Journal of Convex Analysis
Volume 18 (2011), No. 1, 59–65

A Representation of G-Invariant Norms for Eaton Triple

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Received: February 13, 2009
Revised manuscript received: December 9, 2009

We obtain a representation of unitarily invariant norm in terms of Ky Fan norms [1, p. 35]. Indeed we obtain a more general result in the context of Eaton triple with reduced triple. Examples are given.

Keywords: Eaton triple, G-invariant norms, symmetric gauge functions, Ky Fan norms, finite reflection groups

2000 Mathematics Subject Classification: 15A60, 65F35

1. Introduction

A norm $\|\cdot\| : \mathbb{R}^n \to [0,+\infty)$ is called a *symmetric gauge function* if it is invariant under permutation of its entries and sign changes, i.e., for each $x = (x_1,\dots,x_n) \in \mathbb{R}^n$

$$\|(\pm x_{\sigma(1)},\dots,\pm x_{\sigma(n)})\| = \|x\|, \quad \text{for all } \sigma \in S_n,$$

where S_n denotes the symmetric group on $\{1,\dots,n\}$. Such norms play an important role in matrix analysis, because they correspond to *unitarily invariant norms* on $\mathbb{C}_{n\times n}$, due to a well-known result of von Neumann [11]. A norm $|||\cdot||| : \mathbb{C}_{n\times n} \to [0,+\infty)$ is said to be unitarily invariant if for each $A \in \mathbb{C}_{n\times n}$,

$$|||UAV||| = |||A|||, \quad \text{for all } U, V \in U(n).$$

According to von Neumann's result there is one-one correspondence between the unitarily invariant norms $|||\cdot|||$ on $\mathbb{C}_{n\times n}$ and symmetric gauge functions. However the correspondence has not been studied and no explicit representation of $|||\cdot|||$ in terms of "elementary" symmetric gauge functions is given. The goal of this paper is to give such a representation (see [6, 10] for related results on representing invariant norms).

*The first author is supported by the MNiSzW grant N201 026 32/1350.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

The Ky-Fan dominance theorem follows from our representation result. In order to state our main results (in Section 3), we will first review some preliminary material on finite reflection group and Eaton triple.

2. Finite reflection group

Let us recall some rudiments of finite reflection groups and our general references are [2, 4, 7, 8]. Let W be an m-dimensional real vector space with inner product $(\cdot,\cdot)$. A reflection s_α on W is an element of the orthogonal group $O(W)$ on W, which sends some nonzero vector α to its negative and fixes pointwise the hyperplane orthogonal to α, that is,

$$s_\alpha\lambda := \lambda - (\lambda, \alpha^\vee)\alpha, \quad \lambda \in W,$$

where

$$\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}.$$

A finite group H generated by reflections is called a *finite reflection group*. A *root system* of H is a finite set of nonzero vectors in W, denoted by Φ, such that $\{s_\alpha : \alpha \in \Phi\}$ generates H, and satisfies

(R1) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$.
(R2) $s_\alpha\Phi = \Phi$ for all $\alpha \in \Phi$.

The elements of Φ are called *roots*.

Given a *total order* $<$ in W [7, p. 7] (there is one), $\lambda \in W$ is said to be *positive* if $0 < \lambda$. Now $\Phi^+ \subset \Phi$ is called a *positive system* if it consists of all those roots which are positive relative to a given total order. Of course,

$$\Phi = \Phi^+ \cup \Phi^-$$

where $\Phi^- = -\Phi^+$. Now Φ^+ contains [7, p. 8] a unique *simple system* Δ, that is, Δ is a basis for

$$W_1 := \operatorname{span}\Phi \subset W,$$

and each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). The vectors in Δ are called *simple roots* and the corresponding reflections are called *simple reflections*. The finite reflection group H is generated by the simple reflections.

The closed convex cone

$$F := \{\lambda \in W : (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\} \tag{1}$$

is called a (closed) *fundamental domain* for the action of H on W associated with Δ.

Let

$$W_0 := \{x \in W : hx = x \text{ for all } h \in H\}$$

be the set of fixed points in W under the action of H. The reflection group H is said to be **essential** relative to W if $W_0 = 0$, i.e., there is no nonzero fixed point under H. Clearly H is essential relative to $W_0^\perp$ and indeed $W_1 = W_0^\perp$.

The root system Φ is *irreducible*, i.e., there does not exist a nonempty subset $\Delta_1 \subsetneq \Delta$ satisfying $(\alpha, \beta) = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta \setminus \Delta_1$ [8, p. 27–28]. It is equivalent to say that H is irreducible, i.e., H cannot be decomposed as $H = H_1 \times H_2$ where H_1 and H_2 are nontrivial finite reflection groups, or equivalently W contains no proper H-invariant subspace. It is clear that if H is irreducible, then it is essential relative to W except the **trivial** case $\dim W = 1$ and $H = \{id\}$.

We denote the unique element in H of maximal length (relative to $\triangle$, [7, p. 15–16]) by $\omega_0 \in H$ and it sends F onto $-F$ and ω_0 is called the *longest element*. If $-id \in H$, then it is the longest element and H is essential relative to W. For an irreducible reflection group H, $-id \in H$ if and only if Φ is of type $A_1, B_k, D_{2k}, E_7, E_8, F_4, G_2(2k), H_3, H_4$ [8, p. 283]. Thus for a finite reflection group H essential to W, $-id \in H$ if and only if the irreducible components of H are of type A_1, B_k, D_{2k}, E_7, E_8, F_4, $G_2(2k)$, H_3, H_4.

Let

$$\Delta = \{\alpha_1, \dots, \alpha_n\},$$

that is, $\dim W_1 = n$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the basis of W_1 dual to the basis $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, that is, $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. Then

$$F = \left\{ \sum_{i=1}^{n} c_i \lambda_i : c_i \geq 0 \right\} \oplus W_0.$$

3. Eaton triple and representation of G-invariant norms

Let V be a finite dimensional real inner product space with the inner product $(\cdot, \cdot)$. Let G be a closed subgroup of the orthogonal group on V. The triple (V, G, F) is an *Eaton triple* [9, 13, 16, 17] if $F \subset V$ is a nonempty closed convex cone such that

(A1) $Gx \cap F$ is nonempty for each $x \in V$.
(A2) $\max_{g \in G} (x, gy) = (x, y)$ for all $x, y \in F$.

By considering the equality case of Cauchy-Schwarz inequality, it is easy to see that $Gx \cap F$ is a singleton set and we denote by $F(x)$ the unique element in the singleton set. For any nonzero $\alpha \in F$, we define

$$\|A\|_\alpha = (\alpha, F(A)).$$

Obviously $\|\cdot\|_\alpha$ is G-invariant. One can extend the definition of $\|\cdot\|_\alpha$ to nonzero $B \in V$:

$$\|A\|_B = (F(B), F(A)).$$

The Eaton triple (W, H, F) is called a *reduced* triple of the Eaton triple (V, G, F) if it is an Eaton triple and $W := \operatorname{span} F$ and $H := \{g|_W : g \in G,\ gW = W\} \subset O(W)$ [17]. It turns out that H is a finite reflection group [12]. There is a simple system Δ of H such that F in the Eaton triple (W, H, F) coincides with (1). We choose such Δ once and for all.

Theorem 3.1 ([19, Theorem 7]). *Let (V, G, F) be an Eaton triple with reduced triple (W, H, F) in which $\dim W > 1$ and H is not trivial. Suppose that $H \cong H_0 \times H_1 \times$*

$\cdots \times H_k$ is a decomposition of H into its irreducible components, where $H_i = H|_{W_i}$, with $H_0 = \{id\}$, $i = 1, \ldots, k$, and $W = W_0 + W_1 + \cdots + W_k$ is an orthogonal sum. Let $\alpha \in F$ be a nonzero vector. Then $\|\cdot\|_\alpha : V \to \mathbb{R}$ defined by

$$\|x\|_\alpha := (\alpha, F(x))$$

is a norm if and only if H is essential relative to W, $p_i(\alpha) \neq 0$, where $p_i : W \to W_i$ is the orthogonal projection onto W_i, for all $i = 1, \ldots, k$, and $\omega_0(\alpha) = -\alpha$, where ω_0 is the longest element of H.

Theorem 3.1 ensures that if H is irreducible and $-id \in H$, then every nonzero $\alpha \in F$ yields a G-invariant norm $\|\cdot\|_\alpha$.

Recall that we fix a simple system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of H so that $F = \{\sum_{i=1}^n c_i\lambda_i : c_i \geq 0\}$ where $\{\lambda_1, \ldots, \lambda_n\}$ is the basis of W dual to the basis $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$. So $\lambda_1, \ldots, \lambda_n$ form a "skeleton" of the cone F. We will show that $\|\cdot\|_{\lambda_j}$, $j = 1, \ldots, n$, also form a skeleton for the cone generated by the G-invariant norms on V. Indeed it provides a representation for the G-invariant norms in terms of $\|\cdot\|_{\lambda_j}$.

Theorem 3.2. *Let (V, G, F) be an Eaton triple with reduced triple (W, H, F). Suppose that the finite reflection group H is irreducible and $-id \in H$. The function $p : V \to \mathbb{R}$ is a G-invariant norm on V if and only if there exists a sequence of nonzero vectors $\{p^{(k)}\}_{k=1}^\infty$ in $\mathbb{R}^n$ of nonnegative entries such that*

$$p(x) = \sup_{k\geq 1} \sum_{j=1}^n p_j^{(k)} \|x\|_{\lambda_j},$$

where $n := \dim W$.

Proof. By Theorem 3.1, each $\|\cdot\|_{\lambda_j}$ is a norm on V since $\lambda_j \in F$ is nonzero. So the sufficiency is clear.

Suppose that p is a G-invariant norm on V. Fix a nonzero vector $a \in V$. By the Hahn-Banach theorem [15, p. 58], there exists a linear functional $(p_a, \cdot) \in V^*$ $(0 \neq p_a \in V)$ such that $(p_a, x) \leq p(x)$ for all $x \in V$ and $(p_a, a) = p(a)$. Define $L : V \to \mathbb{R}$ by

$$L(x) := (F(p_a), F(x)), \quad \text{for all } x \in V.$$

Since $(p_a, x) \leq p(x)$ for all $x \in V$ and p is G-invariant, $L(x) \leq p(x)$ for all $x \in V$ and $L(a) = p(a)$. Recall that $\{\lambda_1, \ldots, \lambda_n\}$ is the basis of $W = W_1$ dual to the basis $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$. So $z = \sum_{j=1}^n (z, \alpha_j^\vee)\lambda_j$ for each $z \in V$. Therefore

$$L(x) = (F(p_a), F(x)) = \sum_{j=1}^n (F(p_a), \alpha_j^\vee)(\lambda_j, F(x)) = \sum_{j=1}^n (F(p_a), \alpha_j^\vee)\|x\|_{\lambda_j}.$$

So

$$L(x) = \sum_{j=1}^n p_a^{(j)} \|x\|_{\lambda_j} \leq p(x) \text{ for all } x \in V, \qquad L(a) = p(a)$$

where $p_a^{(j)} := (F(p_a), \alpha_j^\vee) \geq 0$ for all j and they cannot be all zero since $p_a \neq 0$.

Now take a dense and countable subset $\{a_k\}_{k=1}^\infty$ of $V \setminus \{0\}$. For $j = 1, \ldots, n$ and for all $k \geq 1$, set

$$p_j^{(k)} := p_{a_k}^{(j)} \geq 0$$

and they are not all zero for a fixed k. Then for $k \geq 1$

$$\sum_{j=1}^n p_j^{(k)} \|x\|_{\lambda_j} \leq p(x) \quad \text{for all } x \in V, \qquad \sum_{j=1}^n p_j^{(k)} \|a_k\|_{\lambda_j} = p(a_k).$$

Set

$$q(x) := \sup_{k \geq 1} \sum_{j=1}^n p_j^{(k)} \|x\|_{\lambda_j}, \quad x \in V.$$

Clearly $q(x) \leq p(x)$ for all $x \in V$. Therefore q is a norm and thus continuous. Finally since $q(a_k) = p(a_k)$ for all $k \geq 1$, we have $q = p$. $\square$

We thus have the following dominance result which extends Ky Fan's dominance theorem [1, p. 93].

Corollary 3.3. *With the assumption of Theorem* 3.2 *let $x, y \in V$. Then $p(x) \leq p(y)$ for all G-invariant norm $p(\cdot)$ if and only if $\|x\|_{\lambda_j} \leq \|y\|_{\lambda_j}$ for all $j = 1, \ldots, n$.*

Example 3.4. (a) Let $G := U(n) \otimes U(n)$ which acts on $\mathbb{C}_{n\times n}$ via $(U \otimes V)(A) = UAV$ for all $U, V \in U(n)$ where $U(n)$ is the unitary group. Let F be the cone of nonnegative diagonal matrices with diagonal entries in descending order. Then $(\mathbb{C}_{n\times n}, G, F)$ is an Eaton triple with reduced triple (W, H, F) where W is the space of diagonal matrices which can be identified with $\mathbb{R}^n$. With the identification, $H = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ the semidirect product of the symmetric group and $(\mathbb{Z}/2\mathbb{Z})^n$ (sign changes), i.e., H is of type B_n. So H acts on $\mathbb{R}^n$ by permuting the entries of $x \in \mathbb{R}^n$ and changing signs.

The G-invariant norm is simply a unitarily invariant norm. With the cone F identified as a subset of $\mathbb{R}^n$, we have

$$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\}$$

and $\lambda_j = \sum_{i=1}^j e_i$, $j = 1, \ldots, n-1$, $\lambda_n = \frac{1}{2}\sum_{i=1}^n e_i$. Thus each unitarily invariant norm $\|\cdot\|$ can be represented as

$$\|A\| = \sup_{k \geq 1} \sum_{j=1}^n p_j^{(k)} \|A\|_j, \quad A \in \mathbb{C}_{n\times n},$$

where $\{p^{(k)}\}_{k=1}^\infty$ is a sequence of nonzero vectors in $\mathbb{R}^n$ of nonnegative entries. Here $\|A\|_j$ is the sum of the largest j singular values of $|A|$, $j = 1, \ldots, n-1$, i.e., Ky Fan j-norm, and $\|A\|_n = \frac{1}{2}\sum_{i=1}^n |s_i(A)|$, where $s_i(A)$ denotes the ith largest singular value of A. Then Corollary 3.3 yields Ky Fan dominance theorem.

We remark that the result is also true for $\mathbb{C}_{p\times q}$.

(b) If we choose $V = \mathbb{R}^n$ $(n \geq 2)$ and $G = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, then F is the set of all nonnegative n-tuples with descending entries. Then $(V, G, F) = (W, G, F)$ and

$$\|x\| = \sup_{k \geq 1} \sum_{j=1}^{n} p_j^{(k)} \|x\|_j, \quad x \in \mathbb{R}^n,$$

where $\{p^{(k)}\}_{k=1}^{\infty}$ is a sequence of nonzero vectors in $\mathbb{R}^n$. Here $\|x\|_j$ is the sum of the largest j entries of $|x|$, $j = 1, \ldots, n-1$ and $\|x\|_n = \frac{1}{2} \sum_{i=1}^{n} |x_i|$.

Example 3.5. When $V = \mathbb{R}^{2n}$ $(n \geq 2)$ and $G = S_{2n} \ltimes (\mathbb{Z}/2\mathbb{Z})^{2n-1}$ the semidirect product of the symmetric group S_{2n} and $(\mathbb{Z}/2\mathbb{Z})^{2n-1}$ (even number of sign changes), i.e., G is of type D_{2n}. So G acts on $\mathbb{R}^{2n}$ by permuting the entries of $x \in \mathbb{R}^{2n}$ and even number of changing signs. Let

$$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{2n-1} - e_{2n}, e_{2n-1} + e_{2n}\}.$$

So $\lambda_j = \sum_{i=1}^{j} e_i$, $j = 1, \ldots, 2n-2$, $\lambda_{2n-1} = \frac{1}{2} \sum_{i=1}^{2n-1} e_i - e_{2n}$, $\lambda_{2n} = \frac{1}{2} \sum_{i=1}^{2n} e_i$. Thus each G-invariant norm $\|\cdot\|$ can be represented as

$$\|x\| = \sup_{k \geq 1} \sum_{j=1}^{2n} p_j^{(k)} \|x\|_j, \quad x \in \mathbb{R}^{2n},$$

where $\|x\|_j$ is the sum of the largest j entries of $|x|$, if $1 \leq j \leq 2n-2$,

$$\|x\|_{2n-1} = \sum_{j=1}^{2n} |x_i| - 2 \min_{i=1,\ldots,2n} |x_i|, \qquad \|x\|_{2n} = \frac{1}{2} \sum_{i=1}^{2n} |x_i|.$$

We have similar result for special orthogonal invariant norm on $\mathbb{R}_{n \times n}$.

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