Journal of Convex Analysis
Volume 18 (2011), No. 2, 513–528

Slices in the Unit Ball of the Symmetric Tensor Product of a Banach Space

María D. Acosta*
Universidad de Granada, Facultad de Ciencias,
Departamento de Análisis Matemático, 18071-Granada, Spain
dacosta@ugr.es

Julio Becerra Guerrero†
Universidad de Granada, Facultad de Ciencias,
Departamento de Matemática Aplicada, 18071-Granada, Spain
juliobg@ugr.es

Received: January 9, 2009
Revised manuscript received: September 27, 2009

We prove that every infinite-dimensional C^*-algebra X satisfies that every slice of the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$ (N-fold projective symmetric tensor product of X) has diameter two. We deduce that every infinite-dimensional Banach space X whose dual is an L_1-space satisfies the same result. As a consequence, if X is either a C^*-algebra or either a predual of an L_1-space, then the space of all N-homogeneous polynomials on X, $\mathcal{P}^N(X)$, is extremely rough, whenever X is infinite-dimensional. If Y is a predual of a von Neumann algebra, then Y is infinite-dimensional if, and only if, every w^*-slice of the unit ball of $\mathcal{P}_I^N(Y)$ (the space of integral N-homogeneous polynomials on Y) has diameter two. As a consequence, under the previous assumptions, the N-fold symmetric injective tensor product of Y is extremely rough. Indeed, this isometric condition characterizes infinite-dimensional spaces in the class of preduals of von Neumann algebras.

Keywords: Banach spaces, slice, homogeneous polynomial, integral polynomial, symmetric projective tensor product, symmetric injective tensor product, C*-algebra

1991 Mathematics Subject Classification: Primary 46B20, Secondary 46B25, 46B28

1. Introduction

Since many of the classical Banach spaces do not satisfy the Radon-Nikodým property, the characterization of such property in terms of slices gives us that some bounded, closed and convex set in such a Banach space do not have arbitrarily small slices. Schachermayer, Sersouri and Werner [31] introduced a modulus of non-dentability. For a (non empty) bounded, closed and convex subset C of a Banach space, its modulus of non-dentability, denoted by $\delta_1(C)$, is given by $\delta_1(C) := \inf\{\operatorname{diam} S : S \text{ is a slice of } C\}$. A well-known characterization states that a Banach space X sat-

*The first named author was supported by MEC projects MTM2006-04837, MTM-2009-07498 and Junta de Andalucía "Proyectos de Excelencia" FQM-01438 and P09-FQM-4911.
†The second named author was partially supported by Junta de Andalucía grants FQM 0199 and FQM 1215, and MTM-2006-15546-C02-02.

ISSN 0944-6532 / $ 2.50

isfies the Radon-Nikodým property if, and only if, every closed, convex and bounded subset C of X satisfies that $\delta_1(C) = 0$. Schachermayer, Sersouri and Werner proved that for every Banach space X not satisfying the Radon-Nikodým property and for each $\varepsilon > 0$, there is a subset C of X with $\operatorname{diam} C = 1$ and $\delta_1(C) > 1 - \varepsilon$.

For some classical Banach spaces X (without the Radon-Nikodým property), it holds that the modulus of non-dentability of the unit ball is two. The above phenomena happens in the following cases:

- X is an infinite-dimensional uniform algebra [25].
- X is a space satisfying the Daugavet property [32].
- $C(K, X)$, where K is an infinite Hausdorff and compact topological space and X is any nontrivial Banach space [7, 26].
- $L_1(\mu, X)$, when μ is an atomless measure and X is a nontrivial Banach space [7].
- X is any infinite-dimensional C*-algebra [9, 8].
- Some L-embedded and M-embedded Banach spaces under some additional assumptions [24].
- $\widehat{\bigotimes}_{N,s,\pi} C(K)$, the N-fold symmetric projective tensor product of $C(K)$, in the case that K is an infinite compact and Hausdorff topological space) [1].
- $\widehat{\bigotimes}_{N,s,\pi} L_1(\mu)$, the N-fold symmetric projective tensor product of $L_1(\mu)$, if μ is a σ-finite and atomless measure [1].
- The interpolation spaces $L_1(\mathbb{R}^+) + L_\infty(\mathbb{R}^+)$ (endowed with two natural norms) and $L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$ (endowed with the maximum norm) [2].

In this paper we obtain new results along the same line for the N-fold symmetric projective and injective tensor product of certain Banach spaces with a good algebraic structure. Let us point out that there are just a few results on isometric properties of the symmetric tensor product of Banach spaces (see [29], [10], [4], [19] and [1]). Let us also notice that for a finite-dimensional normed space, the modulus of non-dentability of the unit ball is zero.

Now we will list the results that we obtained in this paper. In the first section we give a quantitative version of Smulyan test of Fréchet differentiability for $\mathcal{P}^N(X)$, by recalling the modulus of roughness of an element of a Banach space. As a consequence, we give a criteria in terms of X for a polynomial $P \in \mathcal{P}^N(X)$ to have a certain modulus of roughness. In Section 3 we will apply this criteria to deduce that for $X = \mathcal{C}(K)$ (K infinite) the modulus of roughness of every polynomial is 2 and so, $\mathcal{P}^N(X)$ has no points of Fréchet differentiability. Indeed, the same result is true for every infinite predual of an L_1-space. Let us recall that Boyd and Ryan proved that the space $\mathcal{P}^N(X)$ is never smooth, for every Banach space X with dimension greater or equal to 2 and $N \geq 2$ [10, Proposition 17] (see also [29, Corollary 7]).

By localizing the result for $\mathcal{C}(K)$, we will prove in Section 4 that for a JB^*-triple X, if $\mathcal{P}^N(X)$ is not extremely rough, then X is isomorphic (as a Banach space) to a Hilbert space (Theorem 4.2). As a consequence, for every infinite dimensional C^*-algebra X, the modulus of roughness of every N-homogeneous polynomial on X is 2. That is, every slice of the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$ have diameter two.

The last section of the paper contains results of the same kind for the dual of the symmetric injective tensor product of a Banach space X, $\widehat{\bigotimes}_{N,s,\varepsilon}X$. It follows from known results that for every normalized element x in X, if the norm of X is Fréchet differentiable at x, then the symmetric injective tensor norm of $\widehat{\bigotimes}_{N,s,\varepsilon}X$ is Fréchet differentiable at $x\otimes \overset{N}{\ldots} \otimes x$ (see [10, Theorem 11] and [13, Corollary I.1.5]). Hence, for every space X isomorphic to a Hilbert space, the unit sphere of $\widehat{\bigotimes}_{N,s,\varepsilon}X$ has points of Fréchet differentiability of the norm.

In this section we provide a class of Banach spaces for which the converse holds true (Corollary 5.2). As a consequence, if X is a Banach space whose dual is an infinite-dimensional von Neumann algebra, and $N \in \mathbb{N}$, the space $\widehat{\bigotimes}_{N,s,\varepsilon}X$ is extremely rough. That is, every w^*-slice of the space of the integral N-homogeneous polynomials on X, endowed with the usual norm has diameter two. Indeed, this isometric condition characterizes the infinite-dimensional spaces belonging to this class.

We notice that for Banach spaces such that every slice of the unit ball has diameter two, there are not weak-norm points of continuity (of the identity mapping) on the unit ball.

2. Notation and general results

Throughout the paper, X will be a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and X^* will be its topological dual. We will denote by S_X and B_X the unit sphere and the closed unit ball of X, respectively.

For a Banach space X and $N \in \mathbb{N}$, we will consider the *symmetric projective N-tensor product* $\widehat{\bigotimes}_{N,s,\pi}X := X\widehat{\otimes}_{\pi,s} \overset{N}{\ldots} \widehat{\otimes}_{\pi,s}X$. This space is the completion of the linear space generated by $\{x\otimes \overset{N}{\ldots} \otimes x : x \in X\}$ under the norm given by

$$\|z\| = \inf\left\{\sum_{i=1}^{m} |\lambda_i| : z = \sum_{i=1}^{m} \lambda_i x_i\otimes \overset{N}{\ldots} \otimes x_i, m \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i \in S_X, \ \forall 1 \le i \le m\right\}.$$

Its topological dual can be identified with the space of all N-homogeneous (and bounded) polynomials on X, denoted by $\mathcal{P}^N(X)$. Every polynomial $P \in \mathcal{P}^N(X)$ acts as a linear functional $\hat{P}$ on the N-fold symmetric tensor product and the identification is given by $P(x) = \hat{P}(x\otimes \overset{N}{\ldots} \otimes x)$ for every element $x \in X$.

The dual norm of the symmetric projective tensor product on $\mathcal{P}^N(X)$ is the usual *polynomial norm*, given by

$$\|P\| = \sup\{|P(x)| : x \in X, \|x\| \le 1\} \ \ (P \in \mathcal{P}^N(X)).$$

For a Banach space X, given an element $u \in S_X$, we recall that the *modulus of roughness* at u is given by

$$\eta(X,u) := \inf_{\delta>0}\left\{\sup\left\{\frac{\|u+h\| + \|u-h\| - 2}{\|h\|} : h \in X, \|h\| \le \delta\right\}\right\}.$$

Let us notice that $0 \leq \eta(X,u) \leq 2$. The norm of X is Fréchet differentiable at u if, and only if, $\eta(X,u) = 0$ (see [13, Lemma I.1.3]). We say that X is *extremely rough* whenever $\eta(X,u) = 2$ for every $u \in S_X$ ([13, Section I.1]).

Also, for an element $x^* \in X^*$, a slice of a (nonempty) bounded subset $A \subset X$ is given by

$$S(A, x^*, \alpha) := \{x \in A : \operatorname{Re} x^*(x) > \sup_{a \in A} \operatorname{Re} x^*(a) - \alpha\},$$

where $0 < \alpha < \sup \operatorname{Re} x^*(A)$. If $X = Y^*$ for some Banach space Y, $A \subset Y^*$ and $y \in Y$, then we will say that the slice

$$S(A, y, \alpha) := \{z^* \in A : \operatorname{Re} z^*(y) > \sup_{y^* \in A} \operatorname{Re} y^*(y) - \alpha\},$$

is a w^*-slice of A.

The next result is a local quantitative version of the result due to Smulyan stating that the norm of the dual space is Fréchet differentiable at a point x^* if, and only if, x^* strongly exposes the unit ball of X. We recall that an element $x^* \in S_{X^*}$ strongly exposes B_X if any sequence (x_n) in B_X satisfying $(x^*(x_n)) \to 1$ converges in the norm topology.

Lemma 2.1 ([13, Proposition I.1.11]). *Let $x \in S_X$ and $x^* \in S_{X^*}$, then it is satisfied*

$$\eta(X^*, x^*) = \inf\{\operatorname{diam} S(B_X, x^*, \alpha) : 0 < \alpha < 1\},$$

and

$$\eta(X, x) = \inf\{\operatorname{diam} S(B_{X^*}, x, \alpha) : 0 < \alpha < 1\}.$$

We will check that the modulus of roughness can be computed by using slices of a set whose convex hull is dense in the unit ball.

Proposition 2.2. *Assume that $B_X = \overline{\operatorname{co}(A)}$, for some set $A \subset X$ and let $x^* \in S_{X^*}$. Then*

$$\inf_{\alpha > 0} \operatorname{diam} S(B_X, x^*, \alpha) = \inf_{\alpha > 0} \operatorname{diam} S(A, x^*, \alpha).$$

Proof. We can clearly assume that X is a real Banach space. Of course, it holds

$$\operatorname{diam} S(B_X, x^*, \alpha) \geq \operatorname{diam} S(A, x^*, \alpha)$$

and so it is satisfied

$$\inf_{\alpha > 0} \operatorname{diam} S(B_X, x^*, \alpha) \geq \inf_{\alpha > 0} \operatorname{diam} S(A, x^*, \alpha).$$

For the reversed inequality, let us notice that for every $0 < \varepsilon < \frac{1}{2}$ and each $x \in S(B_X, x^*, \varepsilon^2)$, since the closed convex hull of A is dense in the unit ball, we know that there is a convex combination $\sum_{i=1}^n t_i a_i$ of elements $\{a_i : 1 \leq i \leq n\}$ in A satisfying

$$\left\| x - \sum_{i=1}^n t_i a_i \right\| < \varepsilon^2,$$

and so

$$1-\varepsilon^2 \leq x^*(x) \leq \varepsilon^2 + x^*\left(\sum_{i=1}^{n} t_i a_i\right)$$

Let denote by $P := \{i : x^*(a_i) \leq 1-\varepsilon\}$. Hence we obtain that

$$\begin{aligned} 1-\varepsilon^2 &\leq \varepsilon^2 + \sum_{i\in P} t_i(1-\varepsilon) + \sum_{i\notin P} t_i \\ &\leq 1+\varepsilon^2 - \varepsilon \sum_{i\in P} t_i. \end{aligned}$$

From the last inequality we obtain that

$$\sum_{i\in P} t_i \leq 2\varepsilon.$$

We deduce that

$$\left\| x - \sum_{i\notin P} t_i a_i \right\| \leq \left\| x - \sum_{i=1}^{n} t_i a_i \right\| + \left\| \sum_{i\in P} t_i a_i \right\| \leq \varepsilon^2 + 2\varepsilon \leq 3\varepsilon.$$

Since $\sum_{i\notin P} t_i \geq 1-2\varepsilon > 0$, we have

$$\frac{1}{\sum_{i\notin P} t_i} \left\| x - \sum_{i\notin P} t_i a_i \right\| \leq \frac{3\varepsilon}{1-2\varepsilon}.$$

It follows that

$$\begin{aligned} \left\| x - \frac{1}{\sum_{i\notin P} t_i} \sum_{i\notin P} t_i a_i \right\| &\leq \left\| x - \frac{1}{\sum_{i\notin P} t_i} x \right\| + \frac{1}{\sum_{i\notin P} t_i} \left\| x - \sum_{i\notin P} t_i a_i \right\| \\ &\leq \frac{2\varepsilon}{1-2\varepsilon} + \frac{3\varepsilon}{1-2\varepsilon} \\ &= \frac{5\varepsilon}{1-2\varepsilon}. \end{aligned}$$

We showed that, given $x \in S(B_X, x^*, \varepsilon^2)$, there is an element $y \in \mathrm{co}(S(A, x^*, \varepsilon))$ with $\|y-x\| \leq \frac{5\varepsilon}{1-2\varepsilon}$.

Hence

$$\begin{aligned} \operatorname{diam} S(A, x^*, \varepsilon) &= \operatorname{diam} \mathrm{co}(S(A, x^*, \varepsilon)) \\ &\geq \operatorname{diam} S(B_X, x^*, \varepsilon^2) - \frac{10\varepsilon}{1-2\varepsilon} \\ &\geq \inf_{\alpha>0} \operatorname{diam} S(B_X, x^*, \alpha) - \frac{10\varepsilon}{1-2\varepsilon}. \end{aligned}$$

Since the function $\varepsilon \mapsto \operatorname{diam} S(A, x^*, \varepsilon)$ is increasing, by taking limit when $\varepsilon \to 0$, we deduce the inequality

$$\inf_{\alpha>0} \operatorname{diam} S(A, x^*, \alpha) \geq \inf_{\alpha>0} \operatorname{diam} S(B_X, x^*, \alpha).$$

$\square$

By the definition of the norm on the symmetric projective N-tensor product of a Banach space X, the subset A given by

$$A = \{tx \otimes \overset{N}{\cdots} \otimes x : t \in \{+1, -1\}, x \in S_X\},$$

satisfies that its closed convex hull is dense in the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$. By using Lemma 2.1 and Proposition 2.2, we obtain the following quantitative version of [17, Theorem 2.4].

Proposition 2.3. *For $P \in \mathcal{P}^N(X)$ satisfying $\|P\| = 1$ and $\varepsilon > 0$, the following conditions are equivalent:*

i) $\eta(\mathcal{P}^N(X), P) < \varepsilon$

ii) *For every sequences $\{x_n\}, \{y_n\}$ in the unit ball of X with $\lim_n \{P(x_n)\} = s\|P\|$, $\lim_n \{P(y_n)\} = t\|P\|$, where $s, t \in \{+1, -1\}$, we get that*

$$\limsup\{|sQ(x_n) - tQ(y_n)|\} < \varepsilon$$

uniformly for Q in the unit ball of $\mathcal{P}^N(X)$.

Let us observe that

$$\sup\{|sQ(x) - tQ(y)| : Q \in \mathcal{P}^N(X), \|Q\| = 1\} = \|s(x \otimes \overset{N}{\cdots} \otimes x) - t(y \otimes \overset{N}{\cdots} \otimes y)\|,$$

and so the second condition says that the diameter of the slices determined by the functional associated to P on $\widehat{\bigotimes}_{N,s,\pi} X$ is less than ε.

In case that we can apply the previous result for every positive number, taking into account that $\eta(X, x) = 0$ iff x is a point of Fréchet differentiability of the norm, we obtain the result proved by Ferrera (see [17, Theorem 2.4 and Proposition 2.3] and [10, Theorem 14]).

Corollary 2.4. *For $P \in \mathcal{P}^N(X)$, it is satisfied that P is a point of Fréchet differentiability of the norm iff P determines slices of arbitrarily small diameter iff*

$$\forall \{x_n\}, \{y_n\} \subset S_X : \lim\{P(x_n)\} = s\|P\|, \lim_n \{P(y_n)\} = t\|P\|, \text{ where } s, t \in \{+1, -1\}$$
$$\Rightarrow \lim\{|sQ(x_n) - tQ(y_n)|\} = 0 \text{ uniformly on } Q \in B_{\mathcal{P}^N(X)}.$$

In the following result we obtain the largest possible modulus of roughness under certain assumptions that are satisfied by some classical spaces, as we will see later.

Proposition 2.5. *Let X be a Banach space and $P \in S_{\mathcal{P}^N(X)}$ with $1 = \sup_{x \in B_X} \operatorname{Re} P(x)$. Assume that for every $\alpha > 0$ and for every functional $x^* \in S_{X^*}$ such that*

$$S := (\{x \in B_X : \operatorname{Re} P(x) > 1 - \alpha\} \cap S(B_X, x^*, \alpha\}) \neq \emptyset,$$

then $\operatorname{diam} S = 2$. *It holds $\eta(\mathcal{P}^N(X), P) = 2$.*

Proof. For every n, we denote by

$$S_n(P) := \left\{x \in B_X : \operatorname{Re} P(x) > 1 - \frac{1}{n}\right\}.$$

If $u_n \in S_X \cap S_n(P)$ and u_n^* is a norm one functional satisfying $u_n^*(u_n) = 1$, then it is clear that $u_n \in S_n(P) \cap S(B_X, u_n^*, \frac{1}{n})$. Because of the assumption, we know that the previous subset has diameter two. Hence, we can find sequences $\{x_n\}, \{y_n\}$ in S_X and $\{v_n^*\}$ in S_{X^*} satisfying

i) $\lim\{\|x_n - y_n\|\} = 2$.
ii) $\lim\{\mathrm{Re}\ P(x_n)\} = 1 = \lim\{\mathrm{Re}\ P(y_n)\}$.
iii) $\lim\{u_n^*(x_n)\} = 1 = \lim\{u_n^*(y_n)\}$.
iv) $\lim\{v_n^*(x_n)\} = 1 = -\lim\{v_n^*(y_n)\}$.

If N is odd and n is a natural number, we consider the N-homogeneous polynomial given by

$$Q_n = (v_n^*)^N,$$

which is clearly an N-homogeneous polynomial on X. Because of condition iv) it is satisfied $\lim\{\|Q_n\|\} = 1$ and also

$$\lim\left\{\frac{Q_n}{\|Q_n\|}(x_n) - \frac{Q_n}{\|Q_n\|}(y_n)\right\} = 2.$$

If N is even, we consider the polynomial $Q_n = (v_n^*)^{N-1}u_n^*$. By iii) and iv), $\lim\{\|Q_n\|\} = 1$ and we also obtain that

$$\lim\left\{\frac{Q_n}{\|Q_n\|}(x_n) - \frac{Q_n}{\|Q_n\|}(y_n)\right\} = 2.$$

In view of Proposition 2.3, we obtain that $\eta(\mathcal{P}^N(X), P) = 2$. □

3. Results for $\mathcal{C}(K)$

Finite-dimensional spaces have the Radon-Nikodým property. Hence the unit ball of a finite-dimensional space has slices of arbitrarily small diameter. In this section we will show that the symmetric projective tensor product of an isometric predual of an L_1-space is far from satisfying the above condition in the infinite-dimensional case. In order to prove such result we will show first a stronger property for $\mathcal{C}(K)$. To this purpose we will use the following technical result that follows from the Urysohn Lemma.

Lemma 3.1 ([1, Lemma 2.1]). *Let K be a compact and Hausdorff infinite topological space. Then there are two sequences of non-empty open sets $\{V_n\}$ and $\{U_n\}$ satisfying that*

$$\overline{V_n} \subset U_n, \quad U_n \cap U_m = \emptyset \quad (n \neq m),$$

and two sequences of functions $\{g_n\}$ and $\{h_n\}$ in $\mathcal{C}(K)$ satisfying that

$$\{g_n\} \xrightarrow{w} 0, \qquad \{h_n\} \xrightarrow{w} 0,$$

and also

$$0 \leq g_n, h_n \leq 1, \quad \mathrm{supp}\ \ h_n \subset V_n, \quad \|h_n\|_\infty = 1, \quad \mathrm{supp}\ \ g_n \subset U_n, \quad g_n(V_n) = \{1\}.$$

Proposition 3.2. *Let K be an infinite compact Hausdorff topological space and $X := \mathcal{C}(K)$ (either real or complex valued) functions. Given an N-homogeneous polynomial P on X with $\|P\| = 1$, $\alpha > 0$ and $x^* \in S_{X^*}$ such that*

$$S := (\{x \in B_X : \mathrm{Re}\ P(x) > 1 - \alpha\} \cap S(B_X, x^*, \alpha\}) \neq \emptyset,$$

we have that $\operatorname{diam} S = 2$.

Proof. By assumption, since P is N-homogeneous, then there is an element f in $S \cap S_X$. By Lemma 3.1, there are sequences of functions $\{g_n\}$ and $\{h_n\}$ in $\mathcal{C}(K)$ satisfying that

$$\{g_n\} \xrightarrow{w} 0, \qquad \{h_n\} \xrightarrow{w} 0$$

and sequences of pairwise disjoint open sets, $\{V_n\}$ and $\{U_n\}$, satisfying

$$0 \leq g_n, h_n \leq 1, \ \ \text{supp } h_n \subset V_n, \ \ \|h_n\|_\infty = 1, \ \ \text{supp } g_n \subset U_n, \ \ g_n(V_n) = \{1\}. \quad (1)$$

Since $\{g_n\}$ and $\{h_n\}$ converges weakly to zero, then the sequences

$$\{u_n\} = \{f(1 - g_n) + h_n\}, \qquad \{v_n\} = \{f(1 - g_n) - h_n\}$$

converges weakly to f. Also, both sequences are in the unit ball of the space. We check this assertion. If $n \in \mathbb{N}$ and $t \in K$, by using conditions (1), then depending on the fact that $t \in V_n$ or $t \notin V_n$, one of the two following cases holds

$$|(f(1 - g_n) \pm h_n)(t)| = |h_n(t)|, \qquad |(f(1 - g_n) \pm h_n)(t)| = |f(1 - g_n)(t)|,$$

and so, both sequences $\{u_n\}$ and $\{v_n\}$ are in the unit ball of $\mathcal{C}(K)$. Finally, by using that the space $\mathcal{C}(K)$ has the Dunford-Pettis property and, by [28, Theorem 2.1], it has the polynomial Dunford-Pettis property, that is, polynomials on $\mathcal{C}(K)$ preserve weak convergence of sequences, we obtain that

$$\{P(u_n)\} \to P(f), \qquad \{P(v_n)\} \to P(f).$$

Hence, for n large enough, then u_n, v_n satisfy that

$$u_n, v_n \in \{x \in B_X : \mathrm{Re}\ P(x) > 1 - \alpha\} \cap S(B_X, x^*, \alpha\}$$

Now let us note that

$$\|u_n - v_n\|_\infty = \|2h_n\|_\infty = 2,$$

and so $\operatorname{diam} S = 2$. □

In view of Proposition 2.5, we deduce that $\eta(\mathcal{P}^N(\mathcal{C}(K)), P) = 2$ for every N-homogeneous polynomial P on $\mathcal{C}(K)$, whenever K is infinite ([1, Corollary 2.3]).

Proposition 3.3. *Let X be an infinite-dimensional L_1 predual. Given an N-homogeneous polynomial P on X satisfying $\|P\| = 1$, if we assume that $\alpha > 0$, $x^* \in X^*$ and*

$$S := (\{x \in B_X : \mathrm{Re}\ P(x) > 1 - \alpha\} \cap S(B_X, x^*, \alpha\}) \neq \emptyset,$$

then $\operatorname{diam} S = 2$.

Proof. We can clearly assume that X is a real Banach space. Since S is non-empty and P an N-homogeneous polynomial, we can choose $x \in S \cap S_X$. By [3], every homogeneous polynomial P on X can be extended to a polynomial on its bidual. We denote by $\widetilde{P}$ this canonical extension that also satisfies $\|\widetilde{P}\| = 1$. We consider the set S^{**} given by

$$S^{**} := \{x^{**} \in B_{X^{**}} : \widetilde{P}(x^{**}) > 1 - \alpha\} \cap S(B_{X^{**}}, x^*, \alpha\}.$$

Since $x \in S$, then $x \in S^{**}$ and so S^{**} is non empty. Under the assumptions, X^{**} is isometrically isomorphic to $\mathcal{C}(K)$ for suitable infinite compact topological space K. By using Proposition 3.2, given $\varepsilon > 0$, there exist $x^{**}, y^{**} \in S^{**}$ such that $\|x^{**} - y^{**}\| > 2 - \varepsilon$. By [12, Theorem 2], there are nets $\{x_\alpha\}$ and $\{y_\alpha\}$ in B_X satisfying that

$$\{Q(x_\alpha)\} \to \widetilde{Q}(x^{**}), \qquad \{Q(y_\alpha)\} \to \widetilde{Q}(y^{**}), \tag{2}$$

for every k-homogeneous polynomial Q on X with $k \leq N$. In view of the w^*-lower semicontinuity of the norm of X^{**} we have

$$\liminf \|x_\alpha - y_\alpha\| \geq \|x^{**} - y^{**}\|.$$

By condition (2), for α large enough, x_α, y_α belongs to S^{**} and hence in S. We conclude that

$$\operatorname{diam} S \geq \liminf \|x_\alpha - y_\alpha\| > 2 - \varepsilon,$$

for every $\varepsilon > 0$ and so $\operatorname{diam} S = 2$. □

In view of Proposition 2.5 we deduce the following:

Corollary 3.4. *Let X be an infinite-dimensional L_1 predual and P be an N-homogeneous polynomial on X. Then $\eta(\mathcal{P}^N(X), P) = 2$.*

Given a locally compact Hausdorff topological space Ω, we denote by $\mathcal{C}_0(\Omega)$ the Banach space of all scalar (either real or complex) valued continuous functions on Ω vanishing at infinity.

Corollary 3.5. *Let Ω be an infinite locally compact Hausdorff topological space and P be any N-homogeneous polynomial on $\mathcal{C}_0(\Omega)$. Then $\eta(\mathcal{P}^N(\mathcal{C}_0(\Omega)), P) = 2$.*

4. Results for JB*-triples

We recall that a *JB*-triple* [23, 21] is a complex Banach space J with a continuous triple product $\{\ \} : J \times J \times J \to J$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in J, the mapping $y \to \{xxy\}$ from J to J is a hermitian operator on J and has nonnegative spectrum.
2. The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in J.

3. $\|\{xxx\}\| = \|x\|^3$ for every x in J.

JB^*-triples are of capital importance in the study of bounded symmetric domains in complex Banach spaces. Indeed, open balls in JB^*-triples are bounded symmetric domains, and every symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB^*-triple (see [22]).

Several classical spaces are JB^*-triples. For instance, every C^*-algebra and the space of operators between two complex Hilbert spaces are JB^*-triples, endowed with their usual norm and the triple product given by

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

As a consequence, a complex Hilbert space (usual norm) belongs to this class. There are more JB*-triples isomorphic to Hilbert spaces, the so-called spin factors. These are constructed from an arbitrary complex Hilbert space $(H, (\cdot|\cdot))$ of dimension ≥ 3, by taking a conjugate-linear involutive isometry σ on H, and then by defining the triple product and the norm by the formulas

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y) \quad (x, y, z \in H)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2} \quad (x \in H).$$

A *subtriple* of a JB^*-triple J is a vector subspace of J invariant under the triple product. An element $x \in J$ is called *algebraic* if the subtriple of J generated by x is finite-dimensional. In the case that every element in J is algebraic, then the triple J is called algebraic.

The following result shows that two properties of very different nature, one algebraic and one isometric, are related.

Proposition 4.1. *Let X be a complex JB^*-triple and assume that X is not algebraic. If for some normalized N-homogeneous polynomial P on X, $\alpha > 0$ and $x^* \in S_{X^*}$ the set*

$$S := (\{x \in B_X : \operatorname{Re} P(x) > 1 - \alpha\} \cap S(B_X, x^*, \alpha\})$$

is non-empty, then $\operatorname{diam} S = 2$.

Proof. Since S has non-empty norm interior in X and we are assuming that X is not algebraic, S contains an element x that is not algebraic [9, Lemma 2.3]. Denote by J_x the closed subtriple of X generated by x. It is known that there is a unique locally compact subset S_x of $]0, \infty[$ and a surjective triple isomorphism $\phi_x : J_x \to C_0^{\mathbb{C}}(S_x)$ such that $S_x \cup \{0\}$ is compact and $\phi_x(x)$ is the inclusion mapping $S_x \to \mathbb{C}$ (see [18], [21, 4.8] and [22, 1.15]).

Since surjective triple isomorphisms between JB^*-triples are isometries (see again [22]), we can identify J_x and $C_0^{\mathbb{C}}(S_x)$ as Banach spaces. By using that $x \in S$, it is clear that there exists $\beta > 0$ which the following property

$$\{y \in B_{J_x} : \operatorname{Re} P(y) > \|P|J_x\| - \beta\} \cap S(B_{J_x}, x^*{}_{|J_x}, \beta\} \subseteq S.$$

On the other hand, since x is not an algebraic element, J_x is infinite dimensional and by Proposition 3.3, we have

$$\operatorname{diam}(\{y \in B_{J_x} : \operatorname{Re} P(y) > \|P|J_x\| - \beta\} \cap S(B_{J_x}, x^*, \beta)) = 2.$$

We conclude that $\operatorname{diam} S = 2$. □

Assume that X is a Banach space with the Radon-Nikodým property. Then for every $\varepsilon > 0$ there is a functional $x^* \in S_{X^*}$ such that some slice $S(B_X, x^*, \alpha)$ does have diameter less than ε. If N is odd, then $P := (x^*)^N$ is an N-homogeneous polynomial on X such that the diameter of $\{x \in B_X : \operatorname{Re} P(x) > 1 - \alpha\}$ is less than ε. Hence, by using polarization formula we deduce that

$$\operatorname{diam} S\left(B_{\widehat{\bigotimes}_{N,s,\pi} X}, P, \frac{\varepsilon}{2\alpha}\right) \leq \left(1 + \frac{N^{N+1}}{N!}\right) \varepsilon$$

and so $\mathcal{P}^N(X)$ is not extremely rough. As we mentioned before, there are JB^*-triples with the Radon-Nikodým property (indeed isomorphic to Hilbert spaces as Banach spaces) and so $\mathcal{P}^N(X)$ is not extremely rough in these cases. We will show that this is the only exception in this class.

Theorem 4.2. *Let X be a complex JB^*-triple and $N \in \mathbb{N}$. The following assertions are equivalent:*

i) *X is isomorphic (as a Banach space) to a Hilbert space.*
ii) *The norm of $\mathcal{P}^N(X)$ has some points of Fréchet differentiability.*
iii) *$\mathcal{P}^N(X)$ is not extremely rough.*
iv) *There exists a slice of the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$ with diameter less than 2.*

Proof. *i)* $\Rightarrow$ *ii)* A Hilbert space has the Radon-Nikodým property. Hence, by the result of J. Ferrera [17, Theorems 2.4 and 2.6], the points of Fréchet differentiability of the norm in $S_{\mathcal{P}^N(X)}$ is dense.

ii) $\Rightarrow$ *iii)* Trivial (for any Banach space).

iii) $\Rightarrow$ *iv)* It suffices to use Lemma 2.1.

iv) $\Rightarrow$ *i)* By the assumption and Lemma 2.1, there is a normalized N-homogeneous polynomial P on X with $\eta(\mathcal{P}^N(X), P) < 2$. Then by Proposition 2.5, there is $\alpha, x^* \in S_{X^*}$ with

$$0 < \operatorname{diam}\left(\{x \in B_X : \operatorname{Re} P(x) > \|P\| - \alpha\} \cap S(B_X, x^*, \alpha)\right) < 2.$$

By applying the previous result and [9, Lemma 2.3], then X is algebraic. Now, since S_x is finite for every x in X, the result follows easily from the proof of [11, Proposition 4.5]. □

Since C^*-algebras are finite-dimensional spaces whenever they are reflexive [30, Proposition 2], and C^*-algebras are always JB^*-triples, we deduce:

Corollary 4.3. *Let X be a C^*-algebra and $N \in \mathbb{N}$. The following assertions are equivalent:*

i) *X is finite-dimensional.*

ii) *The norm of $\mathcal{P}^N(X)$ is Fréchet differentiable at some point.*

iii) *There exists a slice of the unit ball of $\widehat{\bigotimes}_{N,s,\pi} X$ with diameter less than 2.*

By applying the same argument in the proof of Proposition 3.3, we deduce:

Corollary 4.4. *Let X be a infinite-dimensional complex Banach space such that X^{**} is a C^*-algebra and $N \in \mathbb{N}$. Then $\mathcal{P}^N(X)$ is extremely rough.*

The analogous result of Corollary 4.3 for preduals of C^*-algebras cannot be expected. Indeed there are infinite-dimensional spaces X with the Radon-Nikodým property whose dual are C^*-algebras (for instance ℓ_1). For these spaces $\mathcal{P}^N(X)$ does have points of Fréchet differentiability in view of [17, Theorems 2.4 and 2.6].

5. Big slices in the unit ball of spaces of integral polynomials

An N-homogeneous polynomial P is said to be *integral* [16] if there is a regular Borel measure μ on $(B_{X^*}, \sigma(X^*, X))$ such that

$$P(x) = \int_{B_{X^*}} (x^*(x))^N \, d\mu(x^*), \quad \forall x \in X. \tag{3}$$

We will denote by $\mathcal{P}_I^N(X)$ the space of all N-homogeneous integral polynomials on X. Let us recall that *the integral norm* of an integral polynomial P, $\|P\|_I$, is the infimum of $\|\mu\|$ taken over all regular Borel measures satisfying (3). With the integral norm $\mathcal{P}_I^N(X)$ becomes a Banach space. It is satisfied that $\|P\| \le \|P\|_I$ for every integral polynomial P and so $\mathcal{P}_I^N(X) \subseteq \mathcal{P}^N(X)$. If $x^* \in X^*$, it is immediate that $(x^*)^N$ is an integral N-homogeneous polynomial (represented by the measure δ_{x^*}) and $\|(x^*)^N\|_I = \|(x^*)^N\|$.

Given an element $\sum_{i=1}^k \lambda_i x_i \otimes x_i \otimes ... \otimes x_i$ in $\bigotimes_{N,s} X$, the *symmetric injective tensor norm* is given by

$$\sup_{x^* \in B_{X^*}} \left| \sum_{i=1}^k \lambda_i (x^*(x_i))^N \right| .$$

This norm in the symmetric tensor product of X is inherited from $\mathcal{P}^N(X^*)$. The completion of $\bigotimes_{N,s} X$ with respect to the above norm will be denoted by $\widehat{\bigotimes}_{N,s,\varepsilon} X$. Its dual is isometrically isomorphic to $(\mathcal{P}_I^N(X), \|.\|_I)$ ([14]).

A complex JBW^*-triple is a complex JB^*-triple having a (complete) predual. We note that the predual of a JBW^*-triple is unique [6]. The bidual X^{**} of every JB^*-triple X is a JBW^*-triple under suitable triple product which extends the one of X [15].

It is known that for every reflexive space X, S_X contains points of Fréchet-differentiability of the norm [5]. We will exhibit a class of spaces for which this phenomenon can happen only for reflexive spaces (on the injective symmetric tensor product).

Theorem 5.1. *Let X be the predual of a non reflexive JBW^*-triple and $N \in \mathbb{N}$. Then $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is extremely rough.*

Proof. By Lemma 2.1, $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is extremely rough if the diameter of $S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$ is two for every $z \in S_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ and every $0 < \alpha < 1$.

Let us fix an element z in $\widehat{\bigotimes}_{N,s,\varepsilon} X$ and $0 < \alpha < 1$. Since $\mathcal{P}_I^N(X)$ is a dual Banach space it follows from the Krein-Milman Theorem that $S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$ contains an extreme point of $B_{\mathcal{P}_I^N(X)}$. In view of [10, Proposition 1], there exists $y^* \in S_{X^*}$ such that $(y^*)^N \in S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$. Since $S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$ has nonempty norm interior in $B_{\mathcal{P}_I^N(X)}$, the above slice contains a ball and so, there exist $\rho > 0$ such that for every x^* in S_{X^*} with $\|x^* - y^*\| \le \rho$ we have that $(x^*)^N \in S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$. Since we are assuming that X^* is a non reflexive JB^*-triple, then X^* is not algebraic (see [8, Theorems 2.3 and 3.8]).

It follows from [9, Lemma 2.3], that there exists a non algebraic element x^* in $S_{X^*} \cap (y^* + \rho B_{X^*})$. Hence we can assume without lost of generality that y^* is not algebraic and $(y^*)^N \in S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$. Denote by J_{y^*} the w^*-closed subtriple of X^* generated by y^*. It is known that there is a compact Hausdorff topological space K such that J_{y^*} is linearly isometric to $C^{\mathbb{C}}(K)$ (see [20]). Since y^* is not algebraic, so J_{y^*} is infinite dimensional.

By Lemma 3.1, there are sequences of continuous functions $\{g_n\}$ and $\{h_n\}$ on K satisfying that

$$\{g_n\} \xrightarrow{w} 0, \qquad \{h_n\} \xrightarrow{w} 0$$

and sequences of disjoint open sets, $\{V_n\}$ and $\{U_n\}$, satisfying

$$0 \le g_n, h_n \le 1, \quad \text{supp } h_n \subset V_n, \quad \|h_n\|_\infty = 1, \quad \text{supp } g_n \subset U_n, \quad g_n(V_n) = \{1\}. \tag{4}$$

In view of (4) then for every $f \in B_{C^{\mathbb{C}}(K)}$, $f(1 - g_n) \pm h_n \in B_{C^{\mathbb{C}}(K)}$ and the sequences $\{f(1 - g_n) \pm h_n\}$ converges weakly to f.

Let $T : C^{\mathbb{C}}(K) \longrightarrow J_{y^*}$ be a linear isometry from $C^{\mathbb{C}}(K)$ onto J_{y^*}. Then the sequences

$$\{u_n^*\} = \left\{T\left(T^{-1}(y^*)(1 - g_n) + h_n\right)\right\}, \qquad \{v_n^*\} = \left\{T\left(T^{-1}(y^*)(1 - g_n) - h_n\right)\right\}$$

are in the unit ball of $J_{y^*} \subset X^*$ and converges weakly to y^*. Therefore, $\{u_n^*\}$ and $\{v_n^*\}$ also converges weakly to y^* in X^*. As a consequence, $\{(u_n^*)^N\}$ and $\{(v_n^*)^N\}$ converge to $(y^*)^N$ in the weak-$*$ topology of $\mathcal{P}_I^N(X)$. Since $(y^*)^N \in S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$ and $S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$ is a weak-$*$ open set in $B_{\mathcal{P}_I^N(X)}$, then for n large enough, we have that $(u_n^*)^N$ and $(v_n^*)^N$ belong to $S(B_{\mathcal{P}_I^N(X)}, z, \alpha)$.

From now on, let us fix n large enough such that the previous conditions hold. By using (4) $\|u_n^* - v_n^*\| = 2\|T(h_n)\| = 2\|h_n\|_\infty = 2$. Then there is a sequence $\{x_m\}$ in S_X such that $\{u_n^*(x_m)\}_m \to 1$ and $\{v_n^*(x_m)\}_m \to -1$. For each m, the element $x_m \otimes \overset{N}{\cdots} \otimes x_m$ belongs to $B_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ and we have:

1. If N is odd, then $\|(u_n^*)^N - (v_n^*)^N\|_I \geq |(u_n^*(x_m))^N - (v_n^*(x_m))^N|$, for every $m \in \mathbb{N}$ and so, by taking $m \to \infty$, $\|(u_n^*)^N - (v_n^*)^N\|_I \geq 2$.
2. For the argument in case that N is even we will use the following remark. Given $w := x_1 \otimes \overset{N}{\cdots} \otimes x_N$ in $\bigotimes_N X$, we recall that the element $w_s := \frac{1}{n!}\sum_{\sigma\in\Pi_N} x_{\sigma(1)} \otimes \overset{N}{\cdots} \otimes x_{\sigma(N)}$ belongs to $\widehat{\bigotimes}_{N,s,\varepsilon} X$ (Π_N is the set of all permutations on $\{1, 2, \ldots, N\}$). It is immediate that $w_s \in B_{\widehat{\bigotimes}_{N,s,\varepsilon} X}$ if the elements $x_i \in B_X$ for every $1 \leq i \leq N$.
   We know that $y^* \in S_{X^*}$, so for each $\varepsilon > 0$, let $y \in S_X$ be such that $y^*(y) = \text{Re } y^*(y) > 1 - \varepsilon$. Since $\{u_n^*\}$ and $\{v_n^*\}$ converges weakly to y^*, then for n large enough it holds
   $$\text{Re } u_n^*(y) > 1 - \varepsilon, \qquad \text{Re } v_n^*(y) > 1 - \varepsilon.$$
   By using the fact that N is even and the above remark we deduce that
   $$\begin{aligned}\|(u_n^*)^N - (v_n^*)^N\|_I &\geq |((u_n^*)^N - (v_n^*)^N)(x_m \otimes \overset{N-1}{\cdots} \otimes x_m \otimes y)_s| \\ &\geq |(u_n^*(x_m))^{N-1} u_n^*(y) - (v_n^*(x_m))^{N-1} v_n^*(y)|\end{aligned}$$
   for every $m \in \mathbb{N}$. By taking limit ($m \to \infty$) we deduce that
   $$\|(u_n^*)^N - (v_n^*)^N\|_I \geq |u_n^*(y) + v_n^*(y)| \geq \text{Re } u_n^*(y) + v_n^*(y) > 2 - \varepsilon.$$

In any case, by the choice of n, we have $\operatorname{diam} S(B_{\mathcal{P}_I^N(X)}, z, \alpha) \geq 2$ and so $\operatorname{diam} S(B_{\mathcal{P}_I^N(X)}, z, \alpha) = 2$, as we wanted to show. □

From the previous theorem and some known results, we will establish the following characterization:

Corollary 5.2. *Let X be the predual of a JBW^*-triple and $N \in \mathbb{N}$. Then the following conditions are equivalent:*

i) *X is isomorphic (as a Banach space) to a Hilbert space.*

ii) *The norm of $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is Fréchet differentiable at some element.*

iii) *There exists a w^*-slice of the unit ball of $\mathcal{P}_I^N(X)$ with diameter less than 2.*

iv) *$\widehat{\bigotimes}_{N,s,\varepsilon} X$ is not extremely rough*

Proof. *i)* ⇒ *ii)* Since X is a reflexive space, then the unit sphere of X contains at least one point x of Fréchet differentiability. Hence x determines slices of B_{X^*} with diameter arbitrarily small in view of Lemma 2.1. By [10, Proposition 1] and the argument used in Proposition 2.2, then $x \otimes \overset{N}{\cdots} \otimes x$ determines w^*-slices of the unit ball of its dual $\mathcal{P}_I^N(X)$ with diameter arbitrarily small. Then the norm of $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is Fréchet differentiable at some point by using again Lemma 2.1.

ii) ⇒ *iii)* and *iii)* ⇒ *iv)* are clear in view of Lemma 2.1.

iv) ⇒ *i)* By Theorem 5.1, X is reflexive. Then X^* is a reflexive JB^*-triple and by [9, Proposition 2.4], X is isomorphic (as a Banach space) to a Hilbert space. □

By using again that C^*-algebras are finite-dimensional spaces whenever they are reflexive [30, Proposition 2], and C^*-algebras are always JB^*-triples, we deduce the main result of this section.

Corollary 5.3. *Let X be the predual of a von Neumann algebra and $N \in \mathbb{N}$. Then the following conditions are equivalent:*

1. *X is finite-dimensional.*
2. *The norm of $\widehat{\bigotimes}_{N,s,\varepsilon} X$ is Fréchet differentiable at some element.*
3. *There exists a w^*-slice of the unit ball of $\mathcal{P}_I^N(X)$ with diameter less than* 2.

The analogous result of Theorem 5.1 is not true in the class of JB^*-triples. For instance, if we take $X = c_0$ and $N \in \mathbb{N}$, then $\widehat{\bigotimes}_{2,s,\varepsilon} X$ is an Asplund space by [27, Theorem 1.9].

References

[1] M. D. Acosta, J. Becerra Guerrero: Slices in the unit ball of the symmetric tensor product of $\mathcal{C}(K)$ and $L_1(\mu)$, Ark. Mat. 47 (2009) 1–12.

[2] M. D. Acosta, A. Kamińska: Weak neighborhoods and the Daugavet property of the interpolation spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$, Indiana Univ. Math. J. 57 (2008) 77–96.

[3] R. M. Aron, P. D. Berner: A Hahn-Banach extension theorem for analytic mappings, Bull. Soc. Math. Fr. 106 (1978) 3–24.

[4] R. M. Aron, Y. S. Choi, S. G. Kim, M. Maestre: Local properties of polynomials on a Banach space, Illinois J. Math. 45 (2001) 25–39.

[5] E. Asplund: Fréchet differentiability of convex functions, Acta Math. 121 (1968) 31–47.

[6] T. J. Barton, R. M. Timoney: Weak* continuity of Jordan triple products and applications, Math. Scand. 59 (1986) 177–191.

[7] J. Becerra Guerrero, G. López Pérez: Relatively weakly open subsets of the unit ball in functions spaces, J. Math. Anal. Appl. 315 (2006) 544–554.

[8] J. Becerra, G. López, A. Peralta, A. Rodríguez: Relatively weakly open sets in closed balls of Banach spaces, and real JB^*-triples of finite rank, Math. Ann. 330 (2004) 45–58.

[9] J. Becerra, G. López, A. Rodríguez: Relatively weakly open sets in closed balls of C^*-algebras, J. Lond. Math. Soc., II. Ser. 68 (2003) 753–761.

[10] C. Boyd, R. A. Ryan: Geometric theory of spaces of integral polynomials and symmetric tensor products, J. Funct. Anal. 179 (2001) 18–42.

[11] L. J. Bunce, C. H. Chu: Compact operations, multipliers and Radon-Nikodým property in JB^*-triples, Pacific J. Math. 153 (1992) 249–265.

[12] A. M. Davie, T. W. Gamelin: A theorem on polynomial-star approximation, Proc. Amer. Math. Soc. 106 (1989) 351–356.

[13] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure Appl. Math. 64, Longman Sci. Tech., Harlow (1993).

[14] S. Dineen: Holomorphy types on a Banach space, Studia Math. 39 (1971) 241–288.

[15] S. Dineen: The second dual of a JB^*-triple system, in: Complex Analysis, Functional Analysis and Approximation Theory, J. Mujica (ed.), North-Holland Math. Stud. 125, North-Holland, Amsterdam (1986) 67–69.

[16] S. Dineen: Complex Analysis on Infinite Dimensional Spaces, Springer Monographs in Mathematics, Springer, London (1999).

[17] J. Ferrera: Norm-attaining polynomials and differentiability, Studia Math. 151 (2002) 1–21.

[18] Y. Friedman, B. Russo: Structure of the predual of a JBW^*-triple, J. Reine Angew. Math. 356 (1985) 67–89.

[19] D. García, B. C. Grecu, M. Maestre: Geometry in preduals of spaces of 2-homogeneous polynomials on Hilbert spaces, Monatsh. Math. 157 (2009) 55–67.

[20] G. Horn: Characterization of the predual and ideal structure of a JBW^*-triple, Math. Scand. 61 (1987) 117–133.

[21] W. Kaup: Algebraic characterization of symmetric complex Banach manifolds, Math. Ann. 228 (1977) 39–64.

[22] W. Kaup: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, Math. Z. 183 (1983) 503–529.

[23] O. Loos: Bounded Symmetric Domains and Jordan Pairs, Math. Lectures, University of California, Irvine (1977).

[24] G. López-Pérez: The big slice phenomena in M-embedded and L-embedded spaces, Proc. Amer. Math. Soc. 134 (2005) 273–282.

[25] O. Nygaard, D. Werner: Slices in the unit ball of a uniform algebra, Arch. Math. 76 (2001) 441–444.

[26] T. S. S. R. K. Rao: There are no denting points in the unit ball of $WC(K, X)$, Proc. Amer. Math. Soc. 127 (1999) 2969–2973.

[27] W. M. Ruess, C. P. Stegall: Extreme points in the duals of operator spaces, Math. Ann. 261 (1982) 535–546.

[28] R. Ryan: Dunford-Pettis properties, Bull. Acad. Pol. Sci., Sér. Sci. Math. 27 (1979) 373–379.

[29] R. A. Ryan, B. Turett: Geometry of spaces of polynomials, J. Math. Anal. Appl. 221 (1998) 698–711.

[30] S. Sakai: Weakly compact operators on operator algebras, Pacific J. Math. 14 (1964) 659–664.

[31] W. Schachermayer, A. Sersouri, E. Werner: Moduli of non-dentability and Radon-Nikodým property in Banach spaces, Israel J. Math. 65 (1989) 225–257.

[32] R. V. Shvydkoy: Geometric aspects of the Daugavet property, J. Funct. Anal. 176 (2000) 198–212.