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An Abstract Convex Representation of Maximal Abstract Monotone Operators

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In this paper, we develop a theory of monotone operators in the framework of abstract convexity. We present a definition for abstract monotone operators and some examples of an abstract convex function such that its subdifferential is a maximal abstract monotone operator. Finally, we give an abstract convex representation for maximal abstract monotone operators.

Keywords: Monotone operator, abstract monotonicity, abstract convex function, abstract convexity, IPH function

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1. Introduction

The interest of the theory of monotone operators is propelled by the number of applications, in particular for variational inequalities and partial differential equations (see [1, 3, 9, 23]). Several approaches have established links between maximal monotone operators and convex functions (see [5, 6, 8, 10, 12, 13, 15, 21, 22]). The richness of the theory of monotone operators which has given rise to a great number of works justifies an interest in these links. Recently many authors have explored the use of convex representative functions in the study of monotone operators, e.g., [5, 6, 8, 12, 13, 15]. Roughly speaking the study of monotone operators is reduced to the study of the convexification of the coupling function, restricted to the monotone set. However, the bilinearity of the coupling function is sometimes a restrictive assumption, and therefore the problem arises how to extend the theory of monotone operators outside this context. Generalization of this assertion is in the framework of abstract convexity.

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Abstract convexity has found many applications in the study of problem of mathematical analysis and optimization. Also, it has found interesting applications to the theory of inequalities (see [17, 18, 19, 20]). However, the development of abstract convex analysis was mainly driven by applications to optimization (see [17]). The aim of the present paper is to develop a theory of monotone operators in the framework of abstract convexity. In fact, we give an abstract convex representation for maximal abstract monotone operators, which extends the results of [8, 13].

The structure of the paper is as follows: In Section 2, we provide some preliminary definitions and results related to abstract convexity. Definitions and properties of abstract monotone operators and also some examples of maximal abstract monotone operators are given in Section 3. In Section 4, we give an abstract convex representation for maximal abstract monotone operators.

2. Preliminaries

Let X and Y be two sets. Recall (see [2]) that a set valued mapping (multifunction) from X to Y is a mapping $F : X \longrightarrow 2^Y$, where 2^Y represents the collection of all subsets of Y. We define the domain and graph of F by

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\},$$

and

$$G(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$

respectively.

Let X be a set and L be a set of real valued functions $l : X \longrightarrow \mathbb{R}$, which will be called abstract linear. For each $l \in L$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of l on the constant c

$$h_{l,c}(x) := l(x) - c, \quad (x \in X).$$

The function $h_{l,c}$ is called L-affine. Recall (see [17]) that the set L is called a set of abstract linear functions if $h_{l,c} \notin L$ for all $l \in L$ and all $c \in \mathbb{R} \setminus \{0\}$. The set of all L-affine functions will be denoted by H_L. If L is a set of abstract linear functions, then $h_{l,c} = h_{l_0,c_0}$ if and only if $l = l_0$ and $c = c_0$.

If L is a set of abstract linear functions, then the mapping $(l, c) \longrightarrow h_{l,c}$ is a one-to-one correspondence. In this case, we identify $h_{l,c}$ with (l, c), in other words, we consider an element $(l, c) \in L \times \mathbb{R}$ as a function defined on X by $x \longrightarrow l(x) - c$ $(x \in X)$.

A function $f : X \longrightarrow (-\infty, +\infty]$ is called proper if $\operatorname{dom} f \neq \emptyset$, where $\operatorname{dom} f$ is defined by

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

Let $\mathcal{F}(X)$ be the set of all functions $f : X \longrightarrow (-\infty, +\infty]$ and the function $-\infty$.

Recall (see [17]) that a function $f \in \mathcal{F}(X)$ is called H-convex ($H = L,$ *or* $H = H_L$) if

$$f(x) = \sup\{h(x) : h \in \text{supp }(f, H)\}, \quad \forall\, x \in X,$$

where

$$\text{supp }(f, H) := \{h \in H : h \leq f\}$$

is called the support set of the function f, and $h \leq f$ if and only if $h(x) \leq f(x)$ for all $x \in X$.

Example 2.1. Let X be a locally convex Hausdorff topological vector space. Let L be the set of all real valued continuous linear functionals defined on X. Then, $f : X \longrightarrow (-\infty, +\infty]$ is an L-convex function if and only if f is lower semi-continuous and sublinear. Also, f is an H_L-convex function if and only if f is lower semi-continuous and convex.

Now, we consider the coupling function $\langle .,. \rangle : X \times L \longrightarrow \mathbb{R}$ is defined by $\langle x, l\rangle := l(x)$ for all $x \in X$ and all $l \in L$. For a function $f \in \mathcal{F}(X)$, define the Fenchel-Moreau L-conjugate f_L^* of f (see [17]) by

$$f_L^*(l) := \sup_{x \in X}(l(x) - f(x)), \quad l \in L.$$

The function $f_{L,X}^{**} := (f_L^*)_X^*$ is called the second conjugate (or biconjugate) of f, and by definition we have

$$f_{L,X}^{**}(x) := \sup_{l \in L}(l(x) - f^*(l)), \quad x \in X.$$

The following property of the conjugate function follows directly from the definition.

Fenchel-Young's inequality: for a proper function $f \in \mathcal{F}(X)$, one has

$$f(x) + f_L^*(l) \geq l(x), \quad \forall\, x \in X;\ \forall\, l \in L.$$

Let $f : X \longrightarrow (-\infty, +\infty]$ be a function and $x_0 \in \operatorname{dom} f$. Recall (see [17]) that an element $l \in L$ is called an L-subgradient of f at x_0 if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \quad \forall\, x \in X.$$

The set $\partial_L f(x_0)$ of all L-subgradients of f at x_0 is called L-subdifferential of f at x_0. The subdifferential $\partial_L f(x_0)$ is non-empty (see [17]) if and only if $x_0 \in \operatorname{dom} f$ and

$$f(x_0) = \max\{h(x_0) : h \in \text{supp }(f, H_L)\}.$$

In the following, we gather some results which will be used later.

Lemma 2.2 ([17], Theorem 7.1). *Let $f \in \mathcal{F}(X)$. Then, $f = f_{L,X}^{**}$ if and only if f is an H_L-convex function.*

Lemma 2.3 ([17], Proposition 7.7). *Let $x_0 \in X$, $f \in \mathcal{F}(X)$ and $l_0 \in L$. Then the following assertions are equivalent:*

(*i*) $f(x_0) + f_L^*(l_0) = l_0(x_0)$ *(Fenchel-Young's equality).*
(*ii*) $l_0 \in \partial_L f(x_0)$.

In the sequel, let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone $S \subset X$ (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.

An extended real valued function $f : X \longrightarrow [-\infty, +\infty]$ is called positively homogeneous (of degree one) if $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and all $\lambda > 0$. The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$.

Now, consider the function $l : X \times X \longrightarrow [0, +\infty]$ defined by

$$l(x, y) = \max\{\lambda \geq 0 : \lambda y \leq x\}, \quad (x,\ y \in X),$$

(with the convention $\max \emptyset := 0$).

The function l has the following properties (see [7, 14]). In fact, for every $x,\ y,\ x',\ y' \in X$ and every $\gamma > 0$, one has

$$l(\gamma x, y) = \gamma l(x, y), \tag{1}$$

$$l(x, \gamma y) = \frac{1}{\gamma} l(x, y), \tag{2}$$

$$l(x, y) = +\infty \implies y \in -S, \tag{3}$$

$$l(x, x) = 1 \iff x \notin -S, \tag{4}$$

$$x \in S,\ y \in -S \implies l(x, y) = +\infty, \tag{5}$$

$$x \leq x' \implies l(x, y) \leq l(x', y), \tag{6}$$

$$y \leq y' \implies l(x, y) \geq l(x, y'). \tag{7}$$

Define $L_S := \{l_y : y \in X \setminus (-S)\}$, where $l_y(x) := l(x, y)$ for all $x \in X$ and all $y \in X$. Note that l_y is an increasing positively homogeneous (IPH) function for each $y \in X$. Therefore, L_S is a set of non-negative increasing positively homogeneous (IPH) functions defined on X.

The following results for non-negative IPH functions have been proved in [7, 14].

Lemma 2.4. *Let $f : X \longrightarrow [0, +\infty]$ be a function. Then the following assertions are equivalent:*

(*i*) *f is IPH.*
(*ii*) *$f(x) \geq \lambda f(y)$ for all $x,\ y \in X$ and all $\lambda > 0$ such that $\lambda y \leq x$.*
(*iii*) *$f(x) \geq l_y(x) f(y)$ for all $x,\ y \in X$ with the convention $(+\infty) \times 0 = 0$.*

Lemma 2.5. *Let $f : X \to [0, +\infty]$ be an IPH function and $f(x) \neq 0, +\infty$. Then*

$$\partial_{L_S} f(x) = \{l_y \in L_S \ :\ l_y(x) = f(x),\ f(y) = 1\}.$$

3. Abstract Monotone Operators

Assume that X is a set and L is a set of real valued functions $l : X \longrightarrow \mathbb{R}$, which is called abstract linear with the coupling function $\langle ., . \rangle : X \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle := l(x)$ for all $x \in X$ and all $l \in L$. In the following, we present some definitions and properties of abstract monotone operators (see [4, 11, 16]).

Definition 3.1. A set valued mapping $T : X \longrightarrow 2^L$ is called L-monotone operator (or, abstract monotone operator) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{8}$$

for all $l \in Tx$, $l' \in Tx'$ and all $x,\ x' \in X$.

If X is a Banach space with the dual space X^* and $L := X^*$. Then, T is called monotone operator in the classical case.

Definition 3.2. A set valued mapping $T : X \longrightarrow 2^L$ is called maximal L-monotone operator (or, maximal abstract monotone operator) if T is L-monotone and $T = T'$ for any L-monotone operator $T' : X \longrightarrow 2^L$ such that $G(T) \subseteq G(T')$.

Definition 3.3. A subset S of $X \times L$ is called L-monotone (or, abstract monotone) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0, \quad \forall\ (x, l),\ (x', l') \in S.$$

Definition 3.4. A subset S of $X \times L$ is called maximal L-monotone (or, maximal abstract monotone) if S is L-monotone and $S = S'$ for any L-monotone set S' such that $S \subseteq S'$.

Definition 3.5. Let $T : X \longrightarrow 2^L$ be a set valued mapping. Correspondence to the mapping T define the L-Fitzpatrick function (or, abstract Fitzpatrick function) $\varphi_T : X \times L \longrightarrow \bar{\mathbb{R}}$ by

$$\varphi_T(x, l) := \sup_{l' \in Tx',\ x' \in X} [l(x') + l'(x) - l'(x') - l(x)] + l(x) \tag{9}$$

for all $x \in X$ and all $l \in L$.

Lemma 3.6. *Let $T : X \longrightarrow 2^L$ be a maximal L-monotone operator. Then*

$$\varphi_T(x, l) \geq l(x), \quad \forall\, x \in X;\ \forall\, l \in L, \tag{10}$$

with equality holds if and only if $l \in Tx$.

Proof. Since T is a maximal L-monotone operator, it follows that

$$\sup_{l' \in Tx',\ x' \in X} [l(x') + l'(x) - l'(x') - l(x)] \geq 0 \tag{11}$$

for all $x \in X$ and all $l \in L$. In view of (9), we obtain

$$\varphi_T(x, l) \geq l(x), \quad \forall\, x \in X;\ \forall\, l \in L. \tag{12}$$

Since T is maximal L-monotone, it follows from (8) and (11) that

$$\sup_{l' \in Tx',\ x' \in X} [l(x') + l'(x) - l'(x') - l(x)] = 0$$

if and only if $l \in Tx$. This, together with (9) implies that $\varphi_T(x, l) = l(x)$ if and only if $l \in Tx$, and hence the proof is complete. □

In the following, we give an example of a maximal L-monotone operator. We show that L_S-subdifferential of an IPH function is a maximal L_S-monotone operator, where L_S defined in Section 2.

Theorem 3.7. *Let $f : X \to [0, +\infty]$ be an IPH function and $f(x) \neq 0, +\infty$. Then, $\partial_{L_S} f$ is a maximal L_S-monotone operator.*

Proof. First, we show that $\partial_{L_S} f$ is an L_S-monotone operator. To do this, let $x,\ x_0 \in X$, $l_y \in \partial_{L_S} f(x)$ and $l_{y_0} \in \partial_{L_S} f(x_0)$ $(y,\ y_0 \in X \setminus (-S))$ be arbitrary. Then, by Lemma 2.5, we have

$$l_y(x) = f(x),\ f(y) = 1 \quad \text{and} \quad l_{y_0}(x_0) = f(x_0),\ f(y_0) = 1. \tag{13}$$

In view of Lemma 2.4, we conclude that

$$l_y(t) f(y) \leq f(t), \quad \forall\, t \in X,$$

and

$$l_{y_0}(t) f(y_0) \leq f(t), \quad \forall\, t \in X.$$

This, together with (13) implies that $l_y(t) \leq f(t)$ and $l_{y_0}(t) \leq f(t)$ for all $t \in X$, and hence

$$l_y(x_0) \leq f(x_0) \quad \text{and} \quad l_{y_0}(x) \leq f(x). \tag{14}$$

Now, it follows from (13) and (14) that

$$l_y(x) - l_y(x_0) - l_{y_0}(x) + l_{y_0}(x_0) = [f(x) - l_{y_0}(x)] + [f(x_0) - l_y(x_0)] \geq 0.$$

Hence, $\partial_{L_S} f$ is an L_S-monotone operator.

Now, we show that $\partial_{L_S} f$ is maximal. To this end, let $T : X \longrightarrow 2^{L_S}$ be any L_S-monotone operator such that $G(\partial_{L_S} f) \subseteq G(T)$. We show that $T = \partial_{L_S} f$. It suffices to prove that if $(x_0, l_{y_0}) \in G(T)$, then $(x_0, l_{y_0}) \in G(\partial_{L_S} f)$; that is, $l_{y_0}(x_0) = f(x_0)$ and $f(y_0) = 1$. Assume that $(x_0, l_{y_0}) \in G(T)$ $(x_0 \in X,\ y_0 \in X \setminus (-S))$ be arbitrary. Since T is L_S-monotone and $G(\partial_{L_S} f) \subseteq G(T)$, it follows that

$$l_y(x) - l_y(x_0) - l_{y_0}(x) + l_{y_0}(x_0) \geq 0, \quad \forall\, l_y \in \partial_{L_S} f(x);\ \forall\, x \in X. \tag{15}$$

Let $\lambda > 1$ and $x = \lambda x_0$. Then, in view of (15), we conclude that

$$l_{y_0}(x_0) \leq l_y(x_0). \tag{16}$$

Also, for $0 < \lambda < 1$ and $x = \lambda x_0$, it follows from (15) that

$$l_y(x_0) \leq l_{y_0}(x_0). \tag{17}$$

Therefore, (16) and (17) imply that

$$l_{y_0}(x_0) = l_y(x_0), \quad \forall\, l_y \in \partial_{L_S} f(x_0). \tag{18}$$

Since $l_y(x_0) = f(x_0)$ for each $l_y \in \partial_{L_S} f(x_0)$, we deduce from (18) that $l_{y_0}(x_0) = f(x_0)$. On the other hand, let $x \in X$ be arbitrary and replace x by λx ($\lambda > 0$) in (15). Then, one has

$$[l_y(x) - l_{y_0}(x)] \geq \frac{1}{\lambda}[l_y(x_0) - l_{y_0}(x_0)], \quad \forall\, l_y \in \partial_{L_S} f(x);\ \forall\, x \in X;\ \forall\, \lambda > 0.$$

(Note that $\partial_{L_S} f(\lambda x) = \partial_{L_S} f(x)$ for all $x \in X$ and all $\lambda > 0$.) Therefore, as $\lambda \longrightarrow +\infty$, we conclude that

$$l_{y_0}(x) \leq l_y(x), \quad \forall\, l_y \in \partial_{L_S} f(x);\ \forall\, x \in X.$$

Since for each $x \in X$ and each $l_y \in \partial_{L_S} f(x)$ we have $l_y(x) = f(x)$, then it follows that

$$l_{y_0}(x) \leq f(x), \quad \forall\, x \in X. \tag{19}$$

This implies that $y_0 \notin -S$. Because if $y_0 \in -S$, then in view of (5) we obtain $+\infty = l_{y_0}(0) \leq f(0) = 0$, and this is a contradiction. Thus, by (4) and (19) we have

$$1 = l_{y_0}(y_0) \leq f(y_0). \tag{20}$$

Since $l_{y_0}(x_0) = f(x_0)$, it follows from the definition of l_{y_0} that $f(x_0)y_0 = l_{y_0}(x_0)y_0 \leq x_0$. This implies that $f(x_0)f(y_0) \leq f(x_0)$ because f is increasing. Since $0 < f(x_0) < +\infty$, then we have $f(y_0) \leq 1$, and hence by (20) we get $f(y_0) = 1$. Consequently, $(x_0, l_{y_0}) \in G(\partial_{L_S} f)$, and the proof is complete. $\square$

From now on, let X be a set and L be a set of real valued abstract linear functions $l : X \longrightarrow \mathbb{R}$ defined on X. Assume that $0 \in L$. We consider the coupling function $\langle .,. \rangle : X \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle := l(x)$ for all $x \in X$ and all $l \in L$. Let

$$K := X \times L \quad \text{and} \quad L^* := L \times X. \tag{21}$$

Define the coupling function $\langle .,. \rangle_* : K \times L^* \longrightarrow \mathbb{R}$ by

$$\langle (x', l'), (l, x) \rangle_* := l(x') + l'(x), \quad \forall\, (x', l') \in K;\ \forall\, (l, x) \in L^*. \tag{22}$$

We can consider an element $(l, x) \in L^*$ as a function defined on K by

$$(l, x)(x', l') := \langle (x', l'), (l, x) \rangle_*, \quad \forall\, (x', l') \in K,$$

and an element $(x, l) \in K$ as a function is defined on L^* by

$$(x, l)(l', x') := \langle (x, l), (l', x') \rangle_*, \quad \forall\, (l', x') \in L^*.$$

Note that the coupling function $\langle .,. \rangle_*$ is symmetric, that is

$$\langle (x', l'), (l, x) \rangle_* = \langle (l, x), (x', l') \rangle_*, \quad \text{for all } (x', l') \in K, \text{ and all } (l, x) \in L^*.$$

It is easy to check that L^* and K are sets of abstract linear functions. Indeed, if there exist $(l_0, x_0) \in L^*$ and $c_0 \in \mathbb{R} \setminus \{0\}$ such that $h_{(l_0,x_0),c_0} \in L^*$, where $h_{(l_0,x_0),c_0} := (l_0, x_0) - c_0$, then $h_{(l_0,x_0),c_0} = (l, x)$ for some $(l, x) \in L^*$. It follows that

$$l_0(x') + l'(x_0) - c_0 = l(x') + l'(x), \quad \forall\, (x', l') \in K. \tag{23}$$

Since $0 \in L$, put $l' = 0$ in (23). Thus, we have

$$l_0(x') - c_0 = l(x'), \quad \forall\, x' \in X,$$

and so $l_0 - c_0 = l$ on X. Since L is a set of abstract linear functions, we conclude that $l_0 = l$ and $c_0 = 0$. This is a contradiction, because $c_0 \neq 0$. Hence, $h_{(l,x),c} \notin L^*$ for all $(l, x) \in L^*$ and all $c \in \mathbb{R} \setminus \{0\}$. Therefore, L^* is a set of abstract linear functions. By a similar argument, K is also a set of abstract linear functions.

In the following, we give an example of an H_{L^*}-convex function such that its subdifferential is a maximal L^*- monotone operator.

Let X be a conic set and L be a conic set of positively homogeneous functions $l : X \longrightarrow (-\infty, +\infty]$ defined on X. (A set C is called conic if $\lambda C \subset C$ for all $\lambda > 0$.) Let $K := X \times L$, and $L^* := L \times X$. Define the coupling function $\langle .,. \rangle_*$ on $K \times L^*$ as in (22). It is worth noting that each element of L^* as a function defined on K is a positively homogeneous function. Moreover, L^* is a set of abstract linear functions.

It is easy to check that if a function $h : K \longrightarrow (-\infty, +\infty]$ is an L^*-convex function, then h is a positively homogeneous function. Also, by [17], Proposition 7.15], a positively homogeneous function $h : K \longrightarrow (-\infty, +\infty]$ is L^*-convex if and only if it is H_{L^*}-convex.

Theorem 3.8. *Let $h : K \to (-\infty, +\infty]$ be an L^*-convex function. Then, for each $p \in K$, we have*

$$\partial_{L^*} h(p) = \{l^* \in L^* : \; h(p) = \langle p, l^* \rangle_*; \; \langle q, l^* \rangle_* \leq h(q), \; \forall\, q \in K\}.$$

Proof. By definition we have

$$\partial_{L^*} h(p) = \{l^* \in L^* : \langle q, l^* \rangle_* - \langle p, l^* \rangle_* \leq h(q) - h(p), \; \forall\, q \in K\} \quad (p \in K).$$

Therefore, for each $l^* \in \partial_{L^*} h(p)$, we have

$$\langle q, l^* \rangle_* - \langle p, l^* \rangle_* \leq h(q) - h(p), \quad \forall\, q \in K. \tag{24}$$

Let $\lambda > 1$ and $q = \lambda p$. Then by (24) and positive homogeneity of h and l^* we obtain $h(p) \geq \langle p, l^* \rangle_*$. By a similar argument, for $0 < \lambda < 1$ and $q = \lambda p$, we get $h(p) \leq \langle p, l^* \rangle_*$, and so we have $h(p) = \langle p, l^* \rangle_*$. Thus, the result follows. $\square$

Theorem 3.9. *Let $h : K \longrightarrow (-\infty, +\infty]$ be an L^*-convex function. Then, $\partial_{L^*} h$ is a maximal L^*-monotone operator.*

Proof. Let $p,\, q \in K$, $l^* \in \partial_{L^*} h(p)$ and ${l_0}^* \in \partial_{L^*} h(q)$ be arbitrary. Then we have

$$\langle p, l^* \rangle_* = h(p), \quad \langle r, l^* \rangle_* \leq h(r), \quad \forall\, r \in K, \tag{25}$$

and

$$\langle q, {l_0}^* \rangle_* = h(q), \quad \langle r, {l_0}^* \rangle_* \leq h(r), \quad \forall\, r \in K. \tag{26}$$

It follows from (25) and (26) that $[h(q) - \langle q, l^* \rangle_*] \geq 0$ and $[h(p) - \langle p, {l_0}^* \rangle_*] \geq 0$. This, together with $\langle p, l^* \rangle_* = h(p)$ and $\langle q, {l_0}^* \rangle_* = h(q)$ implies that

$$\begin{aligned} &\langle p, l^* \rangle_* - \langle q, l^* \rangle_* - \langle p, {l_0}^* \rangle_* + \langle q, {l_0}^* \rangle_* \\ &= [h(p) - \langle p, {l_0}^* \rangle_*] + [h(q) - \langle q, l^* \rangle_*] \\ &\geq 0, \end{aligned}$$

and hence $\partial_{L^*} h$ is an L^*-monotone operator.

Now, we show that $\partial_{L^*} h$ is maximal. To this end, let $T : K \longrightarrow 2^{L^*}$ be any L^*-monotone operator such that $G(\partial_{L^*} h) \subseteq G(T)$. We show that $T = \partial_{L^*} h$. It suffices to prove that if $(p_0, {l_0}^*) \in G(T)$, then $(p_0, {l_0}^*) \in G(\partial_{L^*} h)$; that is, $\langle p_0, {l_0}^* \rangle_* = h(p_0)$ and $\langle q, {l_0}^* \rangle_* \leq h(q)$ for all $q \in K$. Assume that $(p_0, {l_0}^*) \in G(T)$ be arbitrary. Since T is L^*-monotone and $G(\partial_{L^*} h) \subseteq G(T)$, it follows that

$$\langle q, l^* \rangle_* - \langle p_0, l^* \rangle_* - \langle q, {l_0}^* \rangle_* + \langle p_0, {l_0}^* \rangle_* \geq 0, \quad \forall\, l^* \in \partial_{L^*} h(q);\ \forall\, q \in K. \tag{27}$$

Let $\lambda > 1$ and $q = \lambda p_0$. Then, in view of (27), we conclude that

$$\langle p_0, {l_0}^* \rangle_* \leq \langle p_0, l^* \rangle_*. \tag{28}$$

Also, for $0 < \lambda < 1$ and $q = \lambda p_0$, it follows from (27) that

$$\langle p_0, {l_0}^* \rangle_* \geq \langle p_0, l^* \rangle_*. \tag{29}$$

Therefore, (28) and (29) imply that

$$\langle p_0, {l_0}^* \rangle_* = \langle p_0, l^* \rangle_*, \quad \forall\, l^* \in \partial_{L^*} h(p_0). \tag{30}$$

Since $\langle p_0, l^* \rangle_* = h(p_0)$ for each $l^* \in \partial_{L^*} h(p_0)$, we deduce from (30) that $\langle p_0, {l_0}^* \rangle_* = h(p_0)$.

On the other hand, let $q \in K$ be arbitrary and replace q by λq ($\lambda > 0$) in (27). Then, one has

$$[\langle q, l^* \rangle_* - \langle q, {l_0}^* \rangle_*] \geq \frac{1}{\lambda} [\langle p_0, l^* \rangle_* - \langle p_0, {l_0}^* \rangle_*], \quad \forall\, l^* \in \partial_{L^*} h(q);\ \forall\, q \in K;\ \forall\, \lambda > 0. \tag{31}$$

(Note that $\partial_{L^*} h(\lambda q) = \partial_{L^*} h(q)$ for all $q \in K$ and all $\lambda > 0$.) Therefore, as $\lambda \longrightarrow +\infty$ in (31), we conclude that

$$\langle q, {l_0}^* \rangle_* \leq \langle q, l^* \rangle_*, \quad \forall\, l^* \in \partial_{L^*} h(q);\ \forall\, q \in K. \tag{32}$$

Since for each $q \in K$ and each $l^* \in \partial_{L^*} h(q)$ we have $\langle q, l^* \rangle_* = h(q)$, then it follows from (32) that $\langle q, {l_0}^* \rangle_* \leq h(q)$ for all $q \in K$, and hence $(p_0, {l_0}^*) \in G(\partial_{L^*} h)$, which completes the proof. $\square$

4. Maximal Abstract Monotone Operators

In this section, we give a representation for maximal abstract monotone operators by abstract convex functions, which extends the results of [8, 13]. Let K, L^* and the coupling function $\langle .,. \rangle_*$ be as defined by (21) and (22), respectively. Denote by

$$\mathcal{P}(H_{L^*}) := \{h : K \longrightarrow (-\infty, +\infty] : h \text{ is a proper } H_{L^*}\text{-convex function}\}$$

the set of all proper H_{L^*}-convex functions defined on K. For each $h \in \mathcal{P}(H_{L^*})$, define

$$T(h) := \{(x, l) \in K : h(x, l) \leq l(x)\},$$

and denote by $\delta_{T(h)}$ the indicator function of $T(h)$ which is defined on K as follows

$$\delta_{T(h)}(x, l) := \begin{cases} 0, & \text{if } (x, l) \in T(h) \\ +\infty, & \text{otherwise,} \end{cases}$$

for each $(x, l) \in K$. Define the transpose operator $t : K \longrightarrow L^*$ by $t(x, l) := (l, x)$ for all $(x, l) \in K$.

Now, let $H(K)$ be defined as follows

$$H(K) := \{h \in \mathcal{P}(H_{L^*}) : h(x, l) = [(\langle .,. \rangle + \delta_{T(h)})^*_{L^*} \circ t](x, l), \ \forall\, (x, l) \in K\}. \tag{33}$$

Note that for each $h \in H(K)$, we have $T(h) \neq \emptyset$. Indeed, if $T(h) = \emptyset$, then $\langle .,. \rangle + \delta_{T(h)} = +\infty$ on K, and hence $h = -\infty$ on K.

Remark 4.1. Let S be any non-empty subset of K. Then the L-Fitzpatrick function φ_S associated with S is an H_{L^*}-convex function on K. Indeed, by Definition 3.5, we have

$$\begin{aligned} & \varphi_S(x, l) \\ = \ & \sup_{(x', l') \in S} [l(x') + l'(x) - l'(x')] \\ = \ & \sup_{(x', l') \in S} [\langle (x, l), (l', x') \rangle_* - l'(x')] \\ = \ & \sup\{\langle (x, l), (l', x') \rangle_* - c : ((l', x'), c) \in \text{supp}\,(\varphi_S, H_{L^*})\} \end{aligned}$$

for all $(x, l) \in K$, and hence the result follows.

Definition 4.2. We say that an H_{L^*}-convex function (or, abstract convex function) $h : X \times L \longrightarrow (-\infty, +\infty]$ represents an L-monotone operator $T : X \longrightarrow 2^L$ if

$$h(x, l) \geq l(x), \quad \forall\, x \in X; \ \forall\, l \in L,$$

with equality holds when $l \in Tx$.

In view of Lemma 3.6 and Remark 4.1, we see that if $T : X \longrightarrow 2^L$ is a maximal L-monotone operator, then φ_T, the L-Fitzpatrick function associated with T, represents T.

Remark 4.3. Let $h \in \mathcal{P}(H_{L^*})$ and $c_{T(h)} := \langle ., . \rangle + \delta_{T(h)}$. Then, $(c_{T(h)})^*_{L^*} \circ t$ is an H_{L^*}-convex function and $(c_{T(h)})^*_{L^*} \circ t = \varphi_{T(h)}$. Indeed, we have

$$\begin{aligned}
&[(c_{T(h)})^*_{L^*} \circ t](x, l) \\
&= (c_{T(h)})^*_{L^*}(l, x) \\
&= \sup_{(x',l') \in K} [\langle (x', l'), (l, x) \rangle_* - l'(x') - \delta_{T(h)}(x', l')] \\
&= \sup_{(x',l') \in T(h)} [l(x') + l'(x) - l'(x')] \\
&= \varphi_{T(h)}(x, l),
\end{aligned}$$

for all $(x, l) \in K$. This, together with Remark 4.1 implies that $(c_{T(h)})^*_{L^*} \circ t$ is an H_{L^*}-convex function and $(c_{T(h)})^*_{L^*} \circ t = \varphi_{T(h)}$.

Lemma 4.4. *Let $S : X \longrightarrow 2^L$ be a maximal L-monotone operator and φ_S be the L-Fitzpatrick function associated with S. Then, $\varphi_S \in H(K)$.*

Proof. Since S is an L-monotone operator, it follows that

$$\varphi_S(x, l) = l(x), \quad \forall\, (x, l) \in G(S).$$

Also, because of S is maximal, we have

$$\varphi_S(x, l) > l(x), \quad \forall\, (x, l) \in K \setminus G(S),$$

and hence $G(S) = T(\varphi_S)$. Therefore, we have

$$\begin{aligned}
&\varphi_S(x, l) \\
&= \sup_{(x',l') \in G(S)} [l(x') + l'(x) - l'(x')] \\
&= \sup_{(x',l') \in T(\varphi_S)} [l(x') + l'(x) - l'(x')] \\
&= \sup_{(x',l') \in K} [l(x') + l'(x) - (\langle ., . \rangle + \delta_{T(\varphi_S)})(x', l')] \\
&= \sup_{(x',l') \in K} [\langle (x', l'), (l, x) \rangle_* - (\langle ., . \rangle + \delta_{T(\varphi_S)})(x', l')] \\
&= (\langle ., . \rangle + \delta_{T(\varphi_S)})^*_{L^*}(l, x) \\
&= [(\langle ., . \rangle + \delta_{T(\varphi_S)})^*_{L^*} \circ t](x, l), \quad \forall\, (x, l) \in K.
\end{aligned}$$

This, together with Remark 4.1 implies that $\varphi_S \in H(K)$. $\square$

Proposition 4.5. *We have*

(1) $h(x, l) \geq l(x)$ *for all* $(x, l) \in K$ *and all* $h \in H(K)$.

(2) $h^*_{L^*}(l, x) \geq l(x)$ *for all* $(x, l) \in K$ *and all* $h \in H(K)$.

Proof. Suppose that $h \in H(K)$ and $(x, l) \in K$ are arbitrary.

(*1*). It is clear that if $(x,l) \notin T(h)$, then (*1*) holds. Assume that $(x,l) \in T(h)$. Therefore, by Fenchel-Young's inequality and $h \in H(K)$, we have

$$\begin{aligned} & 2h(x,l) \\ &= h(x,l) + h(x,l) \\ &= (h + \delta_{T(h)})(x,l) + (\langle .,. \rangle + \delta_{T(h)})^*_{L^*}(l,x) \\ &\geq \langle (x,l),(l,x) \rangle_* \\ &= 2l(x). \end{aligned}$$

(*2*). By (*1*) and $h \in H(K)$, we have

$$\begin{aligned} & h^*_{L^*}(l,x) \\ &= \sup_{(x',l') \in K} [\langle (x',l'),(l,x) \rangle_* - h(x',l')] \\ &\geq \sup_{(x',l') \in T(h)} [\langle (x',l'),(l,x) \rangle_* - h(x',l')] \\ &\geq \sup_{(x',l') \in T(h)} [\langle (x',l'),(l,x) \rangle_* - l'(x')] \\ &= \sup_{(x',l') \in K} [\langle (x',l'),(l,x) \rangle_* - (\langle .,. \rangle + \delta_{T(h)})(x',l')] \\ &= (\langle .,. \rangle + \delta_{T(h)})^*_{L^*}(l,x) \\ &= h(x,l) \\ &\geq l(x), \end{aligned}$$

which completes the proof. □

Theorem 4.6. *Let $h \in H(K)$. Define*

$$S := \{(x,l) \in K : h(x,l) = l(x)\}.$$

Then, we have

(*1*) $S = \{(x,l) \in K : h^*{}_{L^*}(l,x) = l(x)\}$.
(*2*) *S is a maximal L-monotone subset of K.*
(*3*) *Let φ_S be the L-Fitzpatrick function associated with S. Then*
  (*i*) $\varphi_S(x,l) \geq l(x)$ *for all* $(x,l) \in K$.
  (*ii*) $(\varphi_S)^*_{L^*}(l,x) \geq l(x)$ *for all* $(x,l) \in K$.
  (*iii*) $(\varphi_S)^*_{L^*}(l,x) = l(x)$ *if and only if* $(x,l) \in S$.

Proof. (*1*). Let $(x,l) \in S$ be fixed and arbitrary. Then $h(x,l) = l(x)$, and so $(x,l) \in T(h)$. We have also $(l,x) \in L^*$, and so $\langle (x',l'),(l,x) \rangle_* = l'(x) + l(x')$ for all $(x',l') \in K$. Now, define the function g on K by

$$g(x',l') := \langle (x',l'),(l,x) \rangle_* - \langle (x,l),(l,x) \rangle_* + h(x,l), \quad \forall\, (x',l') \in K. \tag{34}$$

It follows that

$$\begin{aligned} & g(x',l') \\ &= \langle (x',l'),(l,x) \rangle_* - l(x) \\ &= l(x') + l'(x) - l(x), \quad \forall\, (x',l') \in K. \end{aligned}$$

Since $(x,l) \in T(h)$, it follows from the definition of $T(h)$ and $h \in H(K)$ that

$$\begin{aligned} & g(x',l') \\ &= l(x') + l'(x) - l(x) \\ &\leq \sup_{(x,l)\in T(h)} [l(x') + l'(x) - l(x)] \\ &= \sup_{(x,l)\in K} [l(x') + l'(x) - (\langle .,. \rangle + \delta_{T(h)})(x,l)] \\ &= \sup_{(x,l)\in K} [\langle (x,l),(l',x') \rangle_* - (\langle .,. \rangle + \delta_{T(h)})(x,l)] \\ &= (\langle .,. \rangle + \delta_{T(h)})^*_{L^*}(l',x') \\ &= h(x',l'), \quad \forall\, (x',l') \in K. \end{aligned}$$

This implies that $g \leq h$ on K. In view of (34) we get $(l,x) \in \partial_{L^*} h(x,l)$. Therefore, by Lemma 2.3, we have

$$h(x,l) + h^*_{L^*}(l,x) = \langle (x,l),(l,x) \rangle_* = 2l(x),$$

and hence $h^*_{L^*}(l,x) = l(x)$.

Conversely, suppose that $(x,l) \in K$ and $h^*_{L^*}(l,x) = l(x)$. Then, by Proposition 4.5(*1*) and the proof of Proposition 4.5(*2*), we conclude that

$$\begin{aligned} & l(x) \\ &= h^*_{L^*}(l,x) \\ &\geq (\langle .,. \rangle + \delta_{T(h)})^*_{L^*}(l,x) \\ &= h(x,l) \\ &\geq l(x). \end{aligned}$$

It follows that $h(x,l) = l(x)$. This completes the proof of (*1*).

(*2*). For L-monotonicity of S, suppose that (x,l), $(x',l') \in S$ are arbitrary. Then, $h(x,l) = l(x)$ and $h(x',l') = l'(x')$, and so (x,l), $(x',l') \in T(h)$. Thus, by Fenchel-Young's inequality we have

$$\begin{aligned} & l(x) - l(x') - l'(x) + l'(x') \\ &= l(x) + h(x',l') - l(x') - l'(x) \\ &= (\langle .,. \rangle + \delta_{T(h)})(x,l) + (\langle .,. \rangle + \delta_{T(h)})^*_{L^*}(l',x') - l(x') - l'(x) \\ &\geq 0, \end{aligned}$$

and hence S is L-monotone.

Now, we show that S is maximal L-monotone. Let $(x,l) \in K$ be arbitrary and

$$l(x) - l(x') - l'(x) + l'(x') \geq 0, \quad \forall\, (x',l') \in S.$$

This, together with Proposition 4.5(1) implies that

$$\begin{aligned} & l(x) \\ \geq & \sup_{(x',l')\in S}[l(x')+l'(x)-l'(x')] \\ = & \sup_{h(x',l')=l'(x')}[l(x')+l'(x)-l'(x')] \\ = & \sup_{h(x',l')\leq l'(x')}[l(x')+l'(x)-l'(x')] \\ = & \sup_{(x',l')\in T(h)}[l(x')+l'(x)-l'(x')] \\ = & \sup_{(x',l')\in K}[l(x')+l'(x)-(\langle .,.\rangle+\delta_{T(h)})(x',l')] \\ = & \sup_{(x',l')\in K}[\langle (x',l'),(l,x)\rangle_* -(\langle .,.\rangle+\delta_{T(h)})(x',l')] \\ = & (\langle .,.\rangle+\delta_{T(h)})^*_{L^*}(l,x) \\ = & h(x,l) \\ \geq & l(x). \end{aligned} \tag{35}$$

Thus, we have $h(x,l)=l(x)$, and hence $(x,l)\in S$. This proves that S is maximal L-monotone, and completes the proof of *(2)*.

(3). By *(2)*, we have S is a maximal L-monotone subset of K. Therefore, it follows from Lemma 4.4 that $\varphi_S\in H(K)$. Hence, by Proposition 4.5, we conclude that (i) and (ii) hold. Now, we prove (iii). By maximality of S, it follows from Lemma 3.6 that

$$\varphi_S(x,l)=l(x) \iff (x,l)\in S. \tag{36}$$

Then, for arbitrary $(x,l)\in K$, we have

$$\begin{aligned} & \varphi_S(x,l) \\ = & \sup_{(x',l')\in S}[l(x')+l'(x)-l'(x')] \\ = & \sup_{(x',l')\in S}[l(x')+l'(x)-\varphi_S(x',l')] \\ \leq & \sup_{(x',l')\in K}[l(x')+l'(x)-\varphi_S(x',l')] \\ = & \sup_{(x',l')\in K}[\langle (x',l'),(l,x)\rangle_* -\varphi_S(x',l')] \\ = & (\varphi_S)^*_{L^*}(l,x). \end{aligned}$$

This, together with (i) implies that

$$(\varphi_S)^*_{L^*}(l,x)\geq \varphi_S(x,l)\geq l(x), \quad \forall\ (x,l)\in K. \tag{37}$$

Further, if $(x,l)\in S$, then the definition of φ_S yields

$$\begin{aligned} & \varphi_S(x',l') \\ \geq & \ l(x')+l'(x)-l(x) \\ = & \ \langle (x',l'),(l,x)\rangle_* - l(x), \quad \forall\ (x',l')\in K. \end{aligned}$$

This implies that

$$\begin{aligned} & l(x) \\ \geq & \sup_{(x',l')\in K} [\langle (x',l'),(l,x)\rangle_* - \varphi_S(x',l')] \\ = & (\varphi_S)^*_{L^*}(l,x), \quad \forall\, (x,l) \in S. \end{aligned}$$

This, together with (36) and (37) implies that

$$(\varphi_S)^*_{L^*}(l,x) = l(x) \iff (x,l) \in S,$$

which completes the proof. □

Theorem 4.7. *Let $S : X \longrightarrow 2^L$ be a maximal L-monotone operator. Then there exists $h \in H(K)$ such that*

$$G(S) = \{(x,l) \in K : h(x,l) = l(x)\}.$$

Proof. Since S is a maximal L-monotone operator, it follows from Lemma 4.4 and the proof of Lemma 4.4 that $\varphi_S \in H(K)$ and $G(S) = T(\varphi_S)$, where φ_S is the L-Fitzpatrick function associated with S. Let $h := \varphi_S$. Thus, in view of Proposition 4.5, we have

$$\begin{aligned} & \{(x,l) \in K : h(x,l) = l(x)\} \\ = & \{(x,l) \in K : \varphi_S(x,l) = l(x)\} \\ = & \{(x,l) \in K : \varphi_S(x,l) \leq l(x)\} \\ = & \{(x,l) \in K : (x,l) \in T(\varphi_S)\} \\ = & \{(x,l) \in K : (x,l) \in G(S)\} \\ = & G(S), \end{aligned}$$

which completes the proof. □

Corollary 4.8. *Let $S : X \longrightarrow 2^L$ be a set valued mapping. Then S is maximal L-monotone if and only if there exists $h \in H(K)$ such that*

$$G(S) = \{(x,l) \in K : h(x,l) = l(x)\}.$$

Proof. This is an immediate consequence of Theorem 4.6 and Theorem 4.7. □

Remark 4.9. Let X be a Banach space with the dual space X^* and the duality product $\langle .,. \rangle : X \times X^* \longrightarrow \mathbb{R}$ is defined by

$$\langle x, x^* \rangle := x^*(x), \quad \forall\, x \in X;\ \forall\, x^* \in X^*.$$

We indentify the dual space of $X \times X^*$ with $X^* \times X$, under the pairing

$$\langle (x,x^*),(y^*,y)\rangle_* := \langle x, y^*\rangle + \langle y, x^*\rangle,$$

for all $(x,x^*) \in X \times X^*$ and all $(y^*,y) \in X^* \times X$.

Let $K := X \times X^*$ and $L^* := X^* \times X$. Let

$$H_{L^*} := \{((l,x),c) : (l,x) \in L^*,\ c \in \mathbb{R}\}$$

be the set of all continuous affine functions defined on K. Therefore, we have every function $h \in \mathcal{P}(H_{L^*})$ is proper, lower semi-continuous and convex (see [17]). Moreover, Theorem 4.6 and Theorem 4.7 remain valid. Consequently, Corollary 4.8 gives us a convex representation for maximal monotone operators which was obtained in [8, 13] in the classical setting.

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