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Nondecreasing Solutions of a Quadratic Integral Equation of Urysohn Type on Unbounded Interval

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Using the technique associated with measures of noncompactness we prove the existence of nondecreasing solutions of a class of quadratic integral equation of Urysohn type in the Fréchet space of real functions defined and continuous on an unbounded interval.

Keywords: Fixed-point theorem, measures of noncompactness, nondecreasing solutions, quadratic integral equation of Urysohn type

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1. Introduction

The theory of integral equations is an important part of nonlinear analysis and it is frequently applicable in other branches of mathematics and mathematical physics, engineering, economics, biology as well in describing problems connected with real world.
In the last years it has been appeared a lot of papers devoted to quadratic integral equations.
A large number of these papers are concerned with the existence of monotonic solutions of the mentioned equations [1, 2, 4–12].

In this paper, we are going to investigate the existence of nondecreasing solutions of the nonlinear integral equation of Urysohn type having the form

$$x(t) = g(t) + f(t, x(t)) \int_0^\infty u(t, \tau, x(\tau))d\tau, \quad t \in \mathbb{R}_+ = [0, \infty). \tag{1}$$

We will look for solutions of those equations in the Fréchet space $C(\mathbb{R}_+)$ of real functions being defined and continuous on $\mathbb{R}_+$. We will apply the measure of noncompactness defined in [13–15] in proving the solvability of the considered equation in the class of nondecreasing functions.
The result obtained in the present paper completes several ones (cf., [1, 2, 4–12]).

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2. Notation, definitions and auxiliary facts

For further purposes, we collect in this section a few auxiliary results which will be needed in the sequel. Assume that $C(\mathbb{R}_+)$ is the Fréchet space consisting of all real functions defined and continuous on $\mathbb{R}_+$ equipped with the family of seminorms

$$|x|_n = \sup\{|x(t)| : t \in [0,n]\}, \quad n \in \mathbb{N}, x \in C(\mathbb{R}_+).$$

It is known that $C(\mathbb{R}_+)$ is a locally convex space. Let us recall the following fact:

(A) a sequence (x_n) is convergent to x in $C(\mathbb{R}_+)$ if and only if (x_n) is uniformly convergent to x on compact intervals of $\mathbb{R}_+$.

If X is a subset of $C(\mathbb{R}_+)$, then $\overline{X}$ and Conv$X$ denote closure and convex closure of X, respectively.
The family of all nonempty subsets of $C(\mathbb{R}_+)$ consisting of functions uniformly bounded on compact intervals of $\mathbb{R}_+$ will be denoted by $\mathfrak{M}_C$, i.e.

$$\mathfrak{M}_C = \{X \subset C(\mathbb{R}_+) : X \neq \emptyset \text{ and } \sup\{|x(t)| : x \in X, t \leq T\} < \infty \text{ for all } T \geq 0\}$$

while subfamily of $\mathfrak{M}_C$ consisting of all relatively compact sets is denoted by $\mathfrak{N}_C$. Throughout this paper, we accept the following definition of the notion of a measure of noncompactness [3].

Definition 2.1. A function $\mu : \mathfrak{M}_C \to [0,\infty]$ is said to be a measure of noncompactness in $C(\mathbb{R}_+)$ if it satisfies the following conditions:

1^o the family $\ker\mu = \{X \in \mathfrak{M}_C : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_C$
2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$
3^o $\mu(\text{Conv}X) = \mu(X)$
4^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$
5^o if (X_n) is a sequence of closed sets from $\mathfrak{M}_C$ such that $X_{n+1} \subset X_n$ $(n = 1,2,...)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

We say that a measure of noncompactness μ is regular provided it satisfies additionally the following conditions:

6^o $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$
7^o $\mu(X+Y) \leq \mu(X) + \mu(Y)$
8^o $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$
9^o $\ker\mu = \mathfrak{N}_C$.

Remark 2.2. Observe that in contrast to the definition of the concept of a measure of noncompactness given in [3], our measures of noncompactness may take the value $+\infty$. This fact is very essential in our considerations in the setting of Fréchet spaces.

For our further purposes we will only need the following fixed point theorem of Darbo type.

Theorem 2.3. *Let Q be nonempty closed convex set from $\mathfrak{M}_C$ such that $\mu(Q) < \infty$ and let $U : Q \to Q$ be a continuous operator satisfying the inequality $\mu(UX) \leq k\mu(X)$ for any nonempty subset X of Q, where $k \in [0,1)$ is a constant. Then, U has a fixed point in the set Q.*

Now, we recall the definition of a measure of noncompactness which will be used in our further investigations. That measure was introduced and studied in [3].
To do this, let us fix a subset $X \in \mathfrak{M}_C$. For $x \in X$, $\varepsilon > 0$ and $T > 0$ let us denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0+} \omega^T(X, \varepsilon).$$

Finally, we define the function μ on the family $\mathfrak{M}_C$ by putting

$$\mu(X) = \sup\{p(T)\omega_0^T(X) : T \geq 0\} \tag{2}$$

where $p(T)$ is arbitrary function defined on the interval $\mathbb{R}_+$ with real positive values. It can be shown [13] that the function μ is a regular measure of noncompactness in the space $C(\mathbb{R}_+)$. The function $p(T)$ (and a set Q from Theorem 2.3) will be defined in the next sections in such a way that $\mu(Q) < \infty$.

3. Main result

In this section we will study the existence of solutions of the quadratic Urysohn integral equation (1). Our considerations are situated in the Fréchet space $C(\mathbb{R}_+)$ described in the previous part. We will investigate Eq. (1) assuming that the following conditions are satisfied:

(i) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function,

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $f(t, x)$ is nondecreasing with respect to each of both nonnegative variables t and x separately, i.e. the function $t \to f(t, x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$ and the function $x \to f(t, x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $t \in \mathbb{R}_+$,

(iii) the function f satisfies the Lipschitz condition with respect to the second variable, i.e. there exists a locally bounded function $m(t) > 0$ such that

$$|f(t, x) - f(t, y)| \leq m(t)|x - y|$$

for $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$,

(iv) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and the function $t \to u(t, \tau, x)$ is nondecreasing on $\mathbb{R}_+$ for arbitrarily fixed $\tau, x \in \mathbb{R}_+$ and moreover there exist a function $p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t, \tau, x)| \leq p(t, \tau)q(|x|)$$

for all $t, \tau \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

(v) there exists a function $r : \mathbb{R}_+ \to (0, \infty)$ such that for every $t \geq 0$ the function $\tau \to p(t,\tau)q(r(\tau))$ is integrable on $\mathbb{R}_+$ and

$$g(t) + f(t, r(t)) \int_0^\infty p(t,\tau)q(r(\tau))d\tau \leq r(t)$$

for all $t \in \mathbb{R}_+$,

(vi) the function $t \to \int_0^\infty p(t,\tau)q(r(\tau))d\tau$ is locally bounded on $\mathbb{R}_+$ i.e.

$$\forall_{T>0} \sup_{t\in[0,T]} \int\limits_0^\infty p(t,\tau)q(r(\tau))d\tau < \infty,$$

(vii) the improper integral $\int_0^\infty p(t,\tau)q(r(\tau))d\tau$ is locally uniformly convergent with respect to t i.e.

$$\forall_{\delta>0}\ \forall_{T>0}\ \exists_{S>0} \sup_{t\in[0,T]} \int_S^\infty p(t,\tau)q(r(\tau))d\tau < \delta,$$

(viii)

$$\sup_{t\geq 0} m(t) \int_0^\infty p(t,\tau)q(r(\tau))d\tau < 1.$$

Now, we can formulate our main existence result.

Theorem 3.1. *Under assumptions* (i)–(viii) *Eq.* (1) *has at least one solution* $x = x(t)$ *which belongs to the space* $C(\mathbb{R}_+)$ *and is nondecreasing on* $\mathbb{R}_+$.

Proof. Let $r(t)$ be a function satisfying assumption (v) and put $\overline{r}(t) = \sup\{r(s) : s \in [0,t]\}$. Moreover, let us define the set

$$Q = \{x \in C(\mathbb{R}_+) : x \text{ is nondecreasing and } 0 \leq x(t) \leq r(t) \text{ for } t \in \mathbb{R}_+\}.$$

Obviously, Q is convex closed subset of $C(\mathbb{R}_+)$ and $Q \in \mathfrak{M}_C$.
Consider the operator U defined on Q by the formula

$$(Ux)(t) = g(t) + f(t, x(t)) \int\limits_0^\infty u(t,\tau,x(\tau))d\tau, \quad t \geq 0.$$

At first we show that the function Ux is continuous on $\mathbb{R}_+$ for any function $x \in Q$. To do this fix arbitrarily $x \in Q$, $T > 0$ and $\varepsilon > 0$. Next, take numbers $t, s \in [0,T]$ such that $|t - s| \leq \varepsilon$. Moreover, let $\delta > 0$ and $S > 0$. Then, using the assumptions (ii)–(iv), we derive the following chain of inequalities:

$$
\begin{aligned}
&|(Ux)(t)-(Ux)(s)| \leq |g(t)-g(s)| \\
&\qquad + \left| f(t,x(t)) \int_0^\infty u(t,\tau,x(\tau))d\tau - f(s,x(s)) \int_0^\infty u(t,\tau,x(\tau))d\tau \right| \\
&\qquad + \left| f(s,x(s)) \int_0^\infty u(t,\tau,x(\tau))d\tau - f(s,x(s)) \int_0^\infty u(s,\tau,x(\tau))d\tau \right| \\
&\leq \omega^T(g,\varepsilon) + |f(t,x(t))-f(s,x(s))| \int_0^\infty u(t,\tau,x(\tau))d\tau \\
&\qquad + f(s,x(s)) \int_0^\infty |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau \\
&\leq \omega^T(g,\varepsilon) + \Big[|f(t,x(t))-f(t,x(s))| + |f(t,x(s))-f(s,x(s))|\Big] \int_0^\infty p(t,\tau)q(x(\tau))d\tau \\
&\qquad + \big[f(s,x(s))-f(s,0)+f(s,0)\big] \int_0^\infty |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau \\
&\leq \omega^T(g,\varepsilon) + \big[m(t)|x(t)-x(s)| + \omega^T_{\overline{r}(T)}(f,\varepsilon)\big] \int_0^\infty p(t,\tau)q(r(\tau))d\tau \\
&\qquad + \big[m(s)x(s)+f(s,0)\big] \int_0^S |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau \\
&\qquad + \big[m(s)x(s)+f(s,0)\big]\Big(\int_S^\infty p(t,\tau)q(r(\tau))d\tau + \int_S^\infty p(s,\tau)q(r(\tau))d\tau\Big), \qquad (3)
\end{aligned}
$$

where we denoted

$$\omega^T_d(f,\varepsilon) = \sup\{|f(t,y)-f(s,y)| : \ t,s \in [0,T], \ |t-s| \leq \varepsilon, \ y \in [0,d]\}.$$

Keeping in mind the assumption (vii) we deduce that there exists S so large that the last term of inequality (3) is less then δ i.e.

$$
\begin{aligned}
|(Ux)(t)-(Ux)(s)| \leq \omega^T(g,\varepsilon) &+ m(t)|x(t)-x(s)| \int_0^\infty p(t,\tau)q(r(\tau))d\tau \\
&+ \omega^T_{\overline{r}(T)}(f,\varepsilon) \int_0^\infty p(t,\tau)q(r(\tau))d\tau \\
&+ \big[m(s)x(s)+f(s,0)\big] \int_0^S |u(t,\tau,x(\tau))-u(s,\tau,x(\tau))|d\tau + \delta.
\end{aligned}
$$

Now, from the above estimate we get:

$$\begin{aligned}\omega^T(Ux,\varepsilon) \le \omega^T(g,\varepsilon) + \omega^T(x,\varepsilon)\sup_{t\le T} m(t)\int_0^\infty p(t,\tau)q(r(\tau))d\tau \\ + \omega^T_{\overline{r}(T)}(f,\varepsilon)\sup_{t\le T}\int_0^\infty p(t,\tau)q(r(\tau))d\tau \\ + \sup_{s\le T}\left(m(s)r(s)+f(s,0)\right) S\omega^{T,S}_{\overline{r}(T)}(u,\varepsilon)+\delta, \qquad (4)\end{aligned}$$

where, similarly as above, we denoted

$$\omega^{T,S}_d(u,\varepsilon) = \sup\{|u(t,\tau,y)-u(s,\tau,y)| : \ t,s\in[0,T],\ \tau\in[0,S],\ |t-s|\le\varepsilon,\ y\in[0,d]\}.$$

Let us notice that $\omega^T_{\overline{r}(T)}(f,\varepsilon)\to 0$ and $\omega^{T,S}_{\overline{r}(T)}(u,\varepsilon)\to 0$ as $\varepsilon\to 0$, which is a consequence of the uniform continuity of the function f on the set $[0,T]\times[0,\overline{r}(T)]$ and the function u on the set $[0,T]\times[0,S]\times[0,\overline{r}(T)]$, respectively. Moreover, the continuity of g yields that $\omega^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$.

Further observe that in virtue of assumption (vi) $\sup_{t\in[0,T]}\int_0^\infty p(t,\tau)q(r(\tau))d\tau<\infty$. Hence, taking into account the facts established above and the choice of $\delta>0$ we infer that the function Ux is continuous on the interval $[0,T]$ for any $T>0$. This implies that Ux is continuous on the whole interval $\mathbb{R}_+$.

Now we show that the mapping U transforms Q into itself. To do this take $x\in Q$ and assume that $t,s\in\mathbb{R}_+$ and $t>s$. Then we get

$$\begin{aligned}(Ux)(t)-(Ux)(s) = (g(t)-g(s)) \ + f(s,x(s))\int_0^\infty\Big(u(t,\tau,x(\tau))-u(s,\tau,x(\tau))\Big)d\tau \\ + \Big[\Big(f(t,x(t))-f(t,x(s))\Big) \\ + \Big(f(t,x(s))-f(s,x(s))\Big)\Big]\int_0^\infty u(t,\tau,x(\tau))d\tau.\end{aligned}$$

Hence, in view of our assumptions (i), (ii) and (iv) it follows that the function Ux is nondecreasing on $\mathbb{R}_+$. Moreover, keeping in mind (v), for arbitrarily fixed $t\in\mathbb{R}_+$ we have

$$\begin{aligned}0\le (Ux)(t) = g(t)+f(t,x(t))\int_0^\infty u(t,s,x(s))ds \\ \le g(t)+f(t,r(t))\int_0^\infty p(t,\tau)q(r(\tau))d\tau \le r(t)\end{aligned}$$

which means that the mapping U transforms Q into itself.

In what follows let us take a nonempty subset X of the set Q. Fix $\varepsilon > 0$ and $T > 0$. Then, using the estimate (4) we obtain:

$$\omega^T(UX,\varepsilon) \leq \omega^T(g,\varepsilon) + \omega^T(X,\varepsilon) \sup_{t\leq T} m(t) \int_0^\infty p(t,\tau)q(r(\tau))d\tau$$
$$+ \omega^T_{\overline{r}(T)}(f,\varepsilon) \sup_{t\leq T} \int_0^\infty p(t,\tau)q(r(\tau))d\tau$$
$$+ \sup_{s\leq T} (m(s)r(s) + f(s,0))\, S\omega^{T,S}_{\overline{r}(T)}(u,\varepsilon) + \delta.$$

Now, taking into account the properties of the components involved in the above inequality we have

$$\omega_0^T(UX) \leq \sup_{t\leq T} m(t) \int_0^\infty p(t,\tau)q(r(\tau))d\tau \cdot \omega_0^T(X).$$

Further, putting in (2) $p(T) = \overline{r}^{-1}(T)$ we obtain

$$\mu(UX) \leq \sup_{t\geq 0} m(t) \int_0^\infty p(t,\tau)q(r(\tau))d\tau \cdot \mu(X). \tag{5}$$

Moreover

$$\mu(Q) \leq 1. \tag{6}$$

Now we show that U is continuous on the set Q.
To do this fix $x \in Q$ and take a sequence of functions $x_n \in Q$ such that $x_n \to x$ in $C(\mathbb{R}_+)$. We will show $Ux_n \to Ux$ in $C(\mathbb{R}_+)$. First, let us observe that

$$|(Ux)(t) - (Ux_n)(t)| \leq |f(t,x(t)) - f(t,x_n(t))| \int_0^\infty |u(t,\tau,x(\tau))|d\tau$$
$$+ f(t,x_n(t)) \int_0^\infty |u(t,\tau,x(\tau)) - u(t,\tau,x_n(\tau))|d\tau$$
$$\leq m(t)|x(t) - x_n(t)| \int_0^\infty p(t,\tau)q(r(\tau))d\tau$$
$$+ f(t,x_n(t)) \int_0^\infty |u(t,\tau,x(\tau)) - u(t,\tau,x_n(\tau))|d\tau. \tag{7}$$

Next, fix $T > 0$. In virtue of (A) it is enough to show that $|(Ux)(t) - (Ux_n)(t)| \to 0$ uniformly on $[0, T]$ for $n \to \infty$.
Using assumption (viii) we have $\sup_{t \leq T} m(t) \int_0^\infty p(t,\tau)q(r(\tau))d\tau < \infty$.
This implies that the first term of the inequality (7) tends to 0 uniformly on $[0, T]$ for $n \to \infty$. Next, let us observe that

$$\sup_{t \leq T} f(t, x_n(t)) \leq \sup_{t \leq T}[m(t)r(t) + f(t, 0)] < \infty.$$

It is enough to show that $\int_0^\infty |u(t, \tau, x(\tau)) - u(t, \tau, x_n(\tau))|d\tau \to 0$ uniformly on $[0, T]$ as $n \to \infty$.
To this end fix $S > 0$ and $\delta > 0$. Then we get:

$$\begin{aligned} &\int_0^\infty |u(t, \tau, x(\tau)) - u(t, \tau, x_n(\tau))|d\tau \\ &\leq \int_0^S |u(t, \tau, x(\tau)) - u(t, \tau, x_n(\tau))|d\tau + 2\int_S^\infty p(t,\tau)q(r(\tau))d\tau. \end{aligned} \tag{8}$$

Hence, in view of the assumption (vii) we can find S so big that the last term of (8) is less then $\frac{\delta}{2}$ for $t \leq T$. Moreover

$$\int_0^S |u(t, \tau, x(\tau)) - u(t, \tau, x_n(\tau))|d\tau \leq S\overline{\omega}^{T,S}_{\overline{r}(T)}(u, \sup_{\tau \leq S}|x(\tau) - x_n(\tau)|), \tag{9}$$

where

$$\overline{\omega}^{T,S}_d(u, \varepsilon) = \sup\{|u(t, \tau, x) - u(t, \tau, y)| : t \in [0, T],\ \tau \in [0, S],\ |x - y| \leq \varepsilon,\ x, y \in [0, d]\}.$$

The convergence of (x_n) to x in $C(\mathbb{R}_+)$ implies $\lim_{n\to\infty} \sup_{\tau \leq S}|x(\tau) - x_n(\tau)| = 0$. Combining this fact, (8), (9) and the uniform continuity of the function u on the set $[0, T] \times [0, S] \times [0, \overline{r}(T)]$ we infer that

$$\sup_{t \leq T} \int_0^\infty |u(t, \tau, x(\tau)) - u(t, \tau, x_n(\tau))|d\tau \leq \delta \quad \text{for } n \text{ sufficiently big.}$$

This ends the proof of continuity the mapping $U : Q \to Q$.

Finally, linking (5), (6), (viii) and using Theorem 2.3 we infer that the operator U has at least one fixed point x in the set Q. Obviously the function $x = x(t)$ is a nondecreasing solution of the integral equation (1). This completes the proof. □

Example 3.2. Consider the following quadratic integral equation

$$x(t) = e^t/4 + \left(e^t/2 + x(t)\right) \int_0^\infty \frac{te^{-3s}x^3(s)}{1 + tx^2(s)}ds, \quad t \in \mathbb{R}_+. \tag{10}$$

Observe that this equation is a special case of Eq. (1), where

$$g(t) = e^t/4, \qquad f(t,x) = e^t/2 + x, \qquad u(t,s,x) = e^{-3s}tx^3/(1+tx^2).$$

It is easily seen that there are satisfied the assumptions of Theorem 3.2. In fact, the function $g(t)$ is positive and nondecreasing on $\mathbb{R}_+$. The function $f(t,x) = e^t/2 + x$ is nondecreasing with respect to each of both variables t and x. Moreover, $m(t) = 1$. Further notice that the function $t \to u(t,s,x)$ is nondecreasing on $\mathbb{R}_+$ for fixed $s, x \in \mathbb{R}_+$.
Next we get

$$\frac{e^{-3s}tx^3}{1+tx^2} \leq e^{-3s}|x|.$$

Thus, the functions $p(t,s)$ and $q(x)$ appearing in the assumption (iv) have the form

$$p(t,s) = e^{-3s}, \qquad q(x) = x.$$

Now we show, that the function $r(t) = e^t$ satisfy assumption (v). Indeed,

$$e^t/4 + (e^t/2 + e^t)\int_0^\infty e^{-3s}\cdot e^s ds \leq e^t/4 + 3/4e^t = e^t = r(t).$$

Keeping in mind the equality $p(t,s)q(r(s)) = e^{-2s}$ we infer that assumptions (vi) and (vii) are satisfied. Finally, we get

$$\sup_{t\geq 0} m(t)\int_0^\infty p(t,s)q(r(s))ds = 1/2 < 1.$$

Taking into account all the above established facts and Theorem 3.1 we conclude that the Eq. (10) has at least one solution $x = x(t)$ defined, continuous and nondecreasing on $\mathbb{R}_+$.

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