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Continuous Selections, Free Vector Lattices and Formal Minkowski Differences

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We investigate the vector lattice of continuous selections of linear functionals on a topological vector space. In particular, we show that it is a free vector lattice on n generators and can be constructed as vector lattice of formal Minkowski differences of polytopes. This can be used to show that every (set-theoretically) minimal representation of polytopes is a representation of a (formal) difference of polytopes.

Keywords: Vector lattice, polyhedral functional, polytope, support functional, Minkowski sum

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1. Continuous Selections

Rådström [10] constructed vector spaces of formal Minkowski differences of non-empty compact convex sets in a topological vector space; later his construction was generalized to non-empty bounded closed convex subsets. In the locally convex situation, Hörmander [7] introduced the method of *support functionals*. Later Pinsker [9] introduced and studied the lattice structure on these vector spaces. Even later this framework was used in quasi-differential calculus (cf. e.g. [2], [4], [6], [8]), in particular for the study of non-smooth optimization problems. Somewhat surprisingly, if we restrict this machinery to polytopes, we obtain a new construction of free vector lattices (cf. [1] and [13]).

A continuous real *functional* on a real topological vector space V (i.e. a continuous map $f : V \to \mathbb{R}$) is called a *continuous selection* of finitely many continuous functionals $f_1, \dots, f_n$ if $f(x) \in \{f_1(x), \dots, f_r(x)\}$ holds for all $x \in V$. The selections of $f_1, \dots, f_r$ always form a *lattice* (in the order-theoretic sense) because the *join* (pointwise maximum) $f \vee f'$ and the *meet* (pointwise minimum) $f \wedge f'$ of two selections f, f' are selections. Since the continuous functionals also form a lattice, the continuous selections of $f_1, \dots, f_r$ form a (possibly empty) sublattice; of course, if $f_1, \dots, f_r$ are continuous themselves, so they belong to the lattice of continuous selections.

Now assume that all f_i are distinct and also *linear*, and consider $x \in V$, $1 \leq j \leq r$ with $f(x) = f_j(x)$, but $f(x) \neq f_i(x)$ for $i \neq j$. Let $K^\circ(x) \in V$ be the subset of all y with $f_i(y) < f(y)$ for all i with $f_i(x) < f(x)$ and $f_i(y) > f(y)$ whenever $f_i(x) > f(x)$.

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Then we have $x \in K^\circ(x)$, and by continuity $K^\circ(x)$ is open. By linearity, $K^\circ(x)$ is also convex and therefore connected, hence f coincides with f_j on $K^\circ(x)$ and we have $K^\circ(y) = K^\circ(x)$ for all $y \in K^\circ(x)$. The closure $K(x)$ of $K^\circ(x)$ is the closed cone of all y with $f_i(y) \le f(y)$ whenever $f_i(x) < f(x)$ and $f_i(y) \ge f(y)$ whenever $f_i(x) > f(x)$. We observe that there are only finitely many cones of the form $K(x)$. Indeed, there are at most n choices for i and for each fixed i there are only finitely many $j \neq i$ and only two possibilities $f_j(x) < f(x)$ and $f_j(x) > f(x)$ for each j; thus there are at most $2^{r-1}r$ such cones. Note that this is also true for $n = 0$; then the lattice is empty. Moreover, the set of all $x \in V$ for which all $f_i(x)$ are distinct is open and dense in V; thus V is the union of the finitely many cones of the form $K(x)$.

Lemma 1.1. *If f is a continuous selection of $f_1, \ldots, f_r$, then for all $x, y \in V$ there exists a $j \in \mathbb{N}_r := \{1, \ldots, r\}$ with $f_j(x) \le f(x)$ and $f_j(y) \ge f(y)$.*

Proof. The above representation of V as a union of finitely many closed cones yields the existence of $s \in \mathbb{N}$, $\lambda_0, \ldots, \lambda_s \in \mathbb{R}$, $i_1, \ldots, i_s \in \mathbb{N}_r$ with $0 = \lambda_0 < \ldots < \lambda_s = 1$ and $f((1-\lambda)x + \lambda y) = f_{i_k}((1-\lambda)x + \lambda y)$ for all $k \in \mathbb{N}_s$, $\lambda \in [\lambda_{i_{k-1}}, \lambda_{i_k}]$. Now we choose $l \in \mathbb{N}_n$ such that $f_{i_l}(y - x)$ is maximal. Then for all $k \in \mathbb{N}_s$ we get

$$
\begin{aligned}
& (f_{i_l}((1-\lambda_{i_k})x + \lambda_{i_k}y) - f((1-\lambda_{i_k})x + \lambda_{i_k}y)) \\
& \qquad - (f_{i_l}((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y) - f((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y)) \\
= \ & f_{i_l}((1-\lambda_{i_k})x + \lambda_{i_k}y) - f_{i_k}((1-\lambda_{i_k})x + \lambda_{i_k}y) \\
& \qquad - f_{i_l}((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y)) + f_{i_k}((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y) \\
= \ & f_{i_l}(((1-\lambda_{i_k})x + \lambda_{i_k}y) - ((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y)) \\
& \qquad - f_{i_k}(((1-\lambda_{i_k})x + \lambda_{i_k}y) - ((1-\lambda_{i_{k-1}})x + \lambda_{i_{k-1}}y)) \\
= \ & f_{i_l}((\lambda_{i_k} - \lambda_{i_{k-1}})(y-x)) - f_{i_k}((\lambda_{i_k} - \lambda_{i_{k-1}})(y-x)) \\
= \ & (\lambda_{i_k} - \lambda_{i_{k-1}})(f_{i_l}(y-x) - f_{i_k}(y-x)) \ge 0
\end{aligned}
$$

because $\lambda_{i_k} > \lambda_{i_{k-1}}$ and $f_{i_l}(y-x)$ is maximal among all $f_{i_k}(y-x)$. So we obtain

$$
\begin{aligned}
& f_{i_l}((1-\lambda_{i_k})x + \lambda_{i_k}y) - f((1-\lambda_{i_k})x + \lambda_{i_k}y) \\
\le \ & f_{i_l}((1-\lambda_{i_{k'}})x + \lambda_{i_{k'}}y) - f((1-\lambda_{i_{k'}})x + \lambda_{i_{k'}}y)
\end{aligned}
$$

for all $k, k' \in \mathbb{N}_s \cup \{0\}$ with $k \le k'$, in particular

$$
\begin{aligned}
f_{i_l}(x) - f(x) &= f_{i_l}((1-\lambda_{i_0})x + \lambda_{i_0}y) - f((1-\lambda_{i_0})x + \lambda_{i_0}y) \\
&\le f_{i_l}((1-\lambda_{i_l})x + \lambda_{i_l}y) - f((1-\lambda_{i_l})x + \lambda_{i_l}y) = 0 \\
&\le f_{i_l}((1-\lambda_{i_s})x + \lambda_{i_s}y) - f((1-\lambda_{i_s})x + \lambda_{i_s}y) = f_{i_l}(y) - f(y).
\end{aligned}
$$

So for $j := i_l$ we have $f_j(x) \le f(x)$ and $f_j(y) \ge f(y)$. □

The following result is essentially due to Bartels, Kuntz, and Scholtes [2] in the finite-dimensional case:

Theorem 1.2. *For finitely many continuous linear functionals, $f_1, \ldots, f_r$, a functional f belongs to the sublattice of V generated by $f_1, \ldots, f_r$ if and only if f is a continuous selection of $f_1, \ldots, f_r$.*

Proof. The "only if" part is obvious. For the "if" part we assume that f is a continuous selection of continuous linear functionals $f_1, \ldots, f_n$. For each $x \in V$ consider the set J_x of all $j \in \mathbb{N}_n$ with $f_j(x) \leq f(x)$ and let h_x be the join of the (finitely many) f_j with $j \in J_x$. Then each h_x belongs to the lattice L generated by $f_1, \ldots, f_r$. As f is a selection, there exists an $i \in \mathbb{N}_n$ with $f(x) = f_i(x)$, and we obviously have $i \in J_x$, hence $h_x(x) = f(x)$. Now Lemma 1.1 gives $h_x(y) \geq f(y)$ for all $y \in V$.

So for all x we have $h_x \geq f$ in L (i.e. pointwise). Since there are only finitely many subsets of $\mathbb{N}_r$, only finitely many functionals occur as h_x (for some x). Thus their meet belongs to L and is greater than or equal to f; it even coincides with f because $f(x) = h_x(x)$, thus we have $f \in L$. □

Note that linearity is essential in the above proof. Indeed, if $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are defined by $f_1(x) := 0$ and $f_2 := \sin x$ for all $x \in \mathbb{R}$, then we get infinitely (even uncountably) many selections f by independently choosing $i_k \in [1, 2]$ and defining $f|_{[\pi k, \pi(k+1)]} := f_{i_k}|_{[\pi k, \pi(k+1)]}$ for each $k \in \mathbb{Z}$. Nevertheless, the lattice generated by f_1, f_2 consists only of the four elements f_1, f_2, $f_1 \wedge f_2$, $f_1 \vee f_2$.

So every continuous selection f of continuous linear functionals $f_1, \ldots, f_r$ is a non-empty meet $\bigwedge_{I \in \mathcal{I}} \bigvee_{i \in I} f_i$ of non-empty joins $\bigvee_{i \in I} f_i$ for some subset $\mathcal{I} \neq \emptyset$ of $\mathcal{P}(\mathbb{N}_r) \setminus \{\emptyset\}$. For $J, J' \in \mathcal{I}$, $J \subset J'$ we always have $\bigvee_{i \in J} f_i \leq \bigvee_{i \in J'} f_i$; if $J \neq J'$, we therefore have $\bigwedge_{I \in \mathcal{I}} \bigvee_{i \in I} f_i = \bigwedge_{I \in \mathcal{I} \setminus \{J'\}} \bigvee_{i \in I} f_i$. If $\mathcal{I}$ has minimal cardinality in the above representation, we see that $\mathcal{I}$ is an *antichain*, i.e. no member of $\mathcal{I}$ is properly contained in another one. Thus the number of continuous selections of $f_1, \ldots, f_n$ is at most as large as the number of non-empty antichains in $\mathcal{P}(\mathbb{N}_n) \setminus \{\emptyset\}$, i.e. of antichains $\mathcal{I} \in \mathcal{P}(\mathbb{N}_n)$, i.e. $f = \bigvee_{I \in \mathcal{I}} \bigwedge i \in I f_i$, these are all antichains except the two trivial ones $\emptyset$ and $\{\emptyset\}$.

It is well-known that in the free distributive lattice on r generators $a_1, \ldots, a_n$ the elements $\bigwedge_{I \in \mathcal{I}} \bigvee_{i \in I} a_i$ are different for different antichains $\mathcal{I}$; therefore the cardinality of the free distributive lattice is the number of non-empty antichains in $\mathcal{P}(\mathbb{N}_r) \setminus \{\emptyset\}$ (cf. [3], p. 273). Obviously the lattice of all functionals V (and even every vector lattice) is distributive; so we immediately get the following

Corollary 1.3. *For linear functionals $f_1, \ldots, f_r$ on V, there is a unique lattice homomorphism from the free distributive lattice on r generators to the lattice of continuous selections of $f_1, \ldots, f_r$, and this homomorphism is always surjective. In particular, the number of continuous selections of $f_1, \ldots, f_r$ is at most the number of non-empty antichains in $\mathcal{P}(\mathbb{N}_r) \setminus \{\emptyset\}$.*

The upper bound from Corollary 1.3 is attained for every r:

Proposition 1.4. *For $n \in \mathbb{N}$ let $p_1, .., p_n : \mathbb{R}^n \to \mathbb{R}$ be the canonical projections, i.e. $p_i((\xi_1, \ldots, \xi_n)^\top) := \xi_i$ for all $i \in \mathbb{N}_n$, $\xi_1, \ldots, \xi_n \in \mathbb{R}$. Moreover, define $p_{n+1} := -\sum_{i=1}^n p_i$, i.e. $p_{n+1}((\xi_1, \ldots, \xi_n)^\top) := -\sum_{i=1}^n \xi_i$ for all $(\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n$. Then the canonical lattice homomorphism from the free distributive lattice on $n+1$ generators to the lattice of all continuous selections for $p_1, \ldots, p_{n+1}$ is an isomorphism, in particular, the number of continuous selections is the number of non-empty antichains* *different from* $\{\emptyset\}$ *in* $\mathcal{P}(\mathbb{N}_{n+1})$.

Proof. It suffices to show $\bigwedge_{I\in\mathcal{I}}\bigvee_{i\in I}p_i \not\leq \bigvee_{i\in J}p_i$ for all $\mathcal{I}\subset\mathcal{P}(\mathbb{N}_{r+1})\setminus\{\emptyset\}$, $J\in\mathcal{P}(\mathbb{N}_{r+1})\setminus\{\emptyset\}$ with $\mathcal{I}\neq\emptyset$, $\emptyset\notin\mathcal{I}$, $J\neq\emptyset$ and $I\not\subset J$ for all $I\in\mathcal{I}$. Now let m be the cardinality of $J\neq\emptyset$ and define $x:=(\xi_1,\ldots,\xi_n)^\top\in V$ by $\xi_i:=-\frac{m}{n}$ for all $i\in J\cap\mathbb{N}_n$ and $\xi_i:=1-\frac{m}{n}$ for all $i\in\mathbb{N}_n\setminus J$. Then we get $p_{n+1}(x)=-\frac{m}{n}$ if $n+1\in J$ and $p_{n+1}(x)=1-\frac{m}{n}$ if $n+1\notin J$. Then for each $I\in\mathcal{I}$ we have $\bigvee_{i\in I}p_i(x)=1-\frac{m}{n}>0$ since $I\not\subset J$, but

$$\bigvee_{i\in J}p_i(x)=-\frac{m}{n}<0<\bigwedge_{I\in\mathcal{I}}\bigvee_{i\in I}p_i(x)=1-\frac{m}{n},$$

proving our claim. □

The sequence of numbers of all non-empty antichains in $\mathcal{P}(\mathbb{N}_r)\setminus\{\emptyset\}$ is no. A007153 in Sloane's list [11]. No explicit formula and not even a recursion formula is known, only some asymptotic statements and the values for $n\leq 8$ are known; these values are:

n	number of antichains
0	0
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

2. Free Vector Lattices

We call a functional on V *polyhedral* if it is a continuous selection of finitely many continuous linear functionals. Some authors as for instance Ewaldt [5] call these functionals piecewise linear. Moreover, every polyhedral functional f is *positively homogeneous*, i.e. $f(\alpha x)=\alpha f(x)$ holds for all $\alpha\in\mathbb{R}^+$, $x\in V$. Every join, meet or sum of two polyhedral functionals and every scalar multiple of a polyhedral functional is polyhedral and the polyhedral functionals on V form a vector lattice under these operations.

We call a functional f on V *sublinear* if f is positively homogeneous and *subadditive* (i.e. $f(x+y)\leq f(x)+f(y)$ for all $x,y\in V$). Obviously, a positively homogeneous functional f on V is subadditive if and only if it is *convex* (i.e. $f((1-\lambda)x+\lambda y\leq(1-\lambda)f(x)+\lambda f(y)$ for all $x,y\in V$, $\lambda\in[0,1]$).

Theorem 2.1. *For a functional f on V, the following statements are equivalent:*

(*i*) *f is polyhedral and subadditive.*

(*ii*) *f is a finite join of continuous linear functionals.*

If these equivalent statements are true, then there exist finitely meany polyhedral cones in such that f is linear on each of them.

Proof. $(i) \Rightarrow (ii)$": By Theorem 1.2, f is a continuous selection of some continuous linear functionals $f_1, \dots, f_r$; w.l.o.g. we can choose r minimal. Then each f_i coincides with f on the polyhedral cone $K_i := \{x \in V | \, f_i(x) \le f_j(x) \, \text{for} \, j = 1, \dots, r\}$, which is regularly closed by minimality, i.e. it is the closure of its interior; in particular it has an interior point a. For a given $x \in V$ we have $a + \alpha^{-1}x \in K_i$ for some (sufficiently large) positive real number α, thus also $\alpha a + x = \alpha(a + \alpha^{-1}x) \in K$. Then we get $f(\alpha a + x) \le f(\alpha a) + f(x)$ by subadditivity and $f(\alpha a) = f_i(\alpha a)$, $f(\alpha a + x) = f_i(\alpha a + x)$ since $\alpha a, \alpha a + x \in K$, hence

$$f(x) \ge f(\alpha a + x) - f(\alpha a) = f_i(\alpha a + x) - f_i(\alpha a) = f_i(\alpha a + x - \alpha a) = f_i(x).$$

This shows $f \ge f_i$ for all $i \in \mathbb{N}_n$, therefore $f \ge \bigvee_{i=1}^n f_i$. Since f is a selection of the f_i, we even have $f = \bigvee_{i=1}^r f_i$.

"$(ii) \Rightarrow (i)$" is obvious because all continuous linear functionals are polyhedral and subadditive and because both polyhedral functionals and subadditive functionals are closed under the formation of non-empty finite joins. Note that for $n \ge 1$ the empty join and the empty meet (i.e. least and largest element) of functionals do not exist. □

Theorem 2.2. *For a functional f on V the following statements are equivalent:*

- (i) *f is polyhedral.*
- (ii) *f belongs to the vector lattice generated by all continuous linear functionals on V.*
- (iii) *f is a difference of two subadditive polyhedral functionals.*
- (iv) *There exist finitely many polyhedral cones such that their union is V and f is linear on each of them.*

Proof. (i) "$\Rightarrow$" (ii): Since f is a continuous selection of finitely many continuous linear functionals, by Theorem 1.2 f belongs to the lattice generated by these linear functionals, and (ii) follows.

"$(ii) \Rightarrow (iii)$": It suffices to show that the mentioned differences form a lattice containing all continuous linear functionals. Obviously all continuous linear functionals belong to this set. The join of two continuous subadditive polyhedral functionals is also continuous, polyhedral and subadditive. Thus for two differences $f = g - h$, $f' = g' - h'$ of continuous subadditive polyhedral functionals g, h, g', h' their join $f \vee g = ((g+h') \vee (g'+h)) - (h+h')$ and their meet $f \wedge f' = (g+g') - ((g+h') \vee (g'+h))$ also belong to the set of these differences.

"$(iii) \Rightarrow (iv)$": Assume $f = g - h'$ with g and g' polyhedral and subadditive. By Theorem 2.1 V is a union of finitely many polyhedral cones $K_1, \dots, K_r$ such that g is linear in each K_i, and V is also the union of finitely many polyhedral $K_1', \dots, K_s$ on which h is linear. Then all $K_i \wedge K_j'$ are also polyhedral cones with union V and $f = g - h$ is linear on each $K_i \wedge K_j'$.

"$(iv) \Rightarrow (i)$" is trivial. □

E. C. Weinberg [13] showed that every identity in the language of vector lattices holds in all vector lattices provided it holds in $\mathbb{R}$. K. Baker [1] concluded that the free vector lattice on a set T is the vector lattice generated by all canonical projections $p_t : \mathbb{R}^T \to \mathbb{R}$, in the vector lattice of all maps from $\mathbb{R}^T$ to $\mathbb{R}$ with pointwise structure. Since the vector subspace generated by the canonical projections is the vector space of all linear functionals $\mathbb{R}^T \to \mathbb{R}$, which are continuous in the product topology of $\mathbb{R}^T$, we see from Theorem 2.2 that the vector sublattice generated by the canonical projections is the vector lattice of continuous polyhedral functionals on $\mathbb{R}^T$, which we shall denote by $F(T)$ throughout the remainder of this paper. So we obtain the following

Corollary 2.3. *For every set T, $F(T)$ is the free vector lattice on T (with the canonical projections as generators).* □

3. Formal Minkowski Differences

H. Rådström [10] introduced vector spaces of formal differences of compact convex sets in topological vector spaces; later this was generalized to bounded closed convex sets. A. G. Pinsker [9] defined and investigated vector lattice structures on these vector spaces. In order to keep this paper as self-contained as possible, we repeat known facts with easy proofs in the locally convex situation; for details see [8]. We shall work with *support functionals* (often called *support functions*) as suggested and used by L. Hörmander [7].

Throughout the remainder of this paper, V will denote a Hausdorff locally convex space. The *Minkowski sum* of two subsets $A, B \subset V$ is the set $A + B := \{x + y |\, x \in A,\, y \in B\}$. If A, B are convex or bounded, then so is $A + B$, and $A, B \neq \emptyset$ implies $A + B \neq \emptyset$. The Minkowski sum of two non-empty bounded closed convex sets A, B need not be closed, but its closure $A \dot{+} B$ is again bounded, non-empty and convex. The *support functional* of a non-empty bounded closed convex subset $A \subset V$ is the functional ρ_A on the dual space V^* (i.e. the vector space of all continuous linear functionals $V \to \mathbb{R}$) defined by $\rho_A(f) := \sup\{f(x)|\; x \in A\}$ for all $f \in V^*$. Observe that it is essential that A is non-empty and bounded in order that the supremum exists in $\mathbb{R}$. By the Hahn-Banach Theorem points can be separated from closed convex sets; therefore every non-empty bounded convex set A can be characterized by linear inequalities; this gives $A = \{x \in V |\, \forall f \in V^*\, f(x) \leq \rho_A(f)\}$.

Proposition 3.1. *The following statements hold for all non-empty bounded closed convex subsets $A, B \subset V$:*

(i) $\quad A \subset B \iff \rho_A \leq \rho_B$.

(ii) $\quad \rho_{A \dot{+} B} = \rho_A + \rho_B$.

Proof. *(i)*: "⇒" is trivial. "⇐": If $\rho_A \leq \rho_B$, then for all $x \in A$ we get $f(x) \leq \rho_A(f) \leq \rho_B(f)$ for all $f \in V^*$, hence $x \in B$.

(ii): For all $x \in A$, $y \in B$ we have $f(x + y) = f(x) + f(y) \leq \rho_A(f) + \rho_B(f) = (\rho_A + \rho_B)(f)$, and by continuity of f we get $f(z) \leq (\rho_A + \rho_B)(f)$ for all $z \in A \dot{+} B$; this gives $\rho_{A \dot{+} B} \leq \rho_A + \rho_B$. On the other hand, for each $f \in V^*$, $\varepsilon > 0$ there are $x \in A$, $y \in B$ with $f(x) \geq \rho_A(f) - \varepsilon$, $f(y) \geq \rho_B(f) - \varepsilon$, thus $\rho_A(f) + \rho_B(f) \leq$

$f(x)+\varepsilon+f(y)+\varepsilon = f(x+y)+2\varepsilon \le \rho_{A\dot{+}B}(f)+2\varepsilon$, since $x+y \in A+B \subset A\dot{+}B$, proving our claim. $\square$

This leads to *Rådström's cancellation law*:

Proposition 3.2. *For all non-empty bounded closed convex sets $A, B, C \subset V$, the following assertions hold:*

(*i*) *If $A\dot{+}B \subset A\dot{+}C$, then $B \subset C$.*
(*ii*) *If $A\dot{+}B = A\dot{+}C$, then $B = C$.*

Proof. (*i*): $A\dot{+}B \subset A\dot{+}C \Rightarrow \rho_A+\rho_B = \rho_{A\dot{+}B} \le \rho_{A\dot{+}C} = \rho_A+\rho_C \Rightarrow \rho_B \le \rho_C \Rightarrow A \subset B$.

(*ii*) follows immediately from (*i*). $\square$

Here $A \neq \emptyset$ is essential, because $\emptyset + B = \emptyset = \emptyset + C$ holds for all $B, C \subset V$. Rådström's original proof also works in topological vector spaces which are not locally convex; our approach based on Hörmander needs local convexity in order to have enough continuous linear functionals for point separation. The above construction allows us to embed the additive monoid $E^+(V)$ of non-empty bounded closed convex sets in V into an abelian group $E(V)$, namely its *Grothendieck group*. The elements of $E(V)$ are all equivalence classes of pairs (A, B) of non-empty bounded closed convex sets, where the equivalence relation "$\sim$" is given by $(A, B) \sim (C, D) \iff A+D = B+C$. The sum of the equivalence classes of (A, B) and of (C, D) is the equivalence class of $(A+C, B+D)$; the zero element is the equivalence class of $(\{0\}, \{0\})$, and the inverse of the equivalence class of (A, B) is the equivalence class of (B, A). Moreover, we obtain an embedding $\iota : E^+(V) \to E(V)$ by $\iota(A)$ always is the equivalence class of $(A, 0)$; then the equivalence class of an arbitrary pair (A, B) is $\iota(A)-\iota(B)$. Pallaschke and Urbański [8] are particularly interested in *minimal pairs*, i.e. pairs (A, B) such that $A = A'$ and $B = B'$ follows for all pairs (A', B') with $(A', B') \sim (A, B)$, $A' \subset A$, $B' \subset B$.

For $\lambda \in \mathbb{R}^+$, we define $\lambda A := \{\lambda x; x \in A\}$ for all $A \in E^+(V)$ and $\lambda(\iota(A)-\iota(B)) := \iota(\lambda A)-\iota(\lambda B)$ and $-\lambda(\iota(A)-\iota(B)) := \lambda(\iota(B)-\iota(A))$ for all pairs (A, B) of non-empty bounded closed convex sets. Then we easily see that $E(V)$ is a real vector space under these operations as already observed by Rådström [10]. But note that the construction does not work if we define $-\iota(A) := \iota(-A)$ where $-A := \{-x | x \in A\}$, because then we do not get $\iota(A)+(-\iota(A)) = 0 := \iota(\{0\})$ in general. For $A, B \in E^+(V)$, there always exists a *join* $A \vee B$ in $E^+(V)$ (i.e. a least non-empty bounded closed convex set containing A and B), namely the closure of the convex hull of $A \cup B$.

A. G. Pinsker [9] proved $A+(B \vee C) = (A+B) \vee (A+C)$ for all $A, B, C \in E^+(V)$ and used this to introduce a lattice structure on $E(V)$. The order can be defined by $\iota(A)-\iota(B) \le \iota(C)-\iota(D) \iff A+D \subset B+C$; it follows from Proposition 3.2 that this gives a well defined order relation. $E(V)$ even becomes a vector lattice with binary joins $(\iota(A)-\iota(B)) \vee (\iota(C)-\iota(D)) = \iota((A+D) \vee (B+C))-\iota(B+D)$ and meets $(\iota(A)-\iota(B)) \wedge (\iota(C)-\iota(D)) = \iota(A+C)-\iota((A+D) \vee (B+C))$.

Theorem 3.3. *There exists a unique linear map ϕ from $E(V)$ to the space W of* *all positively homogenous maps* $V^* \to \mathbb{R}$, *such that* $\phi(\iota(A)) = \rho_A$ *holds for all* $A \in E^+(V)$; ϕ *is even an injective vector lattice homomorphism (where* W *carries the pointwise vector lattice structure). In particular, for* $A, B, C, D \in E^+(V)$ *we always have* $(A, B) \sim (C, D) \iff \rho_A - \rho_B = \rho_C - \rho_D$. *Moreover,* $\phi^{-1}(F(V))$ *is the vector lattice of all* $\iota(A) - \iota(B)$ *for polytopes* A, B; *thus* ϕ *restricts to a vector lattice isomorphism from this vector lattice to* $F(V)$.

Proof. The universal property of the Grothendieck group gives a unique additive homomorphism $\phi_n : E(V) \to W$ such that $\phi(\iota(A)) = \rho_A$ holds for all $A \in E^+(V)$. From the definition of the scalar multiplication we see that ϕ is linear. For all $A, B \in E^+(V)$ we have

$$\begin{aligned} \phi(\iota(A) - \iota(B)) = 0 \iff \rho_A - \rho_B = 0 &\iff \rho_A = \rho_B \\ \iff A = B &\iff \iota(A) - \iota(B) = 0. \end{aligned}$$

So the kernel of ϕ is $\{0\}$, hence ϕ is injective. In particular, for $A, B, C, D \in E^+(V)$ we have

$$\begin{aligned} &(A, B) \sim (C, D) && \iff \iota(A) - \iota(B) = \iota(C) - \iota(D) \\ \iff &\phi(\iota(A) - \iota(B)) = \phi(\iota(C) - \iota(D)) && \iff \rho_A - \rho_B = \rho_C - \rho_D. \end{aligned}$$

For $w \in W$, $\phi(w)$ belongs to $F(V)$ if and only if $\phi(f)$ is a difference of two subadditive polyhedral functionals by Theorem 2.2, i.e. of two finite joins of linear functionals (by Theorem 2.1), i.e. of two support functionals of polytopes. Thus we have $\phi(w) = \rho_A - \rho_B = \phi(\iota(A) - \iota(B))$, hence $w = \iota(A) - \iota(B)$ for some polytopes A, B. By routine inspection, ϕ is also a lattice homomorphism. □

Maybe for $A, B \in E(V)$ one should call $\rho_A - \rho_B$ the *support functional of the pair* (A, B).

Proposition 3.4.

(*i*) ι *preserves binary joins, i.e.* $\iota(A \vee B) = \iota(A) \vee \iota(B)$ *holds for all* $A, B \in E^+(V)$.

(*ii*) *For* $A, B \in E^+(V)$ *with* $A \cap B \neq \emptyset$, $\iota(A \cap B) = \iota(A) \wedge \iota(B)$ *holds if and only if* $A \cup B$ *is convex.*

Proof. (*i*) follows immediately from the construction.

(*ii*): Since $f \vee g + f \wedge g = f + g$ holds in every vector lattice, from (*i*) we get $\iota(A \vee B) + (\iota(A) \wedge \iota(B)) = (\iota(A) \vee \iota(B)) + (\iota(A) \wedge \iota(B)) = \iota(A) + \iota(B)$, hence $\iota(A \cap B) = \iota(A) \wedge \iota(B)$ is equivalent to $\iota(A \vee B) + \iota(A \cap B) = \iota(A) + \iota(B)$. But Urbański [12] showed that the latter holds if and only if $A \cup B$ is convex. □

The intersection of two intervals $A, B \subset \mathbb{R}$ is an interval and thus convex whenever $A \cap B \neq \emptyset$; so for $V = \mathbb{R}$, ι preserves all existing meets. But it is easy to find two compact convex sets (even polytopes) in $\mathbb{R}^2$ with non-empty intersection and non-convex union; therefore ι does not preserve binary meets for $V = \mathbb{R}^2$. This leads to counterexamples to the analogue $A + (B \cap C) = (A + B) \cap (A + C)$ of Pinsker's formula, even with $B \cap C \neq \emptyset$. Moreover, note that for every V and for all non-empty bounded closed convex sets $B, C \subset V$ (possibly with $B \cap C = \emptyset$) there even exists a polytope A with $(A + B) \cap (A + C) \neq \emptyset$.

4. Virtual Polytopes

Now we shall study formal Minkowski differences (i.e. equivalence classes of pairs) of polytopes; G. Ewald [5] calls them *virtual polytopes*. Here we define a polytope in V as a convex hull of an non-empty finite set; in particular, every polytope is non-empty and finite-dimensional, i.e. contained in a finite-dimensional subspace of V. By definition, polytopes in V are non-empty finite unions of single points in $E(V)$, thus also in $E^+(V)$ by Proposition 3.4. In particular, the join of two polytopes is also a polytope. For $A, B \in E^+(V)$ with $A \cap B \neq \emptyset$, it follows that $A \cap B$ is a meet of A and B in $E^+(V)$, i.e. a largest non-empty bounded closed convex set contained in both A and B; if A and B are polytopes, then so is $A \cap B$. For $A, B \in E^+(V)$ with $A \cap B = \emptyset$, A and B have no common lower bound and thus no meet in $E^+(V)$. For a finite-dimensional Hausdorff space V, polytopes are the same as non-empty bounded polyhedra, where a polyhedron is a finite intersection of affine half-spaces, i.e. a set defined by finitely many linear non-strict linear inequalities. But in an infinite-dimensional Hausdorff locally convex space no finite intersection of half-spaces (closed or not) is bounded.

The support functional of a singleton $\{x\}$ (where $x \in V$) is the *evaluation map* $V^* \to \mathbb{R}$, $f \mapsto f(x)$; it is well-known that point evaluations coincide with the linear functionals $V^* \to \mathbb{R}$ which are continuous in the weak*-topology. Thus from Theorems 2.1 and 2.2 we get the following

Proposition 4.1.

(*i*) *A functional on V^* is a support functional of a polytope if and only if it is subadditive and polyhedral in the weak*-topology.*

(*ii*) *A functional on V^* is a difference of two support functionals of polytopes if and only if it is polyhedral in the weak*-topology.* □

For a set T we denote the free vector space on T by $\mathbb{R}^{(T)}$; it has a basis indexed by T. We endow $\mathbb{R}^{(T)}$ with the finest locally convex topology; then all linear functionals on $\mathbb{R}^{(X)}$ are continuous. For every $(x_t)_{t\in T} \in \mathbb{R}^T$ there is a unique linear functional that maps the t-th basis element to x_t; this gives a the canonical vector space isomorphism $\mathbb{R}^T \to (\mathbb{R}^{(T)})^*$; this is even a homeomorphism, where $\mathbb{R}^T$ carries the product topology and $(\mathbb{R}^{(T)})^*$ carries the weak*-topology. Now Proposition 4.1 together with Corollary 2.3 gives the following

Theorem 4.2. *The free vector lattice $F(T)$ on a set T is canonically isomorphic to the subspace of $E(\mathbb{R}^{(T)})$ consisting of all differences of support functionals of polytopes.*

Example 4.3. If V is an irreflexive Banach space, then there exists no topology $\mathcal{T}$ on V^* such that the support functionals of non-empty bounded closed convex sets are exactly those sublinear functionals on V^* which are continuous in $\mathcal{T}$. Indeed, in such a topology the operator norm on V norm must be continuous because it is the support functional of the unit ball of V, which is obviously non-empty, bounded, closed and convex. But then it easily follows that $\mathcal{T}$ must be at least as fine as the topology induced by the operator norm, and every functional which is continuous in the operator norm is also continuous in $\mathcal{T}$. But since V is irreflexive, there exists an element $\theta \in V^{**}$ in the bidual, which is not a point evaluation. Now $\theta : V^* \to \mathbb{R}$ is linear and thus in particular sublinear. Moreover θ is continuous under the operator norm and therefore in $\mathcal{T}$. On the other hand, it is easy to see that θ is not a support functional of a non-empty bounded closed convex subset of V.

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