, Birkhäuser, Basel (1993).

[7] E. De Giorgi: Sulla convergenza di alcune successioni d'integrali del tipo dell'area, Rend. Mat. 8 (1975) 277–294.

[8] E. De Giorgi, T. Franzoni: Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 58 (1975) 842–850.

[9] I. Fonseca, S. Müller: A-quasiconvexity, lower semicontinuity, and Young measures, SIAM J. Math. Anal. 30 (1999) 1355–1390.

[10] A. Garroni, R.V. Kohn: Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity, Proc. R. Soc. Lond., Ser. A 459 (2003) 2613–2625.

[11] A. Garroni, V. Nesi, M. Ponsiglione: Dielectric breakdown: optimal bounds, Proc. R. Soc. Lond., Ser. A 457 (2001) 2317–2335.

[12] G. H. Goldsztein: Rigid perfectly plastic two-dimensional polycrystals, Proc. R. Soc. Lond., Ser. A 457 (2001) 2789–2798.

[13] G. H. Goldsztein: Two-dimensional rigid polycrystals whose grains have one ductile direction, Proc. R. Soc. Lond., Ser. A 459 (2003) 1949–1968.

[14] R. V. Kohn, T. D. Little: Some model problems of polycrystal plasticity with deficient basic crystals, SIAM J. Appl. Math. 59 (1999) 172–197.

[15] F. Murat: Compacité par compensation: condition necessaire et suffisante de continuité faible sous une hypothèse de rang constant, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 8 (1981) 68–102.

[16] G. Sachs: Zur Ableitung einer Fließbedingung, Z. Ver. Dtsch. Ing. 72 (1928) 734–736.

[17] L. Tartar: Compensated compactness and applications to partial differential equations, in: Nonlinear Analysis and Mechanics: Heriot-Watt Symposium. Vol. IV, R. Knops (ed.), Pitman Res. Notes Math. 39, Longman, Harlow (1979) 136–212.

[18] L. Tartar: The compensated compactness method applied to systems of conservation laws, in: Systems of Nonlinear Partial Differential Equations, J. M. Ball (ed.), D. Reidel, Dordrecht (1983) 263–285.

[19] G. Taylor: Plastic strains in metals, J. Inst. Metals 62 (1938) 307–324.