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A Characteristic Intersection Property of Generalized Simplices

Valeriu Soltan
*Department of Mathematical Sciences, George Mason University,
Fairfax, Virginia 22030, USA*
vsoltan@gmu.edu

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Following Rockafellar (1970), a generalized n-simplex in $\mathbb{R}^n$ is defined as the direct sum of an m-simplex and a simplicial $(n-m)$-cone, $0 \leq m \leq n$. Fourneau (1977) showed that a line-free n-dimensional closed convex set $K \subset \mathbb{R}^n$ is a generalized n-simplex if and only if all n-dimensional intersections $K \cap (v+K)$, $v \in \mathbb{R}^n$, are homothetic to K. We extend this characteristic property by proving that for a pair of line-free n-dimensional closed convex sets K_1 and K_2 in $\mathbb{R}^n$ the following two conditions are equivalent: 1) all n-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, belong to a unique homothety class of convex sets, 2) K_1 and K_2 are generalized n-simplices whose n-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, are homothetic to a unique generalized n-simplex.

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1. Introduction and Main Result

A convex set K in a linear space of any dimension is called a *Choquet simplex* if any nonempty intersection of two homothetic copies of K is either a homothetic copy of K or a singleton:

$$(u + \lambda K) \cap (v + \mu K) = w + \nu K, \quad \lambda, \mu > 0, \, \nu \geq 0. \tag{1}$$

We recall that sets S and T are *homothetic* provided $S = x + \lambda T$ for a suitable vector x and a scalar $\lambda > 0$. Various properties of Choquet simplices, widely used in partially ordered linear spaces and the Choquet representation theory (see, e.g., [7, 15]), initiated a comprehensive study of their structure in the Euclidean space $\mathbb{R}^n$ (see, e.g., [13] for an overview).

Rogers and Shephard [9] proved that a convex body $K \subset \mathbb{R}^n$ is a usual n-simplex (i.e., K is the convex hull of $n+1$ affinely independent points) if and only if every nonempty intersection of K and a translate of K is either a homothetic copy of K or a singleton:

$$K \cap (v + K) = w + \lambda K, \quad v, w \in \mathbb{R}^n, \; \lambda \geq 0. \tag{2}$$

This fact obviously implies that a compact convex set in $\mathbb{R}^n$ is a Choquet simplex if and only if it is a usual simplex. A more general result of Gruber [3, Satz 2] shows

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that a line-free closed convex set $K \subset \mathbb{R}^n$ is a Choquet simplex if and only if it is a simplex or a simplicial cone

Since the relation of homothety is an equivalence relation, the family of convex sets in $\mathbb{R}^n$ is a disjoint union of homothety classes (see, e.g., [14] for a study of vector addition and subtraction of homothety classes of convex sets). So, a convex set $K \subset \mathbb{R}^n$ is a Choquet simplex if and only if each nonempty intersection of two homothetic copies of K either belongs to the homothety class containing K or is a singleton. In these terms, the characteristic intersection property (2) of n-simplices was generalized in [11] to the case of two convex bodies as follows. Convex bodies K_1 and K_2 in $\mathbb{R}^n$ are homothetic simplices if and only if they satisfy the following condition (I): every nonempty intersection $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, belongs to a unique homothety class of convex bodies or is a singleton. This characteristic property was used by Schneider [10] to study translation mixtures of convex bodies. As shown in [12], line-free closed convex sets K_1 and K_2 of dimension n in $\mathbb{R}^n$ that satisfy condition (I) above are Choquet simplices, not necessarily homothetic to each other.

To avoid multiple repetitions of the expression "closed convex sets of dimension n in $\mathbb{R}^n$, distinct from $\mathbb{R}^n$ and possibly unbounded", we will call these sets *convex solids*. If K is a convex solid in $\mathbb{R}^n$, then the condition "either a homothetic copy or a singleton" in (1) or (2) implies that any *nonempty* intersection $(u+\lambda K) \cap (v+\mu K)$ or $K \cap (v+K)$ of dimension $n-1$ or less should be a singleton. Fourneau [2] relaxed this condition by considering convex solids $K \subset \mathbb{R}^n$ such that the *n-dimensional* intersections of any two homothetic copies of K are homothetic to K (with no condition on nonempty intersections of dimension $n-1$ or less):

$$(u + \lambda K) \cap (v + \mu K) = w + \nu K, \quad u, v, w \in \mathbb{R}^n, \ \lambda, \mu, \nu > 0. \tag{3}$$

As proved in [2], a line-free convex solid $K \subset \mathbb{R}^n$ satisfies condition (3) if and only if K is a generalized simplex introduced by Rockafellar [8, p. 154], that is, K is the direct sum of an m-simplex and a simplicial $(n-m)$-cone:

$$K = \operatorname{conv}\{x_0, x_1, \ldots, x_m\} + \sum_{i=m+1}^{n} [x_0, x_i), \quad 0 \le m \le n,$$

where $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$ are affinely independent points and $[x_0, x_i)$ means the halfline through x_i originated at x_0. Clearly, usual n-simplices and simplicial n-cones are particular cases of generalized n-simplices. Hinrichsen and Krause [4] showed that any line-free convex polyhedron in $\mathbb{R}^n$ can be partitioned into finitely many generalized simplices.

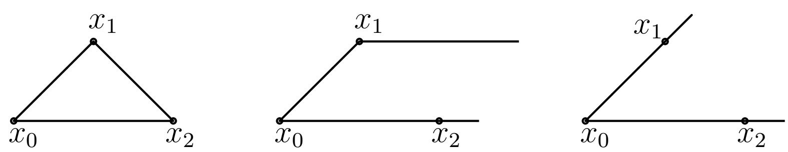


Figure 1.1: Generalized 2-simplices in the plane.

Similarly to [11, 12], we extend below Fourneau's characteristic intersection property of generalized simplices to the case of a pair of convex solids K_1 and K_2 in $\mathbb{R}^n$. Recall that the support cone $C_x(K)$ of a convex solid $K \subset \mathbb{R}^n$ at its boundary point x is the closure of the union of the halflines $[x, z)$, $z \in K \setminus \{x\}$. Our main result is given in the following theorem.

Theorem 1.1. *For line-free convex solids K_1 and K_2 in $\mathbb{R}^n$, conditions 1)–3) below are equivalent:*

1) all n-dimensional intersections $(u + \lambda K_1) \cap (v + \mu K_2)$, $u, v \in \mathbb{R}^n$, $\lambda, \mu > 0$, belong to a unique homothety class of convex solids,

2) all n-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, belong to a unique homothety class of convex solids,

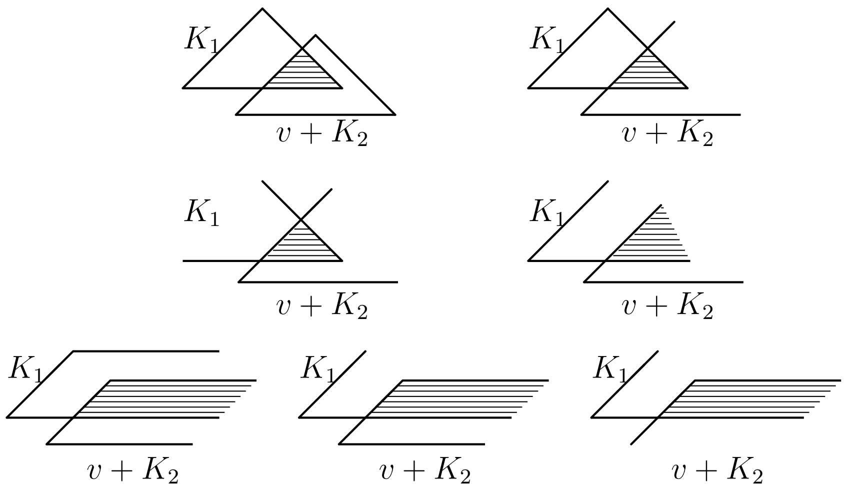
3) both K_1 and K_2 are generalized n-simplices, and there is a generalized n-simplex $K \subset \mathbb{R}^n$ such that all n-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, are homothetic to K. Furthermore, K_1, K_2, and K satisfy either of the following three conditions:

(a) both K_1 and K_2 are homothetic to K,

(b) one of K_1, K_2, say K_1, is homothetic to K, and K_2 is a translate of a support cone $C_x(K)$, where x is a vertex of K,

(c) K_1 and K_2 are translates of support cones $C_x(K)$ and $C_z(K)$, respectively, where x and z are distinct vertices of K.

For $n = 2$, we have the following description of all convex solids K_1 and K_2 that satisfy Theorem 1.1 (the shaded regions are homothetic to K).



We remark that if at least one of the convex solids K_1 and K_2 in Theorem 1.1 is not assumed to be line-free, then we should not expect them to be even polyhedral. For example, if

$$K_1 = \{(x, y, z) \mid z \geq \sqrt{x^2 + y^2} - 1\} \quad \text{and} \quad K_2 = \{(x, y, z) \mid z \leq 0\},$$

then all 3-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^3$, are homothetic copies of the bounded circular cone $K_1 \cap K_2$. The problem to describe all pairs of convex solids $K_1, K_2 \subset \mathbb{R}^n$, not necessarily line-free, that satisfy condition *1*) or *2*) of Theorem 1.1 remains open.

2. Preliminaries

For the convenience of the reader we provide in this section necessary definitions, notation, and auxiliary properties of convex sets in $\mathbb{R}^n$. In what follows, 0 means the origin (zero vector) of $\mathbb{R}^n$. Let us recall that a *face* of a convex set $M \subset \mathbb{R}^n$ is a convex subset F of M such that for any points $x, y \in M$ and a scalar $\lambda \in]0,1[$ the inclusion $(1-\lambda)x + \lambda y \in F$ implies that $x, y \in F$. Zero-dimensional faces of M are called *extreme* points of M and their union is denoted $\operatorname{ext} M$. An *exposed face* of M is the intersection of M with any hyperplane that supports M. Zero-dimensional exposed faces are called *exposed points* of M and their union is denoted $\exp M$.

A convex set $M \subset \mathbb{R}^n$ is called *line-free* provided it contains no lines. Any closed convex set in $\mathbb{R}^n$ can be expressed as the direct sum of a subspace and a closed line-free convex set. The following well-known results of Klee [5, 6] describe the extremal structure of line-free closed convex sets.

1. A closed convex set $M \subset \mathbb{R}^n$ is line-free if and only if it has at least one extreme point (respectively, at least one exposed point).
2. Any line-free closed convex set in $\mathbb{R}^n$ is the convex hull of its extreme points and extreme halflines.
3. Any line-free closed convex set in $\mathbb{R}^n$ is the closed convex hull of its exposed points and exposed halflines.
4. For any line-free closed convex set $M \subset \mathbb{R}^n$, the set $\exp M$ is dense in $\operatorname{ext} M$.

As usual, $\operatorname{bd} K$ and $\operatorname{int} K$ denote, respectively, the boundary and the interior of a convex solid $K \subset \mathbb{R}^n$. Furthermore, $\operatorname{rint} M$ stands for the relative interior of a convex set M in $\mathbb{R}^n$. We say that a closed halfspace P supports a closed convex set M provided the boundary hyperplane of P supports M and the interior of P is disjoint from M. By a convex n-polyhedron (in particular, convex n-polytope, n-simplex, or simplicial n-cone) we mean a closed convex polyhedral set of dimension n.

The recession cone $\operatorname{rec} M$ of a closed convex set $M \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} M = \{y \in \mathbb{R}^n \mid x + \lambda y \in M \text{ for all } x \in M \text{ and } \lambda \geq 0\}.$$

It is well known that $\operatorname{rec} M \neq \{0\}$ if and only if M is unbounded (see, e.g., [8, Theorem 8.4]). Generally, by a *convex cone* with apex p we mean a convex set $C \subset \mathbb{R}^n$ such that $p + \lambda(x - p) \in C$ for all $\lambda \geq 0$ and $x \in C$.

Lemma 2.1. *For any boundary point x of a closed convex set $M \subset \mathbb{R}^n$, the support cone $C_x(M)$ is a closed convex cone with apex x, distinct from $\mathbb{R}^n$. Furthermore, $C_x(M) = C_x(M \cap B_r(x))$ for any closed ball $B_r(x) \subset \mathbb{R}^n$ with center x and radius $r > 0$.* □

Lemma 2.2. *If u and v are distinct exposed points of a convex solid $K \subset \mathbb{R}^n$, then none of the cones $C_u(K)$, $u - v + C_v(K)$ lies in the other.*

Proof. Assume, for contradiction, that $u - v + C_v(K) \subset C_u(K)$ (the opposite inclusion is considered similarly). Let $P \subset \mathbb{R}^n$ be a closed halfspace with the property $K \cap P = \{u\}$. Clearly, P supports $C_u(K)$ and whence supports $u - v + C_v(K)$. Then the halfspace $v - u + P$ supports $C_v(K)$. As a result, $v - u + P$ supports K and $v \in K \cap (v - u + P)$. The parallel halfspaces P and $v - u + P$, both supporting K, should coincide: $P = v - u + P$. Finally, $v \in K \cap (v - u + P) = K \cap P = \{u\}$, in contradiction with $u \neq v$. □

Lemma 2.3. *Let $K \subset \mathbb{R}^n$ be a line-free convex solid and $P \subset \mathbb{R}^n$ be a closed halfspace with the property $P \cap \operatorname{rec} K = \{0\}$. Then any translate of P intersects K along a bounded set, and there is a translate of P that supports K.* □

Lemma 2.4. *Let $K \subset \mathbb{R}^n$ be a line-free convex solid and v be an exposed point of K isolated in $\exp K$. Then K is locally conic at v, that is, there is a closed ball $B_r(v) \subset \mathbb{R}^n$ such that $[v, z] \subset \operatorname{bd} K$ for all $z \in B_r(v) \cap \operatorname{bd} K$.*

Proof. Assume, for contradiction, the existence of a sequence $z_1, z_2, \ldots \in \operatorname{bd} K$ convergent to v such that $[v, z_i] \not\subset \operatorname{bd} K$ for all $i = 1, 2, \ldots$ Choose a closed halfspace $P \subset \mathbb{R}^n$ with the property $P \cap K = \{v\}$. We can write $P = \{x \in \mathbb{R}^n \mid f(x) \le \alpha\}$, where f is a linear functional on $\mathbb{R}^n$ and α is a scalar. Then $f(z_i) \to \alpha$ as $i \to \infty$. Let F_i denote the smallest face of K that contains z_i. Clearly, $F_i \subset \operatorname{bd} K$ because of $z_i \in \operatorname{bd} K$. We have $v \notin F_i$, since otherwise $[v, z_i] \subset F_i \subset \operatorname{bd} K$. Being line-free, F_i contains an extreme point u_i, which is an extreme point of K. Furthermore, u_i can be chosen such that $f(u_i) \le f(z_i)$, since otherwise $F_i \subset \{x \in \mathbb{R}^n \mid f(x) > f(z_i)\}$, in contradiction with $z_i \in F_i$. Due to $P \cap K = \{v\}$, all points $u_1, u_2, \ldots$ are distinct from v and the diameter of $K \cap \{x \in \mathbb{R}^n \mid f(x) \le f(u_i)\}$ tends to 0 as $i \to \infty$. Hence $\lim_{i\to\infty} u_i = v$. Since the set $\exp K$ is dense in $\operatorname{ext} K$, there is a sequence $w_1, w_2, \ldots \in \exp K \setminus \{v\}$ convergent to v, in contradiction with the assumption. □

3. Proof of Theorem 1.1

3) ⇒ *1)* If convex solids K_1, K_2, and K satisfy condition *3)* of the theorem, then either K is a simplicial cone, which can be described in suitable coordinates as

$$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge 0, \ldots, x_n \ge 0\}, \tag{4}$$

or K is a generalized simplex, which can be described in suitable coordinates as

$$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge 0, \ldots, x_n \ge 0,\ x_1 + \cdots + x_m \le 1\}, \tag{5}$$

where m is an integer between 1 and n (see [2]).

In case (4), condition *3)* implies that both K_1 and K_2 are translates of K: $K_1 = p + K$ and $K_2 = q + K$ for suitable vectors $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$. Then for any vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ and scalars $\lambda, \mu > 0$, we have

$$\begin{aligned}(u + \lambda K_1) \cap (v + \mu K_2) &= (u + p + K) \cap (v + q + K) \\ &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \ge \max(u_1 + p_1, v_1 + q_1), \ldots, \\ &\qquad x_n \ge \max(u_n + p_n, v_n + q_n)\}.\end{aligned}$$

Equivalently, $(u + \lambda K_1) \cap (v + \mu K_2) = w + K$, where

$$w = (\max(u_1 + p_1, v_1 + q_1), \ldots, \max(u_n + p_n, v_n + q_n)).$$

Dealing with case (5), we consider only the most elaborated condition *3*)(c) of the theorem, since conditions *3*)(a) and *3*)(b) are similar. Then K_1 and K_2 are translates of support cones $C_x(K)$ and $C_z(K)$, where x and z are distinct vertices of K. Let $K_1 = p + C_x(K)$ and $K_2 = q + C_z(K)$ for some vectors $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ in $\mathbb{R}^n$. Without loss of generality, we may assume that $x = (1, 0, \dots, 0)$ and $z = (0, 0, \dots, 0)$. Then

$$\begin{aligned} K_1 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_2 \ge p_2, \dots, x_n \ge p_n, \\ x_1 + \dots + x_m \le 1 + p_1 + \dots + p_m\}, \\ K_2 = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \ge q_1, \dots, x_n \ge q_n\}. \end{aligned}$$

For any vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in $\mathbb{R}^n$ and scalars $\lambda, \mu > 0$, the set $(u + \lambda K_1) \cap (v + \mu K_2)$ equals

$$\begin{aligned} \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \ge v_1 + \mu q_1, x_2 \ge \max(u_2 + \lambda p_2, v_2 + \mu q_2), \dots, \\ x_n \ge \max(u_n + \lambda p_n, v_n + \mu q_n),\ x_1 + \dots + x_m \le \xi_m\}. \end{aligned}$$

where

$$\xi_m = u_1 + \dots + u_m + \lambda(1 + p_1 + \dots + p_m).$$

The intersection $(u+\lambda K_1) \cap (v+\mu K_2)$ is n-dimensional if and only if $\xi_m > \eta_m$, where

$$\eta_m = v_1 + \mu q_1 + \max(u_2 + \lambda p_2, v_2 + \mu q_2) + \dots + \max(u_m + \lambda p_m, v_m + \mu q_m).$$

Hence, if the set $(u + \lambda K_1) \cap (v + \mu K_2)$ is n-dimensional, then it can be expressed as $w + \nu K$, with $\nu = \xi_m - \eta_m$ and

$$w = (v_1 + \mu q_1, \max(u_2 + \lambda p_2, v_2 + \mu q_2), \dots, \max(u_n + \lambda p_n, v_n + \mu q_n)).$$

2) $\Rightarrow$ *3*) Since *1*) trivially implies *2*), it remains to show that *2*) implies *3*). Let K_1 and K_2 be convex solids in $\mathbb{R}^n$ that satisfy condition *2*) of the theorem. Denote by K a convex solid that generates the homothety class containing all n-dimensional intersections $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$. The following three lemmas will be of use later.

Lemma 3.1. *If there is a vector $v \in \mathbb{R}^n$ such that* $\operatorname{int}(v + K_2)$ *contains two distinct exposed points of K_1 or an exposed line segment of K_1, then $K_1 \subset \operatorname{int}(v + K_2)$.*

Proof. Assume the existence of a vector $v \in \mathbb{R}^n$ such that $K_1 \not\subset \operatorname{int}(v + K_2)$ while $\operatorname{int}(v+K_2)$ contains a pair of distinct points $x, z \in \exp K_1$ or an exposed line segment $[x, z]$ of K_1. Choose a point $u \in K_1 \setminus \operatorname{int}(v + K_2)$. Then there is a scalar $\varepsilon > 0$ so small that the convex solid $v + \varepsilon(x - u) + K_2$ still contains both points x and z in its interior. Clearly, $u \notin v + \varepsilon(x - u) + K_2$. If $x, z \in \exp K_1$, then let P_x and P_z be closed halfspaces in $\mathbb{R}^n$ which satisfy the conditions $K_1 \cap P_x = \{x\}$ and $K_1 \cap P_z = \{z\}$; otherwise let P be a closed halfspace such that $K_1 \cap P = [x, z]$. Put

$$K' = K_1 \cap (v + K_2), \qquad K'' = K_1 \cap (v + \varepsilon(x - u) + K_2).$$

By condition *2*), both K' and K'' are homothetic to K. If $x, z \in \exp K_1$, then from

$$K' \cap P_x = K'' \cap P_x = \{x\}, \qquad K' \cap P_z = K'' \cap P_z = \{z\}$$

we conclude that $K' = K''$. Similarly, if $[x, z]$ is an exposed line segment of K_1, then the homotheticity of K' and K'' and the relation $K' \cap P = K'' \cap P = [x, z]$ again imply that $K' = K''$. In both cases, the line segments $[x, u] \cap K'$ and $[x, u] \cap K''$ should be of the same length, which is impossible by the choice of ε. $\square$

Given convex solids K and M in $\mathbb{R}^n$, we say that exposed points $x \in \exp K$ and $z \in \exp M$ *correspond to each other* provided $C_x(K)$ is a translate of $C_z(M)$.

Lemma 3.2. *Any exposed point of K_1 (respectively, of K_2) corresponds to an exposed point of K.*

Proof. Let $z \in \exp K_1$ (the case $z \in \exp K_2$ is similar). Choose a vector $v \in \mathbb{R}^n$ such that $z \in \operatorname{int}(v + K_2)$ and let $K' = K_1 \cap (v + K_2)$. Clearly, $z \in \exp K'$ and Lemma 2.1 implies that $C_z(K') = C_z(K_1)$. Since $K' = w + \lambda K$ for suitable $w \in \mathbb{R}^n$ and $\lambda > 0$, we obtain that z corresponds to the exposed point $u = \lambda^{-1}(z - w)$ of K. $\square$

We recall that a boundary point x of a convex solid $M \subset \mathbb{R}^n$ is *regular* provided there is a unique hyperplane supporting M at x. Denote by $N(M)$ the family of outward unit normals to the convex solid M at its regular points. Clearly, a convex solid $M \subset \mathbb{R}^n$ is a convex n-polyhedron if and only if the set $N(M)$ is finite.

Lemma 3.3. *If K is a convex n-polyhedron, then both K_1 and K_2 are convex n-polyhedra and $N(K_1) \cup N(K_2) \subset N(K)$.*

Proof. Let x be a regular boundary point of K_1. Denote by e the outward unit normal to K_1 at x. Choose a point $v \in \mathbb{R}^n$ such that $x \in \operatorname{int}(v + K_2)$. Clearly, x is a regular point of the convex solid $K' = K_1 \cap (v + K_2)$ and e is the outward unit normal to K' at x. Since K' is homothetic to K, we have $e \in N(K') = N(K)$. Hence $N(K_1) \subset N(K)$. Similarly, $N(K_2) \subset N(K)$.

If K is a convex n-polyhedron, then $N(K)$ is finite and both sets $N(K_1)$ and $N(K_2)$ are finite. This obviously implies that K_1 and K_2 are convex n-polyhedra. $\square$

Before proceeding with the proof of *2*) $\Rightarrow$ *3*), we make several remarks.

A) First of all, we eliminate the case when K is bounded, since this case is considered in [11] when both K_1 and K_2 are bounded, and it can be easily deduced from the proof in [12] when at least one of K_1, K_2 is unbounded. Thus we may assume, in what follows, that all three convex solids K_1, K_2, and K are unbounded.

B) Furthermore, we may suppose that $\dim(\operatorname{rec} K) < n$. Indeed, if the cone $\operatorname{rec} K$ is n-dimensional, then any nonempty intersection $K' = K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, is n-dimensional and, by condition *2*), is homothetic to K. In this case, K is a simplicial cone and both K_1 and K_2 are translates of K (see the proof in [12]).

C) Finally, we state that $\operatorname{rec} K = \operatorname{rec} K_1 \cap \operatorname{rec} K_2$. Indeed, choose a vector $v \in \mathbb{R}^n$ such that the set $K' = K_1 \cap (v + K_2)$ is n-dimensional. Since K and K' are homothetic, we obtain from [8, Corollary 8.3.3] that

$$\operatorname{rec} K = \operatorname{rec} K' = \operatorname{rec} K_1 \cap \operatorname{rec}(v + K_2) = \operatorname{rec} K_1 \cap \operatorname{rec} K_2. \tag{6}$$

We divide the proof of *2)* $\Rightarrow$ *3)* into the following three cases.

I. $\operatorname{rec} K_1 = \operatorname{rec} K_2$.
II. $\operatorname{rec} K_1 \subsetneq \operatorname{rec} K_2$ (the case $\operatorname{rec} K_2 \subsetneq \operatorname{rec} K_1$ is similar).
III. $\operatorname{rec} K_1 \not\subset \operatorname{rec} K_2$ and $\operatorname{rec} K_2 \not\subset \operatorname{rec} K_1$.

Case I. $\operatorname{rec} K_1 = \operatorname{rec} K_2$. By induction on n, we are going to prove that K is a generalized n-simplex and both K_1 and K_2 are homothetic to K. Since the case $n = 1$ trivially holds, we assume that $n \geq 2$ and that *2)* implies *3)* for all $\mathbb{R}^d$ with $d \leq n-1$. We divide the proof of *Case I* into Lemmas 3.4–3.7.

Lemma 3.4. *If* $\operatorname{rec} K_1 = \operatorname{rec} K_2$*, then any exposed point of* K *corresponds to an exposed point of* K_1 (*respectively, of* K_2).

Proof. Let x be an exposed point of K and $P \subset \mathbb{R}^n$ be a closed halfspace such that $P \cap K = \{x\}$. If P_0 is a translate of P with the property $0 \in \operatorname{bd} P_0$, then $P_0 \cap \operatorname{rec} K = \{0\}$. Since $\operatorname{rec} K_1 = \operatorname{rec} K$, Lemma 2.3 implies the existence of a translate P_1 of P which supports K_1. Choose a point $x_1 \in P_1 \cap K_1$, and let $v \in \mathbb{R}^n$ be such that $x_1 \in \operatorname{int}(v + K_2)$. By condition *2)*, the convex solid $K' = K_1 \cap (v + K_2)$ is homothetic to K. Then $P_1 \cap K' = \{x_1\}$, which shows that x_1 is an exposed point of K' such that $C_{x_1}(K')$ is a translate of $C_x(K)$. Since $C_{x_1}(K_1) = C_{x_1}(K')$, we conclude that x corresponds to $x_1 \in \exp K_1$. $\square$

Lemma 3.5. *If* $\operatorname{rec} K_1 = \operatorname{rec} K_2$*, then* K *has finitely many exposed points.*

Proof. Let x and z be distinct exposed points of K. By Lemma 3.4, there are points $x_1 \in \exp K_1$ and $z_2 \in \exp K_2$ such that x corresponds to x_1 and z corresponds to z_2. Consider the intersection $K' = K_1 \cap (x_1 - z_2 + K_2)$. Clearly, x_1 is an exposed point of K'.

We claim that $\dim K' < n$. Indeed, assume for a moment that $\dim K' = n$. Since K' is homothetic to K, the point $x_1 \in \exp K'$ corresponds to a point $v \in \exp K$. Because $x \neq z$, the point v is distinct from one of x and z. Let $v \neq x$. Then

$$C_v(K) = v - x_1 + C_{x_1}(K') \subset v - x_1 + C_{x_1}(K_1) = v - x + C_x(K),$$

in contradiction with Lemma 2.2. Hence $\dim K' < n$.

Choose a hyperplane H that contains K' such that K_1 and $x_1 - z_2 + K_2$ lie in distinct closed halfspaces determined by H. Then both hyperplanes which are parallel to H and contain the points x and z, respectively, support K from opposite sides. Let h be a halfline that lies in $\operatorname{rec} K$ ($\operatorname{rec} K \neq \{0\}$ because K is unbounded by A) above), and let l be the line that contains h. Clearly, l is parallel to H. Choose a subspace G of dimension $n-1$ complementary to l. For any point $x \in \exp K$, denote by x' the point of intersection of G and the line through x parallel to l. Put $X = \{x' \mid x \in \exp K\}$. Since $x + h \subset K$ for any point $x \in \exp K$, the set X is in one-to-one correspondence with $\exp K$.

Choose any points $u', v' \in X$, and let u, v be the points in $\exp K$ corresponding to them. By the above, there is a pair of distinct parallel hyperplanes M_u and M_v both supporting K and containing u and v, respectively. Since l is parallel to both M_u

and M_v, the intersections $G \cap M_u$ and $G \cap M_v$ are parallel $(n-2)$-dimensional planes in G such that $u' \in G \cap M_u$, $v' \in G \cap M_v$, and X lies between $G \cap M_u$ and $G \cap M_v$. As proved in [1], X has 2^{n-1} or fewer elements. Hence $|\exp K| = |X| \leq 2^{n-1}$. □

Lemma 3.6. *If* $\operatorname{rec} K_1 \subset \operatorname{rec} K_2$ *and* K *has finitely many exposed points, then* K *is a convex polyhedron.*

Proof. Due to Lemma 3.5, it is sufficient to show that K has finitely many exposed halflines. Assume, for contradiction, that K has infinitely many exposed halflines. Since the endpoints of exposed halflines of K are extreme points of K and since the set $\operatorname{ext} K$ is finite (because $\exp K$ is dense in $\operatorname{ext} K$ and finite), there is an extreme point x of K such that infinitely many exposed halflines of K, say $h_1, h_2, \ldots$, have x as a common endpoint. By a compactness argument, we may assume that the sequence $h_1, h_2, \ldots$ converges to a halfline $h \subset K$ with endpoint x. Furthermore, because the halflines $h_i - x$, $i = 1, 2, \ldots$, lie in $\operatorname{rec} K$ and because the cone $\operatorname{rec} K$ is closed, the halfline $h - x$ also lies in $\operatorname{rec} K$.

From (6) it follows that $h - x$ lies in $\operatorname{rec} K_1 \cap \operatorname{rec} K_2 = \operatorname{rec} K_1$. Since K_2 is line-free, the opposite halfline $x - h$ does not lie in $\operatorname{rec} K_2$. Hence there is a vector $v \in \mathbb{R}^n$ such that $x \notin v + K_2$ and h intersects $\operatorname{int}(v + K_2)$. By a continuity argument, there is an integer m such that the halfline h_i intersects $\operatorname{int}(v + K_2)$ for all $i \geq m$. Denote by z_i the point of intersection of h_i and $\operatorname{bd} K$, $i \geq m$, and let z be the point of intersection of h and $\operatorname{bd} K$. Clearly, $z_m, z_{m+1}, \ldots$ are distinct extreme points of the convex solid $K' = K_1 \cap (v + K_2)$ and $z_i \to z$ when $i \to \infty$. Since K is homothetic to K', the convex solid K has infinitely many extreme points, contradicting Lemma 3.5. Hence K is a convex polyhedron. □

Lemma 3.7. *If* $\operatorname{rec} K_1 = \operatorname{rec} K_2$, *then* K *is a generalized* n*-simplex and both* K_1 *and* K_2 *are homothetic to* K.

Proof. By Lemmas 3.3, 3.5, and 3.6, K_1, K_2, and K are convex n-polyhedra with $N(K_1) \cup N(K_2) \subset N(K)$. We state that $N(K_1) = N(K_2) = N(K)$. Indeed, let $e \in N(K)$ and F be the facet of K with the outward unit normal e. Since F is line-free (as a face of the line-free convex polyhedron K), it has a vertex, x. Clearly, F lies in a facet of the convex polyhedral cone $C_x(K)$. By Lemma 3.4, there is a vertex z of K_1 such that $C_z(K_1)$ is a translate of $C_x(K)$. Hence K_1 has a facet with the outward unit normal e. Thus $N(K) \subset N(K_1)$. Similarly, $N(K) \subset N(K_2)$.

Next we state that any facets $F_1 \subset K_1$ and $F_2 \subset K_2$ with the same outward unit normal, f, are homothetic generalized $(n-1)$-simplices. Indeed, denote by H the hyperplane that contains F_1 and consider a vector $v \in \mathbb{R}^n$ such that $v + F_2 \subset H$ and $\operatorname{rint} F_1 \cap \operatorname{rint}(v + F_2) \neq \varnothing$. Then $\operatorname{int} K_1 \cap \operatorname{int}(v + K_2) \neq \varnothing$, implying that $K_1 \cap (v + K_2)$ is homothetic to K. Hence $F_1 \cap (v + F_2)$ is homothetic to the facet F of K with the outward unit normal f. If G is the $(n-1)$-dimensional subspace parallel to H, then $\operatorname{rec} K_1 = \operatorname{rec} K_2 = \operatorname{rec} K$ implies

$$\operatorname{rec} F_i = G \cap \operatorname{rec} K_i = G \cap \operatorname{rec} K = \operatorname{rec} F, \quad i = 1, 2.$$

By the inductive assumption, F is a generalized $(n-1)$-simplex and both F_1 and F_2 are homothetic to F.

The argument above shows that any facet of K is homothetic to each of the respective facets of K_1 and K_2. This obviously implies that all three convex polyhedra K_1, K_2, and K are homothetic. Finally, from [2] it follows that K is a generalized n-simplex. $\square$

Case II. $\operatorname{rec} K_1 \subsetneq \operatorname{rec} K_2$. By induction on n, we are going to prove that K is a generalized n-simplex distinct from a cone, K_1 is homothetic to K, and K_2 is a simplicial n-cone which is a translate of a support cone $C_x(K)$ for a suitable vertex x of K. Since the case $n = 1$ trivially holds, we assume that $n \geq 2$ and that *2*) implies *3*) for all $\mathbb{R}^d$ with $d \leq n - 1$. We divide the proof of *Case II* into Lemmas 3.8–3.10.

Lemma 3.8. *If* $\operatorname{rec} K_2 \not\subset \operatorname{rec} K_1$*, then* K_2 *is a convex cone.*

Proof. Assume, for contradiction, that K_2 is not a cone. Then there is an exposed point q of K_2 such that $C_q(K_2) \neq K_2$. From $\operatorname{rec} K_2 \not\subset \operatorname{rec} K_1$ we conclude that no translate of K_2 entirely lies in K_1. If the interior of a translate $u + K_1$ contains q, then, by Lemma 3.1, q is the only exposed point of K_2 that lies in $\operatorname{int}(u + K_1)$. Lemma 2.4 implies that K_2 is locally conic at q. Hence there is a non-trivial exposed line segment $[q, t]$ of K_2 that lies in an exposed halfline h of the cone $C_q(K_2)$ such that $h \not\subset K_2$. Choose an arbitrary point $v \in \operatorname{int} K_1$ and consider the convex solid $K_2' = v - q + K_2$. By the above, K_2' is locally conic at v and $[v, w] = v - q + [q, t]$ is an exposed line segment of K_2'. Let H be a hyperplane in $\mathbb{R}^n$ with the property $H \cap K_2' = [v, w]$.

For any scalar $\varepsilon \geq 0$, put

$$\begin{aligned} K_2'(\varepsilon) &= \varepsilon(v - w) + K_2', \qquad & L(\varepsilon) &= \varepsilon(v - w) + [v, w], \\ v(\varepsilon) &= \varepsilon(v - w) + v, \qquad & w(\varepsilon) &= \varepsilon(v - w) + w. \end{aligned}$$

Then $L(\varepsilon)$ is an exposed line segment of $K_2'(\varepsilon)$ with endpoints $v(\varepsilon)$ and $w(\varepsilon)$ such that $K_2'(\varepsilon) \cap H = L(\varepsilon)$.

We claim that the line segment $L(0) = [v, w]$ does not lie in K_1. Indeed, assume for a moment that $L(0) \subset K_1$. Since $v(0) = v \in \operatorname{int} K_1$, one can choose $\varepsilon > 0$ so small that $L(\varepsilon) \subset \operatorname{int} K_1$. By Lemma 3.1, we should have $K_2' \subset \operatorname{int} K_1$, which is impossible due to $\operatorname{rec} K_2' = \operatorname{rec} K_2 \not\subset \operatorname{rec} K_1$.

By a continuity argument, there is a scalar $\varepsilon_0 > 0$ such that $v(\varepsilon_0) \in \operatorname{bd} K_1$ and $L(\varepsilon_0)$ intersects $\operatorname{int} K_1$. As above, $L(\varepsilon_0) \not\subset K_1$; whence $w(\varepsilon_0) \notin K_1$. Continuously increasing $\varepsilon\, (\geq \varepsilon_0)$, we find a scalar ε_1 such that $w(\varepsilon_1) \in \operatorname{bd} K_1$ and $L(\varepsilon_1)$ intersects $\operatorname{int} K_1$. Further increasing $\varepsilon\, (\geq \varepsilon_1)$, we find a scalar ε_2 such that $K_1 \cap L(\varepsilon_2) = w(\varepsilon_2)$. As a result, the line through v and w intersects $\operatorname{bd} K_1$ at two points, v^* and w^*, such that $v^* = v(\varepsilon_0) = w(\varepsilon_2)$ and $w^* = w(\varepsilon_1)$.

Now consider the intersections

$$K'(\varepsilon) = K_1 \cap K_2'(\varepsilon), \quad 0 \leq \varepsilon < \varepsilon_2.$$

By condition *2*), every $K'(\varepsilon)$ is a convex solid homothetic to K. Clearly, $T(\varepsilon) =$

$H \cap K'(\varepsilon)$ is an exposed line segment of $K'(\varepsilon)$ and

$$T(\varepsilon) = \begin{cases} [v(\varepsilon), w^*] & \text{if } 0 \le \varepsilon \le \varepsilon_0, \\ [v^*, w^*] & \text{if } \varepsilon_0 \le \varepsilon \le \varepsilon_1, \\ [v^*, w(\varepsilon)] & \text{if } \varepsilon_1 \le \varepsilon < \varepsilon_2. \end{cases}$$

Because $K'(\varepsilon)$ is homothetic to K for all $0 \le \varepsilon < \varepsilon_2$, the homothety factor of $K'(\varepsilon)$ is proportional to the length of $T(\varepsilon)$. Hence the homothety factor of $K'(\varepsilon)$ increases when ε increases from 0 to ε_0; it remains constant when ε increases from ε_0 to ε_1; and it decreases when ε increases from ε_1 to ε_2.

Choose a halfline m with the endpoint *0* that lies in $\operatorname{rec} K_2 \setminus \operatorname{rec} K_1$ and put

$$M(\varepsilon) = \begin{cases} K_1 \cap (v(\varepsilon) + m) & \text{if } 0 \le \varepsilon \le \varepsilon_0, \\ K_1 \cap (v^* + m) & \text{if } \varepsilon_0 \le \varepsilon < \varepsilon_2. \end{cases}$$

Since $m \not\subset \operatorname{rec} K_1$, each $M(\varepsilon)$ is a chord of $K'(\varepsilon)$ in the direction m and its position is uniquely determined by $T(\varepsilon)$. Moreover, $M(\varepsilon)$ and $T(\varepsilon)$ have a common endpoint: it is $v(\varepsilon)$ if $0 \le \varepsilon \le \varepsilon_0$, and it is v^* if $\varepsilon_0 \le \varepsilon < \varepsilon_2$. Since $v(0) = v \in \operatorname{int} K_1$, the length $|M(0)|$ of $M(0)$ is positive. From the definition of $M(\varepsilon)$ it follows that $|M(\varepsilon)|$ should be proportional to $|T(\varepsilon)|$. This is not the case because $|M(\varepsilon)| \to |M(\varepsilon_0)|$ when ε increases from 0 to ε_0, and then $|M(\varepsilon)|$ remains constant when ε increases from ε_0 to ε_2 (contrary to the decrease of $|T(\varepsilon)|$ when ε increases from ε_1 to ε_2). The obtained contradiction shows that K_2 is a convex cone. □

Lemma 3.9. *If* $\operatorname{rec} K_1 \subsetneq \operatorname{rec} K_2$, *then* K_1 *is a convex* n*-polyhedron distinct from a cone,* K *is homothetic to* K_1, *and* K_2 *is a convex polyhedral* n*-cone that is a translate of a support cone* $C_x(K_1)$, *where* x *is a vertex of* K_1.

Proof. By Lemma 3.8, K_2 is a convex cone. Denote by q_2 the apex of K_2. We claim the existence of a point $q_1 \in \exp K_1$ such that K_2 is a translate of $C_{q_1}(K_1)$. Indeed, choose a point $v \in \mathbb{R}^n$ such that $v + q_2 \in \operatorname{int} K_1$ and let $K' = K_1 \cap (v + K_2)$. Obviously, $v + q_2$ is an exposed point of the convex solid K' that corresponds to q_2. Choose a closed halfspace P_2 such that $P_2 \cap K_2 = \{q_2\}$. Because $\operatorname{rec} K_1 \subset \operatorname{rec} K_2$, there is a translate P_1 of P_2 such that P_1 supports K_1 (see Lemma 2.3). Choose any points $q_1 \in P_1 \cap K_1$ and $x \in \operatorname{int} K_2$, and consider the intersection $K'' = K_1 \cap (q_1 - x + K_2)$. Clearly, $C_{q_1}(K'') = C_{q_1}(K_1)$. Since K' and K'' are homothetic, the points $q_1 \in \exp K_1$ and $q_2 \in \exp K_2$ correspond to each other. Hence K_2 is a translate of $C_{q_1}(K_1)$. Furthermore, Lemma 2.4 implies that K_1 is locally conic at q_1.

The facts proved above imply that K_1 is not a cone, since otherwise it should be a cone with apex q_1, resulting in $\operatorname{rec} K_1 = \operatorname{rec} K_2$. Hence K_1 has exposed points distinct from q_1.

We claim that K and K_1 are homothetic. Indeed, choose any distinct points $x, z \in \exp K_1$. Since K_2 is a cone, there is a translate $v + K_2$ that contains both x and z. By Lemma 3.1, $K_1 \subset v + K_2$, and condition *2*) implies that K is homothetic to $K_1 \cap (v + K_2) = K_1$.

Next we state that K_2 is a convex polyhedral n-cone with apex q_2. Indeed, assume, for contradiction, that the cone K_2 is not polyhedral. Then K_2 has infinitely many exposed halflines with common apex q_2. By a compactness argument, we can choose a sequence of such halflines $h_1, h_2, \ldots$ that converge to a boundary halfline h of K_2. Choose a vector $w \in \mathbb{R}^n$ such that $w + q_2 \notin K_1$ and the halfline $w + h$ intersects $\operatorname{int} K_1$ (this is possible since $\operatorname{rec} K_1 \subset \operatorname{rec} K_2$). Then there is a positive integer m so large that the halflines $w + h_m$ and $w + h_{m+1}$ also intersect $\operatorname{int} K_1$.

Now consider the intersection $K_0'' = K_1 \cap (w + K_2)$. By the above, the sets $(w + h_m) \cap K_0''$ and $(w + h_{m+1}) \cap K_0''$ are disjoint 1-dimensional exposed faces of K_0''. On the other hand, $C_{q_1}(K_1)$ is a translate of K_2 and K_1 is locally conic at q_1. Furthermore, $(q_1 - q_2 + h_m) \cap K_1$ and $(q_1 - q_2 + h_{m+1}) \cap K_1$ are 1-dimensional exposed faces of K_1 that have q_1 as a unique common point. This is in contradiction with the fact that K_0'' and K_1 are homothetic. Hence $K_2 = C_{q_1}(K_1)$ is a convex polyhedral n-cone.

Our next claim is that each exposed point of K_1 lies on an exposed halfline of $C_{q_1}(K_1)$. Indeed, assume, for contradiction, the existence of a point $x \in \exp K_1$ that does not lie on an exposed halfline of $C_{q_1}(K_1)$. Choose a point $z \in K_1 \setminus [x, q_1]$ such that $x \in z - q_1 + C_{q_1}(K_1)$, and let $K_0' = K_1 \cap (z - q_2 + K_2)$. Clearly, K_0' is a convex solid. By condition *2*), K_0' is homothetic to K, and whence K_0' is homothetic to K_1. Let P and Q be closed halfspaces that satisfy the conditions $P \cap K_1 = \{q_1\}$ and $Q \cap K_1 = \{x\}$. Put $P' = z - q_1 + P$. Clearly, $P' \cap K_0' = \{z\}$. Since K_1 and K_0' are homothetic, the line segments $[x, q_1]$ and $[x, z]$ should be collinear, which is impossible by the choice of z. The obtained contradiction shows that each exposed point of K_1 lies on an exposed halfline of $C_{q_1}(K_1)$.

Since all exposed points of K_1 lie on exposed halflines of $C_{q_1}(K_1)$, and since $K_2 = C_{q_1}(K_1)$ is a convex polyhedral n-cone, K_1 has finitely many exposed points. Because K and K_1 are homothetic, Lemma 3.6 implies that K_1 has finitely many exposed halflines, which shows that K_1 is a convex n-polyhedron. □

Lemma 3.10. *If* $\operatorname{rec} K_1 \subsetneq \operatorname{rec} K_2$*, then* K_1 *is a generalized* n*-simplex distinct from a cone and* K_2 *is a simplicial* n*-cone.*

Proof. By Lemma 3.9, K_1 is a convex n-polyhedron distinct from a cone and K_2 is a convex polyhedral n-cone with $N(K_2) \subset N(K_1)$. Furthermore, if q_2 is the apex of K_2, then there is a vertex q_1 of K_1 such that K_2 is a translate of the support cone $C_{q_1}(K_1)$. Let F_1 be a facet of K_1 that contains q_1 and is distinct from a cone (such a facet exists since otherwise K_1 would be a convex cone with apex q_1). Denote by e the outward unit normal to K_1 at F_1. Let F_2 be the facet of K_2 that has the same outward unit normal e. Clearly, F_2 is a $(n-1)$-cone with apex q_2. This implies the inclusion $\operatorname{rec} F_1 \subsetneq \operatorname{rec} F_2$, since otherwise F_1 would also be a cone with apex q_1.

We are going to show that F_2 is a simplicial $(n-1)$-cone and F_1 is a generalized $(n-1)$-simplex. Indeed, denote by H the hyperplane containing F_1. Consider the family of translates $v + K_2$, $v \in \mathbb{R}^n$, whose facets $v + F_2$ lie in H such that $\operatorname{rint} F_1 \cap \operatorname{rint}(v + F_2) \neq \varnothing$. Clearly, $\operatorname{int} K_1 \cap \operatorname{int}(v + K_2) \neq \varnothing$ for any such a translate $v + K_2$. By condition *2*), the convex solids $K_1 \cap (v + K_2)$, with $\operatorname{rint} F_1 \cap \operatorname{rint}(v + F_2) \neq \varnothing$, are homothetic to K. Hence the respective $(n-1)$-dimensional intersections $F_1 \cap (v + F_2)$, are homothetic to the facet of K that is parallel to H. By the inductive assumption, F_1 is a generalized $(n-1)$-simplex and F_2 is a simplicial $(n-1)$-cone.

We claim that K_2 itself is a simplicial n-cone with apex q_2. Since F_2 is a simplicial $(n-1)$-cone, it is sufficient to show that K_2 has exactly one extreme halfline with apex q_2 that does not lie in F_2. Assume for a moment that K_2 has two such halflines, say h_1 and h_2. Choose a vector $v \in \mathbb{R}^n$ such that $v + q_2 \notin K_1$ and both halflines $h'_1 = v + h_1$ and $h'_2 = v + h_2$ intersect the relative interior of F_1 (this is possible since $\operatorname{rec} K_1 \subset \operatorname{rec} K_2$). Clearly, $h'_1 \cap K_1$ and $h'_2 \cap K_1$ are disjoint exposed halflines of the convex n-polyhedron $K' = K_1 \cap (v + K_2)$. At the same time, K' should be homothetic to K_1, which has exposed halflines parallel to h'_1 and h'_2 and having a common endpoint. The obtained contradiction shows that K_2 is a simplicial n-cone.

It remains to prove that K_1 is a generalized n-simplex. Denote by h the exposed halfline of $v + K_2$ that does not lie in H. Since $C_{q_1}(K_1)$ is a translate of K_2, the convex solid K_1 has exactly one 1-dimensional face, m, parallel to h and containing q_1. If m is a halfline with endpoint q_1, then the obvious equality $K_1 = \operatorname{conv}(F_1 \cup m)$ implies that K_1 is a generalized n-simplex. Assume now that m is a line segment, say $[q_1, r_1]$. Then $\operatorname{rec} K_1 = \operatorname{rec} F_1$. Since r_1 is the only vertex of K_1 that does not lie in F_1 (because every vertex of K_1 lies on an edge of the cone $C_{q_1}(K_1)$), the cone $G = r_1 + \operatorname{rec} F_1$ is a face of K_1 and

$$K_1 = \operatorname{conv}(F_1 \cup G) = \operatorname{cl}(\operatorname{conv}(F_1 \cup m)).$$

Hence K_1 is a generalized n-simplex. □

Case III. $\operatorname{rec} K_1 \not\subset \operatorname{rec} K_2$ and $\operatorname{rec} K_2 \not\subset \operatorname{rec} K_1$. By induction on n, we are going to prove that K is a generalized n-simplex distinct from a cone and K_1 and K_2 are translates of support cones $C_x(K)$ and $C_z(K)$, respectively, where x and z are distinct vertices of K. Since the case $n = 1$ trivially holds, we assume that $n \geq 2$ and that *2*) implies *3*) for all $\mathbb{R}^d$ with $d \leq n - 1$.

By Lemma 3.8, both K_1 and K_2 are convex cones. Without loss of generality, we may assume that *0* is their common apex. Then $\operatorname{rec} K_i = K_i$, $i = 1, 2$, which implies that $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$. We state that $K_1 \cap K_2$ cannot be n-dimensional. Indeed, if $\dim(K_1 \cap K_2) = n$, then each nonempty intersection $K_1 \cap (v + K_2)$, $v \in \mathbb{R}^n$, is n-dimensional and whence is homothetic to K. In this case, K is a simplicial n-cone and both K_1 and K_2 are translates of K (see [11] for the proof). We divide the proof of *Case III* into Lemmas 3.11–3.14.

Lemma 3.11. *If $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$, then* $0 < \dim(K_1 \cap (-K_2)) < n$.

Proof. Assume for a moment that the convex cone $K_1 \cap (-K_2)$ is n-dimensional. Then $\operatorname{int}(K_1 \cap (-K_2))$ contains distinct unit vectors v_1 and v_2. Put $K' = K_1 \cap (v_1 + K_2)$ and $K'' = K_1 \cap (v_2 + K_2)$. If P_1 and P_2 are closed halfspaces such that $P_1 \cap K_1 = P_2 \cap K_2 = \{0\}$, then

$$P_1 \cap K' = \{0\}, \qquad (v_1 + P_2) \cap K' = \{v_1\},$$
$$P_2 \cap K'' = \{0\}, \qquad (v_2 + P_1) \cap K'' = \{v_2\}.$$

By condition *2*), both K' and K'' are convex solids homothetic to K. This implies that the line segments $[v_1, 0]$ and $[v_2, 0]$ should be collinear, which is impossible by the choice of v_1 and v_2. Hence $\dim(K_1 \cap (-K_2)) < n$.

Now assume that $K_1 \cap (-K_2) = \{0\}$. Then there is a hyperplane $G \subset \mathbb{R}^n$ separating K_1 and $-K_2$ such that $K_1 \cap G = (-K_2) \cap G = \{0\}$. Equivalently, both K_1 and K_2 lie in the same closed half-space determined by G such that $K_1 \cap G = K_2 \cap G = \{0\}$. Choose a point $v \in \operatorname{int} K_1$ and consider the set $K_0' = K_1 \cap (v + K_2)$. Obviously, the support cone $C_v(K_0')$ equals $v + K_2$. Similarly, given a point $w \in \operatorname{int} K_2$, the support cone $C_0(K_0'')$ of the intersection $K_0'' = K_1 \cap (K_2 - w)$ equals K_1. Since the convex solids K_0' and K_0'' are homothetic, we easily obtain that v and 0 should correspond to each other, and $C_v(K_0')$ should be a translate of $C_0(K_0'')$. The last is impossible since K_1 is not a translate of K_2. The obtained contradiction shows that $K_1 \cap (-K_2) \neq \{0\}$. □

Choose an extreme halfline h_1 of the closed convex cone $K_1 \cap (-K_2)$. By Lemma 3.11, $h_1 \subset \operatorname{bd} K_1 \cap \operatorname{bd}(-K_2)$. Put $h_2 = -h_1$, and let S_i be the set of points $x \in \operatorname{bd} K_i$ such that $x + h_i$ intersects $\operatorname{int} K_i$, $i = 1, 2$. Clearly, S_i is an open part of $\operatorname{bd} K_i$. Put $T_i = \operatorname{cl} S_i$, $i = 1, 2$.

Lemma 3.12. *If $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$, then T_i is an $(n-1)$-dimensional face of K_i, $i = 1, 2$.*

Proof. Indeed, assume for a moment that S_1 does not lie in a hyperplane. Then there are regular points v, w of K_1 that lie in S_1 such that the outward unit normals e_v and e_w to K_1 at these points are distinct. Denote by e_1 the unit vector in the direction of h_1. Due to the choice of S_1, both sets

$$K' = K_1 \cap (e_1 + v + K_2), \qquad K'' = K_1 \cap (e_1 + w + K_2)$$

are n-dimensional. By condition *2*), K' and K'' are homothetic. Clearly, the points $e_1 + v \in \exp K'$ and $e_1 + w \in \exp K''$ correspond to each other. As a result, the line segment $[v, e_1 + v] \subset K'$ should correspond to $[w, e_1 + w] \subset K''$ and some neighborhoods of K' and K'' at v and w, respectively, should be homothetic, which is impossible since $e_v \neq e_w$. The obtained contradiction implies that S_1 lies in a hyperplane through 0. Similarly, S_2 lies in a hyperplane trough 0. Hence the sets T_1 and T_2 are $(n-1)$-dimensional faces of K_1 and K_2, respectively. □

Lemma 3.13. *If $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$, then $K_i = \operatorname{conv}(h_i \cup T_i)$, $i = 1, 2$.*

Proof. Since K_1 is a closed convex cone with apex 0, the set T_1 is also a closed convex cone with apex 0. Because T_1 is the projection of K_1 on the hyperplane that contains T_1 in the direction of h_1, the closed convex cone $C_1 = \operatorname{conv}(h_1 \cup T_1)$ lies in K_1.

Assume, for contradiction, that $K_1 \not\subset C_1$. Since K_1 is the convex hull of its extreme halflines with common apex 0, there is an extreme halfline m of K_1 that does not lie in C_1. Choose a point $z \in m \setminus \{0\}$. Clearly, there is a point $x \in S_1$ so close to z that $x \notin C_1$, a contradiction to the choice of C_1. Hence $K_1 = C_1$. Similarly, $K_2 = \operatorname{conv}(h_2 \cup T_2)$. □

Lemma 3.14. *If $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$, then both K_1 and K_2 are simplicial n-cones, and K is a generalized n-simplex.*

Proof. Denote by N the hyperplane containing the facet T_1 of K_1. Choose a point $v \in \operatorname{int} K_1$. Then the set $K_1 \cap (v + K_2)$ is n-dimensional and $Q = T_1 \cap (v + K_2)$ is an $(n-1)$-dimensional set that lies in $\operatorname{rint} T_1$. Since $0 \in N$, we have $w + Q = N \cap (v+w+K_2)$ for any vector $w \in N$, and the set $T_1 \cap (w+Q)$ is $(n-1)$-dimensional if and only if $K_1 \cap (v + w + K_2)$ is n-dimensional

If the set $K_1 \cap (v+w+K_2)$ is n-dimensional, then, by condition *2*), it is homothetic to K; hence all $(n-1)$-dimensional intersections $T_1 \cap (w+Q)$, $w \in N$, are homothetic to Q. By the inductive assumption, T_1 is a simplicial $(n-1)$-cone and Q is a generalized $(n-1)$-simplex. Then $K_1 = \operatorname{conv}(h_1 \cup T_1)$ is a simplicial n-cone, and $K = \operatorname{cl}(\operatorname{conv}(v \cup Q))$ is a generalized n-simplex. Similarly, K_2 is a simplicial n-cone. □

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