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Second-Order Asymptotic Directions of Unbounded Sets with Application to Optimization*

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In this paper, we introduce a new concept of second-order asymptotic directions of unbounded sets and apply it to establish global optimality conditions of nonlinear functions on unbounded sets, existence conditions for efficient points in a partially ordered space, and to extend Dieudonné's theorem on the closedness of the sum of two closed sets.

Keywords: First-order and second-order asymptotic directions, optimality conditions, efficient points

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1. Introduction

The concept of asymptotic directions, called also recession directions, was introduced at the beginning of the last century by Steinitz [20] in order to study unbounded convex sets. It is an indispensable tool in globally characterizing the behavior of convex sets and convex functions, and in establishing global optimality conditions of convex problems. Later on, this concept was generalized by Debreu in his famous book [5] on value theory in which economic models may have no convex structure. Since then a great number of researchers are involved in development and use of asymptotic directions to various fields such as optimization, economics, mechanics, engineering, finance etc. (see [1]–[4], [6]–[21]).

It is well-known that unbounded polyhedral convex sets are completely controlled by their asymptotic directions, that is, any element of an unbounded polyhedral set can be reached by following an asymptotic direction starting from a bounded region. This is a representation theorem which says that a polyhedral convex set is the sum of its asymptotic cone and a bounded polyhedron (see Theorem 19.1, [18]). When a set is not polyhedral, such a representation is no longer true. For instance, the graph of the function $\log(x)$ with $x \geq 1$ in the plane has the positive abscises as asymptotic cone,

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although the distance from an element $(x, \log(x))$ of the graph to the asymptotic cone grows up to infinity as x goes to infinity. In order to manage such gaps between an asymptotic ray and those elements of the set that generate it, we set up our aim to develop new asymptotic directions, called second-order asymptotic directions, and apply them to some problems of optimization theory.

The paper is structured as follows. In the next section we recall the concept of first-order asymptotic directions and some elementary properties. Then we introduce a new concept of asymptotic directions: second-order asymptotic directions, and establish calculus rules on them. In the remaining sections we are concerned with applications. In Section 3, we construct second-order asymptotic functions and derive necessary and sufficient conditions for existence of global minima of a function on an unbounded region. In Section 4 we deal with existence of efficient points in vector optimization. Our conditions are expressed in terms of the second-order asymptotic directions and fairly deepen the existing first-order criteria. In the last application (Section 5) we study a question on the closedness of the sum of two closed sets which is very important in the study of existence and stability of optimization problems (see Chapter 2, Section 2.3 of [4]). The result of this section refines a well-known theorem by Dieudonné [7] in which only first-order asymptotic directions are employed. The applications presented in the three last sections show the importance that second-order asymptotic directions play in optimization. Applications to other areas are also conceivable and need further attention.

2. Asymptotic directions

Given a nonempty set $A \subseteq \mathbb{R}^k$, we say that a vector u is a first-order asymptotic direction (or asymptotic direction for short) of A if there are a sequence of elements a_n of A and positive numbers t_n converging to 0 such that

$$u = \lim_{n\to\infty} t_n a_n.$$

The set of all asymptotic directions of A is denoted by $R'(A)$ (the letter R is referred to recession direction, as it is called in convex analysis). This set is a closed cone, and is convex when the set A is convex. It can also be written as outer limit in the sense of Kuratowski-Painlevé

$$R'(A) = \limsup_{t\downarrow 0} tA.$$

Sometimes a smaller cone, the inner limit, is considered

$$R'_{inn}(A) = \liminf_{t\downarrow 0} tA$$

which consists of all vectors u such that for any sequence of positive numbers t_n converging to 0, there is a sequence of elements a_n of A with $u = \lim_{n\to\infty} t_n a_n$. When these two cones coincide, the set A is called regular in [4] or asymptotable in [13]. This is the case for instance when the set A is convex. We shall distinguish those asymptotic directions which contain only a bounded part of A in their neighborhoods. Namely a nonzero asymptotic direction u of A is said to be isolated if there is no real number $r > 0$ such that the intersection of A with $\mathbb{R}_+\{u\}+B(0,r)$ is unbounded; here $B(0,r)$ denotes the closed ball centered at the origin and of radius r, and $\mathbb{R}_+\{u\}$ the set of vectors tu with $t \geq 0$. It is clear that convex sets have no isolated directions. A parabola in a plane has all asymptotic directions isolated. To progress further let us mention some elementary properties of asymptotic cones that can be found in [4] and [9].

(1) $R'(A) = \{0\}$ if and only if A is bounded;
(2) $R'(A \cup B) = R'(A) \cup R'(B)$;
(3) $R'(A \cap B) \subseteq R'(A) \cap R'(B)$;
(4) $R'(A + B) \subseteq R'(A) + R'(B)$ if $R'(A) \cap -R'(B) = \{0\}$ and $R'(A) + R'(B) \subseteq R'(A + B)$ if A is regular;
(5) $R'(A \times B) \subseteq R'(A) \times R'(B)$. Equality holds if A is regular.
(6) If $L : \mathbb{R}^k \to \mathbb{R}^m$ is a linear operator, then $L(R'(A)) \subseteq R'(L(A))$. Equality holds provided that $R'(A) \cap \mathrm{Ker}(L) = \{0\}$.

We now introduce second-order asymptotic directions. They are based on limits of sequences.

Definition 2.1. Let u be a vector of $\mathbb{R}^k$. We say that a vector v is a second-order asymptotic direction of A at u if there are a sequence of elements a_n of A and positive numbers t_n and s_n such that

$$\lim_{n\to\infty} \left(\frac{a_n}{s_n} - t_n u \right) = v, \tag{1}$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \infty.$$

The set of all second-order asymptotic directions of A at u is denoted by $R''(A; u)$. This set is a cone if nonempty. It can be expressed by outer limit

$$R''(A; u) = \limsup_{t,s\uparrow\infty} \left(\frac{A}{s} - tu \right),$$

hence it is closed. Let us give some elementary properties of second-order asymptotic directions.

Proposition 2.2. *For nonempty sets A, B and a vector u in $\mathbb{R}^k$, the following assertions hold.*

(*i*) $R''(A; 0) = R'(A)$;
(*ii*) $R''(A; u)$ *is nonempty if and only if the vector u is a first-order asymptotic direction of A;*
(*iii*) $R''(A; u) \subseteq R''(B; u)$ *if* $A \subseteq B$;
(*iv*) $R''(A \cup B; u) = R''(A; u) \cup R''(B; u)$;
(*v*) $R''(A \cap B; u) \subseteq R''(A; u) \cap R''(B; u)$;
(*vi*) $R''(\alpha A; \beta u) = R''(A; u)$ *for every* $\alpha > 0$ *and* $\beta > 0$;
(*vii*) $R''(A + B; u) = R''(A; u)$ *whenever B is bounded;*
(*viii*) $R''(A; u) + \mathbb{R}\{u\} = R''(A; u)$. *In particular, if u is a first-order asymptotic direction, then* $\mathbb{R}\{u\} \subseteq R''(A; u)$. *Moreover,* $R'(A) \subseteq \mathbb{R}\{u\} = R''(A; u)$ *holds if*

and only if there exits a ball $B(0,r)$ *with* $r>0$ *such that* $A \subseteq \mathbb{R}\{u\} + B(0,r)$. *In particular, if* u *is an isolated direction, then* $R''(A;u) \neq \mathbb{R}\{u\}$.

(*ix*) *If* $L : \mathbb{R}^k \to \mathbb{R}^m$ *is a linear operator and* $u \in R'(A)$, *then* $L(R''(A;u)) \subseteq R''(L(A);L(u))$. *Equality holds provided that* L *is injective on* $R'(A)$ *and* $R''(A;u) \cap \operatorname{Ker}(L) = \{0\}$.

Proof. The first assertion is immediate from the definition because the outer limit expression of $R''(A;0)$ is exactly that of the cone $R'(A)$. To prove the second assertion, if v is an element of $R''(A;u)$, then (1) implies

$$\lim_{n\to\infty}\left(\frac{a_n}{t_n s_n} - u\right) = \lim_{n\to\infty}\frac{v}{t_n} = 0$$

which shows that u is a first-order asymptotic direction of A. Conversely, let u be an element of $R'(A)$, that is

$$u = \lim_{n\to\infty}\frac{a_n}{\alpha_n}$$

for some $a_n \in A$ and $\alpha_n > 0$ converging to ∞. Consider the sequence of the terms $a_n - \alpha_n u$. If it is bounded, then choose $t_n = s_n = \sqrt{\alpha_n}$. We deduce

$$\lim_{n\to\infty}\left(\frac{a_n}{s_n} - t_n u\right) = \lim_{n\to\infty}\frac{1}{\sqrt{\alpha_n}}(a_n - \alpha_n u) = 0,$$

with t_n and s_n converging to ∞, by which the null vector belongs to $R''(A;u)$. If that sequence is unbounded, say with norm converging to ∞, then we may assume that

$$\lim_{n\to\infty}\frac{a_n - \alpha_n u}{\|a_n - \alpha_n u\|} = w$$

for some unit norm vector w. Choosing $s_n = \|a_n - \alpha_n u\|$ and $t_n = \frac{\alpha_n}{s_n}$ we see that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left\|\frac{a_n}{\alpha_n} - u\right\|^{-1} = \infty$$

and conclude that w belongs to the set $R''(A;u)$.

Assertions (*iii*)–(*v*) are clear. For assertion (*vi*) it suffices to notice that given v as in Definition 2.1 and $\alpha > 0$ and $\beta > 0$, by setting $s'_n = \alpha s_n$ and $t'_n = t_n/\beta$ one derives at once that $v \in R''(\alpha A;\beta u)$.

Assertion (*vii*) follows from the fact that

$$\lim_{n\to\infty}\left(\frac{a_n + b_n}{s_n} - t_n u\right) = \lim_{n\to\infty}\left(\frac{a_n}{s_n} - t_n u\right)$$

for every bounded sequence $\{b_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 0}$ converging to ∞.

Let us prove assertion (*viii*). If $R''(A;u) = \emptyset$, then the equality $R''(A;u) + \mathbb{R}\{u\} = R''(A;u)$ is obvious. Consider the case $R''(A;u) \neq \emptyset$. Since $\mathbb{R}\{u\}$ contains the null vector, the inclusion $R''(A;u) \subseteq R''(A;u) + \mathbb{R}\{u\}$ is clear. For the converse inclusion let $v \in R''(A;u)$ defined by (1) and $\alpha \in \mathbb{R}$. Then $t'_n = t_n - \alpha$ converges to ∞ and so

$$v + \alpha u = \lim_{n\to\infty} \left(\frac{a_n}{s_n} - t'_n u \right) \in R''(A;u),$$

which shows that $v + \alpha u$ belongs to $R''(A;u)$. Thus, the first equality of $(viii)$ follows. Moreover, being nonempty the set $R''(A;u)$ is a cone, hence it contains zero, and therefore $\mathbb{R}\{u\} \subseteq R''(A;u)$.

Now assume $A \subseteq \mathbb{R}\{u\} + B(0,r)$ for some $r \geq 0$. It is clear that $R'(A) \subseteq \mathbb{R}\{u\}$. If $v \in R''(A;u)$ is given by (1), then the elements a_n can be written in the form $a_n = \alpha_n u + b_n$ where the sequence $\{b_n\}_{n\in\mathbb{N}} \subset B(0,r)$ is bounded. Hence

$$v = \lim_{n\to\infty} \left(\frac{\alpha_n u + b_n}{s_n} - t_n u \right) = \lim_{n\to\infty} \left(\frac{\alpha_n u}{s_n} - t_n u \right) \in \mathbb{R}\{u\}$$

and $R''(A;u) \subseteq \mathbb{R}\{u\}$, which is in fact equality because the converse inclusion is evident.

Conversely, assume that for all $r > 0$, the inclusion $A \subseteq \mathbb{R}\{u\} + B(0,r)$ does not hold. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ such that the distance $d(a_n, \mathbb{R}\{u\})$ from a_n to $\mathbb{R}\{u\}$ tends to ∞ as n goes to ∞. Let $t_n u$ be the projection of a_n onto $\mathbb{R}\{u\}$. We may assume that t_n are nonnegative, otherwise consider $-u$ instead of u. Then $\lim_{n\to\infty} \|a_n - t_n u\| = \infty$ and $\langle a_n - t_n u, u\rangle = 0$. We may suppose without loss of generality that $\frac{a_n - t_n u}{\|a_n - t_n u\|}$ converges to a nonzero vector w and show that $\frac{t_n}{\|a_n - t_n u\|}$ converges to ∞. Indeed, if not, say it converges to a finite number α, then $\frac{a_n}{\|a_n - t_n u\|}$ converges to a first-order asymptotic direction $\alpha u + w$. Since w is orthogonal to u we deduce that the limit $\alpha u + w$ does not belong to the straight line $\mathbb{R}\{u\}$ which contradicts the hypothesis.

Finally, let $v \in R''(A;u)$ be given by (1). Then

$$L(v) = \lim_{n\to\infty} \left(\frac{L(a_n)}{s_n} - t_n L(u) \right) \tag{2}$$

with s_n and t_n converging to ∞. Hence $L(v) \in R''(L(A); L(u))$. Conversely, let $p \in R''(L(A); L(u))$ be given by the limit of the right hand side of (2). Note that (2) implies that $\lim_{n\to\infty} L(\frac{a_n}{s_n t_n}) = L(u)$. We show first that

$$\lim_{n\to\infty} \frac{a_n}{s_n t_n} = u. \tag{3}$$

Indeed, if not, by taking a subsequence if necessary, we may assume that either $\lim_{n\to\infty} \|\frac{a_n}{s_n t_n}\| = \infty$ or $\lim_{n\to\infty} \frac{a_n}{s_n t_n} = u' \neq u$. In the first case, by setting $\mu_n = \|\frac{a_n}{s_n t_n}\|$ we may consider that the sequence $\{\frac{a_n}{s_n t_n \mu_n}\}_{n\geq 1}$ converges to some unit norm vector $w \in R'(A)$. Then

$$L(w) = \lim_{n\to\infty} \frac{L(a_n)}{s_n t_n \mu_n} = 0,$$

which is impossible by our assumption. In the second case, $L(u') = L(u)$, which is also excluded. Thus, (3) is true. Now consider the sequence $\{\frac{a_n}{s_n} - t_n u\}_{n\geq 1}$. If it is bounded, then it has a cluster vector, say v. We have $v \in R''(A;u)$ and $L(v) = p$. If it is unbounded, then by setting $\alpha_n = \|\frac{a_n}{s_n} - t_n u\|$ we may assume that the bounded sequence $\{(\frac{a_n}{s_n} - t_n u)/\alpha_n\}_{n \geq 1}$ converges to some nonzero vector q. In view of (3), t_n/α_n converges to ∞, which yields the inclusion $q \in R''(A;u)$. This, however, together with

$$L(q) = \lim_{n\to\infty} \frac{\frac{L(a_n)}{s_n} - t_n L(u)}{\alpha_n} = \lim_{n\to\infty} \frac{p}{\alpha_n} = 0$$

contradicts the hypothesis. The proof is complete. □

Notice that in the above proposition the inclusions of (v) and (ix) are not equality in general. This is seen in the next examples.

Example 2.3. Consider the following sets

$$A = \{(0,y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : y = x^2, x \geq 0\}$$
$$B = \{(0,y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : y = x^3, x \geq 0\}.$$

Then it is easy to see that for $u = (0,1) \in R'(A \cap B)$ one has

$$R''(A \cap B; u) = \{(0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\},$$
$$R''(A;u) = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \in \mathbb{R}\},$$
$$R''(B;u) = R''(A;u).$$

And so the inclusion of (v) is strict.

Example 2.4. Define a set A in $\mathbb{R}^3$ and a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^2$ as follows

$$A = \{(n, n^2, n^3) \in \mathbb{R}^3 : n = 1, 2, ...\},$$
$$L(x,y,z) = (x,z) \text{ for } (x,y,z) \in \mathbb{R}^3.$$

Then the first-order asymptotic cone of A is generated by the vector $u = (0,0,1)$. The second-order asymptotic cone of A at u is given by $R''(A;u) = \{0\} \times \mathbb{R}_+ \times \mathbb{R}$, which implies that $L(R''(A;u)) = \{0\} \times \mathbb{R}$. Moreover, $L(u) = (0,1)$ and so $R''(L(A); L(u)) = \mathbb{R}_+ \times \mathbb{R}$. Consequently inclusion in (ix) of Proposition 2.2 is strict. In this example the kernel of L and the second-order asymptotic cone of A at u have nonzero vectors in common.

We notice also that in the assertion $(viii)$ of Proposition 2.2 equality $\mathbb{R}\{u\} = R''(A;u)$ alone does not guarantee the inclusion $A \subseteq \mathbb{R}\{u\} + B(0,r)$. For instance a set A composed of two crossing straight lines $\mathbb{R}\{u\}$ and $\mathbb{R}\{v\}$ does satisfy the equality but not the inclusion.

Among second-order asymptotic directions we are particularly interested in those for which t_n, s_n and a_n satisfy an additional relation.

Definition 2.5. A second-order asymptotic direction v is said to be parabolic if $s_n = t_n$ for every n in (1), that is

$$v = \lim_{n\to\infty} \left(\frac{a_n}{t_n} - t_n u \right) \quad \text{with } t_n \uparrow \infty. \tag{4}$$

And it is said to be canonic if $\|u\| \neq 0$ and $s_n = \frac{\|a_n\|}{t_n}$, that is

$$v = \lim_{n\to\infty} t_n \left(\frac{a_n}{\|a_n\|} - \frac{u}{\|u\|} \right) \quad \text{with } t_n \uparrow \infty, \frac{t_n}{\|a_n\|} \to 0. \tag{5}$$

The sets of all parabolic and canonic second-order asymptotic directions of A at u are denoted respectively by $R^2(A;u)$ and $R^\nu(A;u)$. By convention, if $u = 0$, the set $R^\nu(A;u)$ is empty. The set $R^\nu(A;u)$ is a closed cone. In the terminology of outer limit, $R^2(A;u)$ is expressed as

$$R^2(A;u) = \limsup_{t\uparrow\infty} \left(\frac{A}{t} - tu \right).$$

Thus, $R^2(A;u)$ is closed too. It is not a cone, however. To see the distinction between parabolic and canonic second-order asymptotic directions let us consider the following examples.

Example 2.6. Let A be the same set as in Example 2.3 and $u = (0,1)$. If $v = (p,q) \in R^2(A;u)$ is given by relation (4) with a_n on the curve $y = x^2$, then $a_n = (x_n, x_n^2)$ with $x_n \to \infty$. It follows that

$$p = \lim_{n\to\infty} \frac{x_n}{t_n}, \qquad q = \lim_{n\to\infty} t_n \left(\frac{x_n^2}{t_n^2} - 1 \right).$$

Thus, $p = 1$ since $\lim_{n\to\infty} t_n = \infty$. Hence such parabolic second-order asymptotic directions v belong to $\{1\} \times \mathbb{R}$. Conversely, given $q \in \mathbb{R}$ set, for n sufficiently large, $x_n = (n^2 + nq)^{1/2}$ and $a_n = (x_n, x_n^2)$. Then

$$\lim_{n\to\infty} \left(\frac{a_n}{n} - nu \right) = (1,q).$$

Hence the set of parabolic second-order asymptotic directions defined through sequences a_n belonging to the curve $y = x^2$ is exactly $\{1\} \times \mathbb{R}$. It is easy to see that the set of parabolic asymptotic directions v given by (4) with a_n belonging to $\{0\} \times \mathbb{R}_+$ is $\{0\} \times \mathbb{R}$. Thus, we deduce that $R^2(A;u) = \{0,1\} \times \mathbb{R}$ which is not a cone. Let us find $R^\nu(A;u)$. It is also not difficult to see that 0 is the only canonic direction given by relation (5) with a_n belonging to $\{0\} \times \mathbb{R}_+$. Now assume that $a_n = (x_n, x_n^2)$, $x_n \to \infty$. If $v = (p,q)$ then

$$p = \lim_{n\to\infty} \frac{t_n x_n}{(x_n^2 + x_n^4)^{1/2}}, \tag{6}$$

$$q = \lim_{n\to\infty} t_n \frac{x_n^2 - (x_n^2 + x_n^4)^{1/2}}{(x_n^2 + x_n^4)^{1/2}} = \lim_{n\to\infty} \frac{-t_n x_n^2}{(x_n^2 + x_n^4)^{1/2}(x_n^2 + (x_n^2 + x_n^4)^{1/2})}. \tag{7}$$

Using (6) we deduce that $q = 0$. Since $p \geq 0$, we have $R^\nu(A;u) \subseteq \mathbb{R}_+ \times \{0\}$. Conversely given $p > 0$, by taking $x_n = n$, $a_n = (n, n^2)$ and $t_n = pn$ we see that $(p,0) \in R^\nu(A;u)$. Since $(0,0) \in R^\nu(A;u)$ we infer that $R^\nu(A;u) = \mathbb{R}_+ \times \{0\}$. Thus, we see that $R''(A;u)$, $R^2(A;u)$ and $R^\nu(A;u)$ are distinct.

Example 2.7. Similar calculations applied to the set

$$C = \left\{ \left(n, \frac{n}{\ln(n+1)} \right) \in \mathbb{R}^2 : n = 1, 2, ... \right\}$$

show that in this case, $R'(C) = \mathbb{R}_+ \times \{0\}$ and for $u = (1,0)$, $R''(C;u) = \mathbb{R} \times \mathbb{R}_+$, $R^\nu(C;u) = \{0\} \times \mathbb{R}_+$ while $R^2(C;u)$ is empty.

We now establish some properties of $R^\nu(A;u)$ and its relationship with $R''(A;u)$.

Proposition 2.8. *Let A be a nonempty set in $\mathbb{R}^k$. The following assertions hold.*

(i) *$R^\nu(A;u)$ is nonempty if and only if u is a nonzero first-order asymptotic direction of A.*
(ii) *$R''(A;u) = \mathbb{R}\{u\} + R^\nu(A;u)$ if u is nonzero.*
(iii) *Elements of $R^\nu(A;u)$ are orthogonal to the vector u.*
(iv) *Assume that the first-order asymptotic cone of A is generated by a unit norm vector u, that is $R'(A) = \mathbb{R}_+\{u\}$. Then $R^\nu(A;u) = \{0\}$ if and only if there is a ball $B(0,r)$ with $r > 0$ such that $A \subseteq \mathbb{R}_+\{u\} + B(0,r)$. In particular, if u is an isolated direction, then $R^\nu(A;u) \neq \{0\}$.*

Proof. For (i), if $R^\nu(A;u)$ contains an element v given by relation (4), then $u \neq 0$ by our convention. From Proposition 2.2(ii) and the inclusion $R^\nu(A;u) \subseteq R''(A;u)$ we obtain that $R''(A;u)$ is nonempty, thus $u \in R'(A)$.

Conversely, let u be a unit norm vector that is the limit of a sequence $\{\frac{a_n}{\|a_n\|}\}_{n\geq 1}$ in A with $\lim_{n\to\infty} \|a_n\| = \infty$. Consider the sequence $\{(a_n - \|a_n\|u)\}_{n\geq 0}$. If it is bounded, set $t_n = \sqrt{\|a_n\|}$; otherwise set

$$t_n = \frac{\|a_n\|}{\|(a_n - \|a_n\|u)\|}.$$

Then the sequence of the terms $t_n(\frac{a_n}{\|a_n\|} - u)$ is bounded, and hence we may assume that it converges to some vector v. By construction, t_n converges to ∞ and $\frac{t_n}{\|a_n\|}$ converges to 0. Consequently, v belongs to $R^\nu(A;u)$.

To prove (ii), we notice first that $R^\nu(A;u) \subseteq R''(A;u)$. Using Proposition 2.2($viii$), we deduce

$$\mathbb{R}\{u\} + R^\nu(A;u) \subseteq \mathbb{R}\{u\} + R''(A;u) = R''(A;u).$$

Conversely, let $v \in R''(A;u)$ be given by relation (1) with $u \neq 0$. We write

$$\frac{a_n}{s_n} - t_n u = \frac{\|a_n\|}{s_n} \left(\frac{a_n}{\|a_n\|} - \frac{u}{\|u\|} \right) + \left(\frac{\|a_n\|}{s_n} - t_n \|u\| \right) \frac{u}{\|u\|}. \tag{8}$$

Since $\{\frac{a_n}{s_n} - t_n u\}_{n\geq 0}$ converges to v,

$$\limsup_{n\to\infty} \left| \frac{\|a_n\|}{s_n} - t_n \|u\| \right| \leq \|v\|.$$

Hence we can assume without loss of generality that $\{\frac{\|a_n\|}{s_n} - t_n \|u\|\}_{n\geq 0}$ converges to some $\alpha \in \mathbb{R}$, and by equation (8), $\{\frac{\|a_n\|}{s_n}(\frac{a_n}{\|a_n\|} - \frac{u}{\|u\|})\}_{n\geq 0}$ converges to some $w \in \mathbb{R}^k$. Obviously, $w \in R^\nu(A;u)$ and so $v = w + \alpha u \in R^\nu(A;u) + \mathbb{R}\{u\}$.

To prove (iii), we may assume without loss of generality that $u \in R'(A)$ and $v \in R^\nu(A;u)$ are vectors of unit norm. Thus,

$$u = \lim_{n\to\infty} \frac{a_n}{\|a_n\|}, \tag{9}$$

$$v = \lim_{n\to\infty} \frac{a_n - \|a_n\|u}{\|(a_n - \|a_n\|u)\|}. \tag{10}$$

Using $\|(a_n - \|a_n\|\, u)\|^2 = -2\,\|a_n\|\,(\langle a_n, u\rangle - \|a_n\|)$, we deduce from (10) that

$$\begin{aligned}
\langle v, u\rangle &= \lim_{n\to\infty} \frac{\langle a_n - \|a_n\|u, u\rangle}{\|(a_n - \|a_n\|u)\|} \\
&= \lim_{n\to\infty} \frac{\langle a_n, u\rangle - \|a_n\|}{\|(a_n - \|a_n\|u)\|} \\
&= \lim_{n\to\infty} \frac{-\|(a_n - \|a_n\|u)\|}{2\|a_n\|} \\
&= 0 \quad \text{(by (9))}.
\end{aligned}$$

To prove assertion (iv) we notice that if for every positive integer n, there is some element $a_n \in A \setminus (\mathbb{R}_+\{u\} + B(0,n))$, then by hypothesis one may assume that the sequence $\{\frac{a_n}{\|a_n\|}\}_{n\geq 0}$ converges to u. Moreover, the sequence $\{(a_n - \|a_n\|u)\}_{n\geq 0}$ is unbounded. Defining t_n as in the first part, we obtain a unit norm vector v that belongs to $R^\nu(A;u)$. Conversely, if A is contained in $\mathbb{R}_+\{u\} + B(0,r)$ for some $r > 0$, then by Proposition 2.2$(viii)$ we obtain $R''(A;u) = \mathbb{R}\{u\}$. Combining with parts (ii) and (iii) of the present proposition we deduce $R^\nu(A;u) = \{0\}$. This completes the proof. □

According to (i), the set $R^\nu(A;u)$ is always nonempty when u is a nonzero first-order asymptotic direction of A. This, however, is not the case for $R^2(A;u)$ as we have seen in Example 2.7. Observe also that linear operators being not necessarily isometric, inclusion like that of Proposition 2.2(ix) is, in general, not available for canonic second-order asymptotic directions. This is shown by the next example.

Example 2.9. Let us consider the set A given in Example 2.4 and define a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^2$ by $L(x,y,z) = (x+z, y+z)$. Then $R'(A) = \mathbb{R}\{(0,0,1)\}$ and for the vector $u = (0,0,1)$ we have

$$\begin{gathered}
R^\nu(A;u) = \{0\} \times \mathbb{R}_+ \times \{0\}, \\
L(R^\nu(A;u)) = \{0\} \times \mathbb{R}_+, \\
R^\nu(L(A); L(u)) = \mathbb{R}_+\{(-1,-1)\}.
\end{gathered}$$

From these formulae it is clear that the inclusion (ix) of Proposition 2.2 does not hold for the canonic directions.

Below are some properties of parabolic asymptotic directions.

Proposition 2.10. *For a nonempty set A, the following assertions hold.*

(i) $R^2(A;0) = R'(A)$;

(ii) $R^2(tA+B;tu) = tR^2(A;u)$ *for every* $t \in \mathbb{R}$ *and for every* B *bounded;*

(iii) $R^2(A;u) \subseteq R^2(B;u)$ *if* $A \subseteq B$;

(iv) $R^2(A \cap B;u) \subseteq R^2(A;u) \cap R^2(B;u)$;

(v) $\lambda R^2(A;u) = R^2(A;\lambda^2 u)$ *for every* $\lambda > 0$;

(vi) *If there is some* $a \in A$ *such that the ray* $\{a+tu : t \geq 0\}$ *lies in* A*, then* $u \in R^2(A;u)$*. In particular, if* A *is closed and convex and* $u \in R'(A)$*, then* $u \in R^2(A;u)$.

(vii) $R^2(A;u) + \mathbb{R}\{u\} = R^2(A;u)$*. In particular,* $u \in R^2(A;u)$ *if and only if* $0 \in R^2(A;u)$.

$(viii)$ *If* A *is closed and convex and* $u \in R'(A)$*, then* $R^2(A;u)$ *is nonempty and radiant in the sense that it contains the segment* $[0,v]$ *for every* $v \in R^2(A;u)$.

(ix) *If* $L : \mathbb{R}^k \to \mathbb{R}^m$ *is a linear operator and* $u \in R'(A)$*, then* $L(R^2(A;u)) \subseteq R^2(L(A);L(u))$*. Equality holds provided that* L *is injective on* $R'(A)$ *and* $R''(A;u) \cap \operatorname{Ker}(L) = \{0\}$.

Proof. The four first assertions being clear, we prove the remaining ones. For every $\lambda > 0$ and $v \in R^2(A;u)$ given by equation (4), set $s_n = \frac{t_n}{\lambda}$; then

$$\lim_{n\to\infty}\left(\frac{a_n}{s_n} - \lambda^2 s_n u\right) = \lambda \lim_{n\to\infty}\left(\frac{a_n}{t_n} - t_n u\right) = \lambda v.$$

This proves assertion (v). For (vi), in view of (ii), we may assume that $a = 0$, so that $tu \in A$ for every $t \geq 0$. Choose $t_n = n$, $a_n = (n^2+n)u$. Then $\lim(\frac{a_n}{t_n} - t_n u) = \lim((n+1)u - nu) = u$. Thus, $u \in R^2(A;u)$.

To show (vii), let $v \in R^2(A;u)$ be given by equation (4) and $\lambda \in \mathbb{R}$. Set $s_n = 2t_n^2/(2t_n+\lambda)$. Then one can readily verify that $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} s_n/t_n = 1$ and $\lim_{n\to\infty}(t_n - s_n^2/t_n) = \lambda$. It follows that

$$\lim_{n\to\infty}\left(\frac{a_n}{s_n} - s_n u\right) = \lim_{n\to\infty}\frac{t_n}{s_n}\left(\frac{a_n}{t_n} - t_n u + \left(t_n - \frac{s_n^2}{t_n}\right)u\right) = v + \lambda u.$$

Thus, $v + \lambda u \in R^2(A;u)$. For assertion $(viii)$, it is clear from (vii) that $R^2(A;u)$ is nonempty and it contains 0. Let $v \in R^2(A;u)$ be given by equation (4) and $\alpha \in (0,1)$. To show that $\alpha v \in R^2(A;u)$ we may assume by assertion (ii) that $0 \in A$. Then

$$\alpha v = \lim_{n\to\infty}\left(\frac{\alpha^2 a_n}{\alpha t_n} - \alpha t_n u\right).$$

But $\alpha^2 a_n \in A$ since $0 \in A$, and $\alpha t_n \to \infty$, hence $\alpha v \in R^2(A;u)$.

For the last assertion the inclusion is clear. To prove equality, let $p \in R^2(L(A), L(u))$, say

$$p = \lim_{n\to\infty}\left(\frac{L(a_n)}{t_n} - t_n L(u)\right) \tag{11}$$

for some $a_n \in A$ and $t_n > 0$ converging to ∞. Consider the sequence $\{(\frac{a_n}{t_n} - t_n u)\}_{n\geq 1}$. Exactly as in the proof of assertion (ix) of Proposition 2.2, either that sequence is bounded, in which case it has a cluster point $v \in R^2(A;u)$ with $L(v) = p$, or it is unbounded, in which case we obtain a nonzero vector $q \in R''(A;u)$ with $L(q) = 0$. In both cases we arrive at a contradiction with our assumption. The proof is complete. □

3. Global optimality conditions

In this section we define a second-order asymptotic function for a given real-valued function and derive conditions for its minima via second-order asymptotic directions. Let $X \subseteq \mathbb{R}^k$ be a nonempty set and let f be a real valued function on X. It is known that the first-order asymptotic cone of the epigraph of f is an epigraph. The first-order asymptotic function of f is given by

$$\mathrm{epi}(R'f) = R'(\mathrm{epi}(f)).$$

It follows from the definition that for every $u \in R'(X)$, the point $(u, R'f(u))$ belongs to the asymptotic cone $R'(\mathrm{epi}(f))$, in which $R'f(u)$ is computed by

$$R'f(u) = \inf\left\{\liminf_{n\to\infty} \frac{f(x_n)}{t_n} : x_n \in X, t_n \uparrow \infty, \frac{x_n}{t_n} \to u\right\}.$$

Let us fix a nonzero direction $u \in R'(X)$ for which $R'f(u)$ is finite, and define the lower and upper second-order asymptotic functions of f at $p \in R''(X;u)$ as follows. We first define the set of sequences

$$K(u) = \left\{\{x_n\}_{n\geq 1} : x_n \in X, \|x_n\| \to \infty, \frac{x_n}{\|x_n\|} \to \frac{u}{\|u\|}\right\}$$

and for $\{x_n\}_{n\geq 1} \in K(u)$ and $p \in R''(X;u)$,

$$\begin{aligned} &L(\{x_n\}_{n\geq 1}, p) \\ &= \left\{\{(s_n,t_n)\}_{n\geq 1} : s_n, t_n \uparrow \infty, \frac{x_n}{s_n} - t_n u \to p, \lim_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right) \text{ exists}\right\}. \end{aligned}$$

Then we define

$$\begin{aligned} &R''_- f(u;p) \\ &= \inf_{\substack{\{x_n\}\in K(u) \\ L(\{x_n\},p)\neq\emptyset}} \ \inf_{\{(s_n,t_n)\}\in L(\{x_n\},p)} \ \lim_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right) \\ &= \inf\left\{\liminf_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right) :\ x_n \in X, s_n, t_n \uparrow \infty, \frac{x_n}{s_n} - t_n u \to p\right\}. \end{aligned}$$

Also, we define

$$R''_+ f(u;p) = \inf_{\substack{\{x_n\}\in K(u) \\ L(\{x_n\},p)\neq\emptyset}} \ \sup_{\{(s_n,t_n)\}\in L(\{x_n\},p)} \ \lim_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right).$$

Remark 3.1. It can be easily seen that for any nonzero vector $u \in R'(X)$ with $R'f(u)$ finite, one has $\mathrm{epi}(R''_-f(u;\cdot)) = R''(\mathrm{epi}(f);(u,R'f(u))$. Indeed, if $(p,\alpha) \in R''(\mathrm{epi}(f);(u,R'f(u))$ then there exist sequences $(x_n,\alpha_n) \in \mathrm{epi}(f)$, $t_n, s_n \uparrow \infty$ such that

$$\frac{x_n}{s_n} - t_n u \to p \quad \text{and} \quad \frac{\alpha_n}{s_n} - t_n R'f(u) \to \alpha.$$

Since $f(x_n) \le \alpha_n$ it follows that $R''_-f(u;p) \le \alpha$, i.e., $(p,\alpha) \in \mathrm{epi}(R''_-f(u;\cdot))$. Conversely, if $(p,\alpha) \in \mathrm{epi}(R''_-f(u;\cdot))$ then for each $\varepsilon > 0$ one has $R''_-f(u;p) < \alpha + \varepsilon$; hence there exist sequences $x_n \in X$, $t_n, s_n \uparrow \infty$ such that

$$\frac{x_n}{s_n} - t_n u \to p \quad \text{and} \quad \beta := \lim_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right) \le \alpha + \varepsilon.$$

It follows that for each $\varepsilon > 0$ there exists $\beta \in [\alpha, \alpha+\varepsilon]$ such that $(p,\beta) \in R''(\mathrm{epi}(f);(u,R'f(u))$. Since the second-order asymptotic cone is closed, we deduce that $(p,\alpha) \in R''(\mathrm{epi}(f);(u,R'f(u))$.

Similar definitions can be given for parabolic and canonic second-order asymptotic directions. However, for application purposes to be developed in the this section, we shall focus on the asymptotic functions R''_-f and R''_+f only.

Here are some elementary properties of the second-order asymptotic functions.

Proposition 3.2. *For a nonzero vector $u \in R'(X)$ with $R'f(u)$ finite, the following assertions hold.*

- (i) *For strictly positive numbers α and β one has $R''_-f(\alpha u;\beta p) = \beta R''_-f(u;p)$ and $R''_+f(\alpha u;\beta p) = \beta R''_+f(u;p)$.*
- (ii) *For every strictly positive number λ, one has $R''_-(\lambda f)(u;p) = \lambda R''_-f(u;p)$ and $R''_+(\lambda f)(u;p) = \lambda R''_+f(u;p)$.*
- (iii) *If $R'(f+g)(u) = R'f(u) + R'g(u)$, then*

  $$R''_-(f+g)(u;p) \ge R''_-f(u;p) + R''_-g(u;p).$$

  In particular, this is always the case when f and g are convex and lower semicontinuous on a convex set X.

Proof. To prove (i) let us compute $R''_-f(\alpha u;\beta p)$:

$$\begin{aligned}
&R''_-f(\alpha u;\beta p)\\
&= \inf\left\{\liminf_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(\alpha u)\right) : x_n \in X, s_n, t_n \uparrow \infty, \left(\frac{x_n}{s_n} - t_n\alpha u\right) \to \beta p\right\}\\
&= \inf\Bigg\{\liminf_{n\to\infty}\beta\left(\frac{f(x_n)}{\beta s_n} - \frac{\alpha t_n}{\beta}R'f(u)\right) :\\
&\qquad x_n \in X, (\beta s_n), \left(\frac{\alpha t_n}{\beta}\right) \uparrow \infty, \left(\frac{x_n}{\beta s_n} - \frac{\alpha t_n}{\beta}u\right) \to p\Bigg\}\\
&= \beta R''_-f(u;p).
\end{aligned}$$

The proof for the upper second-order asymptotic function follows the same argument. The second assertion follows from the fact that $R'(\lambda f) = \lambda R'f$. The last assertion is clear from the definition. □

Let us now consider the following optimization problem denoted by (P):

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

where X is a nonempty subset of $\mathbb{R}^k$ and f is a real valued function on $\mathbb{R}^k$.

Theorem 3.3 (necessary condition). *The following conditions are necessary for (P) to have an optimal solution:*

(i) $R'f(u) \geq 0$ *for every* $u \in R'(X)$.

(ii) $R''_{-}f(u;p) \geq 0$ *for every nonzero vector* $u \in R'(X)$ *with* $R'f(u) = 0$ *and* $p \in R''(X;u)$.

Proof. The first condition is already known. Let us prove the second condition. Let u be a nonzero first-order asymptotic direction of X with $R'f(u) = 0$. Let p be a second-order asymptotic vector defined by $p = \lim_{n\to\infty}(\frac{x_n}{s_n} - t_n u)$ where s_n and t_n are positive numbers converging to ∞, and $x_n \in X$. Since $f(x_n)$ is bounded from below by the minimum of f on X we have

$$\liminf_{n\to\infty} \frac{f(x_n)}{s_n} \geq 0.$$

This being true for all sequences $\{x_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ defining p as above, we conclude that $R''_{-}f(u;p) \geq 0$. □

Example 3.4. Consider the function $f(x) = -\sqrt{|x|}$ on $X = \mathbb{R}$. The asymptotic cone of the epigraph of f is the upper half plane. So that

$$R'f(u) = 0 \text{ for every } u \in X.$$

Let us check the second-order condition. It follows from Proposition 2.2(*viii*) that for $u = 1$, the set $R''_{-}(X;u) = X$. Pick $p = 1$ and we look for an estimate of $R''_{-}f(u;p)$. Choose $x_n = n + n^2, s_n = n$ and $t_n = n$. Then

$$\lim_{n\to\infty}\left(\frac{x_n}{s_n} - t_n u\right) = p.$$

Then

$$\liminf_{n\to\infty}\left(\frac{f(x_n)}{s_n} - t_n R'f(u)\right) = \liminf_{n\to\infty}\frac{-\sqrt{n+n^2}}{n} < 0.$$

Hence $R''_{-}f(1;1) < 0$, and the second-order necessary condition is violated. Problem (P) cannot have optimal solutions, as expected.

To obtain a sufficient condition, we will need the following lemma which improves the inclusion $\mathbb{R}\{u\} \subseteq R''(A;u)$ of Proposition 2.2(*viii*):

Lemma 3.5. *Let $\{x_n\}_{n\geq 1}$ be a sequence in X such that* $\lim_{n\to\infty}\|x_n\| = \infty$ *and* $\lim_{n\to\infty}\frac{x_n}{\|x_n\|} = u$. *Then there exist sequences $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ converging to ∞ such that* $\lim_{n\to\infty}(\frac{x_n}{s_n} - t_n u) = u$.

Proof. Without loss of generality we may assume that the vectors x_n are nonzero for every $n \geq 1$. Define λ_n by

$$\frac{1}{\lambda_n} = \max\left\{\frac{1}{\|x_n\|^{1/2}}, \|\frac{x_n}{\|x_n\|} - u\|^{1/2}\right\}.$$

Then $\lambda_n \to \infty$, $\lambda_n(\frac{x_n}{\|x_n\|} - u) \to 0$ and $\frac{\|x_n\|}{\lambda_n} \to \infty$. If we set $s_n = \frac{\|x_n\|}{\lambda_n}$ and $t_n = \lambda_n - 1$, then we obtain

$$\lim_{n\to\infty}\left(\frac{x_n}{s_n} - t_n u\right) = \lim_{n\to\infty}\left(\lambda_n\left(\frac{x_n}{\|x_n\|} - u\right) + u\right) = u$$

as requested. □

Theorem 3.6 (sufficient condition). *Assume that f is lower semicontinuous and X is closed. The following conditions are sufficient for (P) to have an optimal solution:*

(i) $R'f(u) \geq 0$ *for every* $u \in R'(X)$.
(ii) $R''_+f(u;u) > 0$ *for every* $u \in R'(X), u \neq 0$ *with* $R'f(u) = 0$.

Proof. It suffices to show that f is coercive in the sense that $f(x_n)$ tends to ∞ as $x_n \in X$ and $\|x_n\| \to \infty$. Let x_n be a sequence in X with $\lim_{n\to\infty}\|x_n\| = \infty$. We may assume that $\frac{x_n}{\|x_n\|}$ converges to some $u \in R'(X)$. If $R'f(u) > 0$ then clearly $\lim_{n\to\infty} f(x_n) = \infty$. Assume that $R'f(u) = 0$. By Lemma 3.5 there exist sequences s_n and t_n converging to ∞ such that $\lim_{n\to\infty}\frac{x_n}{s_n} - t_n u = u$. This shows that the set $L(\{x_n\}_{n\geq 1}, u)$ is nonempty. Since $R''_+f(u;u) > 0$ by our assumption, we deduce that for some $\{(s_n,t_n)\}_{n\geq 1} \in L(\{x_n\}_{n\geq 1}, u\})$, one has $\lim_{n\to\infty}(\frac{f(x_n)}{s_n} - t_n R'f(u)) = \lim_{n\to\infty}\frac{f(x_n)}{s_n} > 0$. It follows that $\lim_{n\to\infty} f(x_n) = \infty$. □

Note that the assumption $R''_+f(u;u) > 0$ is much weaker than $R''_-f(u;u) > 0$. This becomes clear in the following example.

Example 3.7. Consider the function $f(x) = \sqrt{|x|}$ on $X = \mathbb{R}$. The asymptotic cone of the epigraph of f is the upper half plane. We have $R'f(u) = 0$ for every vector u. Let $u \neq 0$, say $u > 0$. Let $\{x_n\}_{n\geq 1}$ be any sequence of X converging to ∞ (so that $\lim_{n\to\infty} x_n/|x_n| = u/|u|$). Choose

$$t_n = \frac{-u + \sqrt{u^2 + 4ux_n}}{2u}$$

for every $n \geq 1$. Then

$$\lim_{n\to\infty}\left(\frac{x_n}{t_n} - t_n u\right) = u,$$

$$\lim_{n\to\infty}\frac{f(x_n)}{t_n} = \sqrt{u}.$$

Hence $R''_+f(u;u) > 0$. By Theorem 3.6, (P) has optimal solutions. Note that $R''_-f(u;u) = 0$.

4. Efficient points

Let A be a nonempty set and C a closed, convex and pointed cone with a nonempty interior in $\mathbb{R}^k$. A point $x_0 \in A$ is said to be efficient (with respect to the ordering cone C) if

$$A \cap (x_0 + C) = \{x_0\}$$

and it is said to be weakly efficient if

$$A \cap (x_0 + \operatorname{int} C) = \emptyset.$$

It is known (Theorem 3.17, page 52, [9]) that if the set A has weakly efficient points, then

$$R'(A) \cap \operatorname{int} C = \emptyset. \tag{12}$$

When A is closed, a sufficient condition for existence of efficient points can be given by

$$R'(A) \cap C = \{0\}. \tag{13}$$

Actually under the latter condition the set A has the domination property, which means that for every point x in A there is an efficient point a of A such that a dominates x in the sense that $a \in x + C$, that is $a \geq_C x$, where "$\geq_C$" is the usual partial order generated by C. The interested reader is referred to [9] for details on the domination property. In this section we will establish new conditions for efficient points that deepen conditions (12) and (13), by making use of second-order asymptotic directions.

Theorem 4.1. *If A has weakly efficient points, then for every nonzero vector $u \in R'(A) \cap C$ one has*

$$R''(A; u) \cap \operatorname{int} C = \emptyset.$$

Proof. Suppose to the contrary that there is some vector v from the intersection $R''(A; u) \cap \operatorname{int} C$. Then there exist elements $a_n \in A$ and positive numbers t_n and s_n converging to ∞ such that v is the limit of $\frac{a_n}{s_n} - t_n u$ as n tends to ∞. We claim that for each element $a \in A$ there is some a_n such that

$$a_n \in a + \operatorname{int} C. \tag{14}$$

Indeed, if not, one has

$$\frac{a_n}{s_n} - t_n u \notin \frac{a}{s_n} - t_n u + \operatorname{int} C$$

for every $k \geq 1$. Since $u \in C$ we deduce

$$\frac{a_n}{s_n} - t_n u \notin \frac{a}{s_n} + \operatorname{int} C. \tag{15}$$

By passing to the limit in (15) we obtain $v \notin \operatorname{int} C$ which contradicts the hypothesis. Inclusion (14) shows that A has no weakly efficient points and the proof is complete.

□

Notice that since the second-order asymptotic set contains the straight line $\mathbb{R}\{u\}$ by Proposition 2.2($viii$), the conclusion of Theorem 4.1 produces the first-order necessary condition (12) as well. Next we give an example to show that the second-order condition above is, in fact, better than the first-order condition in detecting a set without efficient points.

Example 4.2. Consider the set

$$A := \{(x, y) \in \mathbb{R}^2 : y = x^2, x \geq 0\}$$

and the Pareto ordering cone $C = \mathbb{R}^2_+$. Then

$$R'(A) = \{(0, y) \in \mathbb{R}^2 : y \geq 0\},$$

so that the first-order necessary condition (12) is satisfied. Let us look at the second-order condition. Pick a nonzero first-order asymptotic direction u of A, say $u = (0, 1)$. Then a direct calculation yields

$$R''(A; u) = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}.$$

In particular the second-order asymptotic set $R''(A; u)$ meets the interior of the cone C, and hence the second-order necessary condition is violated, by which A has no weakly efficient points as expected.

Now we turn to second-order sufficient conditions. Given a subset D of $\mathbb{R}^k$ with $0 \in D$, the tangent cone of D at 0 is defined by $T(D) = \limsup_{t\to\infty} tD$.

Theorem 4.3. *Let A be a nonempty closed set in $\mathbb{R}^k$. If every nonzero direction $u \in R'(A) \cap C$ is isolated and satisfies the condition*

$$R^{\nu}(A; u) \cap T(C - u) = \{0\},$$

then the set A has the domination property.

Proof. Our aim is to show that for every point x in A, the section $A_x := A \cap (x + C)$ is compact. This will achieve the proof because being nonempty and compact, the section A_x admits efficient points which are also efficient points of the set A (Theorem 3.3, page 46 [9]) and evidently dominate x. Suppose to the contrary that for some $x \in A$ the section A_x is unbounded. Pick a sequence $\{a_n\}_{n\geq 1}$ in A_x with $\|a_n\| \to \infty$ and such that the sequence $\{\frac{a_n}{\|a_n\|}\}_{n\geq 1}$ converges to some unit norm vector u. It is clear that u is an asymptotic direction of A and belongs to C. Consider the projection u_n of a_n on $\mathbb{R}\{u\}$. Let α_n be a positive number such that $u_n = \alpha_n u$. By hypothesis u is an isolated direction, the sequence $\{a_n - \alpha_n u\}_{n\geq 0}$ is unbounded, say with $s_n := \|a_n - \alpha_n u\| \to \infty$ as $n \to \infty$. Without loss of generality we may assume that $\{\frac{a_n - \alpha_n u}{s_n}\}_{n\geq 0}$ converges to a unit norm vector v. By construction, the vectors $a_n - \alpha_n u$ are orthogonal to u, therefore the limit v is orthogonal to u too. We claim that

$$v \in R''(A; u) \cap T(C - u). \tag{16}$$

Indeed, set $t_n = \frac{\alpha_n}{s_n}$. Then as we have already mentioned above the numbers s_n converge to ∞. We prove that t_n converges to ∞ too. Suppose that this is not the case, say t_n converges to some positive number t_0. Then

$$\begin{aligned} v &= \lim_{n\to\infty} \left(\frac{a_n}{s_n} - t_n u\right) \\ &= \lim_{n\to\infty} \frac{a_n}{s_n} - t_0 u \\ &= \alpha u \end{aligned}$$

for some real number α because u is an asymptotic direction defined by the sequence $\{a_n\}_{n\geq 1}$. But this contradicts the fact that v is orthogonal to u. Thus, both s_n and t_n converge to ∞ and hence v is a second-order asymptotic direction of A at u. Furthermore, since $a_n \in x + C$, we have $\frac{a_n - x}{s_n t_n} \in C$. Let us express v as

$$v = \lim_{n\to\infty} t_n \left(\frac{a_n}{s_n t_n} - u\right) = \lim_{n\to\infty} t_n \left(\frac{a_n - x}{s_n t_n} - u\right)$$

which shows that v is a first-order tangential direction of the cone C at u. This establishes (16). Since v is orthogonal to u, Proposition 2.8 shows that $v \in R^{\nu}(A;u) \cap T(C-u)$ and so we arrive at a contradiction with the hypothesis of the theorem. The proof is complete. $\square$

Let us derive a sufficient condition making use of the full second-order asymptotic cone $R''(A;u)$.

Corollary 4.4. *If every nonzero direction $u \in R'(A) \cap C$ is isolated and satisfies the condition*

$$R''(A;u) \cap T(C-u) \subseteq \mathbb{R}\{u\},$$

then the set A has the domination property.

Proof. It is plain that the hypothesis of this corollary implies that of Theorem 4.3, and so the conclusion follows immediately. $\square$

Needless to say that the condition given in Corollary 4.4 is stronger than the condition required in Theorem 4.3, and both of them are a consequence of the first-order condition (13). Below we present an example to show that the second-order condition of Corollary 4.4 is a real refinement of the first-order condition (13).

Example 4.5. Consider the set $A := \{(x, x^2) \in \mathbb{R}^2 : x \leq 0\}$ and the ordering cone $C := \mathbb{R}^2_+$. Then the first-order asymptotic cone of A is given by

$$R'(A) = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}.$$

We see that the first-order condition (13) is not fulfilled. Pick any nonzero first-order asymptotic direction u of A. It is clear that u is isolated. Moreover, the second-order asymptotic set is given by

$$R''(A;u) = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$$

and the first-order tangent cone of C at u is the set

$$T(C-u) = \{(x,y) \in \mathbb{R}^2 : x \geq 0\}.$$

Hence the hypothesis of Theorem 4.3 is satisfied and by this A has the domination property.

Notice that without the hypothesis on isolated directions, the result may fail. To see this consider the set $A = \{(0,y) \in \mathbb{R}^2 : y \geq 0\}$. Then $R'(A) = A$ and $R''(A;u) = \mathbb{R}\{u\}$ for every nonzero $u \in R'(A)$. We have $R''(A;u) \cap T(C-u) = \mathbb{R}\{u\}$. However, A has no efficient points. Its asymptotic directions are not isolated.

The sufficient condition of Theorem 4.3 can be simplified when the cone C is polyhedral.

Corollary 4.6. *Assume that C is a polyhedral cone and that every nonzero direction $u \in R'(A) \cap C$ is isolated and satisfies the condition*

$$R^{\nu}(A;u) \cap (C-u) = \{0\}.$$

Then the set A has the domination property.

Proof. When the cone C is polyhedral, the tangent cone to C at u is the cone generated by the polyhedral set $C-u$. Thus, a nonzero vector v belongs to $T(C-u)$ if and only if there is a strictly positive number t and a nonzero vector $w \in C-u$ such that $v = tw$. Due to this observation the corollary follows from Theorem 4.3. $\square$

We end this section by an example to show that the last corollary does not work when the cone C is not polyhedral.

Example 4.7. Let us define an ordering cone C in $\mathbb{R}^3$ by

$$C = \{(x,y,z) \in \mathbb{R}^3 : (x-z)^2 + y^2 \leq z^2, z \geq 0\}.$$

Consider the set A defined by

$$A = \{(n, n\sqrt{2n-1}, n^2) \in \mathbb{R}^3 : n = 0, 1, ...\}.$$

It is easy to see that

$$R'(A) = \{(0,0,z) \in \mathbb{R}^3 : z \geq 0\}$$

and so

$$R'(A) \cap C = R'(A).$$

We claim that every asymptotic direction $u = (0,0,\alpha)$ with $\alpha > 0$ is isolated. Indeed, the projection u_n of $a_n := (n, n\sqrt{2n-1}, n^2)$ on $\mathbb{R}\{u\}$ is given by $u_n = \frac{n^2}{\alpha}u$. Hence $a_n - u_n = (n, n\sqrt{2n-1}, 0)$ which implies that for every positive number r, a_n does not belong to $\mathbb{R}\{u\} + B(0,r)$ as soon as $n \geq r$. Moreover, one can see without difficulty that the second-order asymptotic set is given by

$$R''(A;u) = \{(0,y,z) \in \mathbb{R}^3 : y \geq 0\}.$$

hence

$$R^\nu(A;u) = \{(0,y,0) \in \mathbb{R}^3 : y \geq 0\}.$$

This yields

$$R^\nu(A;u) \cap (C-u) = \{0\},$$

i.e., the condition of Corollary 4.6 is satisfied. However, the set A does not have the domination property; even more, it has no efficient points because for each $n \geq 1$,

$$a_{n+1} - a_n = \left(1, (n+1)\sqrt{2(n+1)-1} - n\sqrt{2n-1}, (n+1)^2 - n^2\right) \in C.$$

The assumption on the cone C in Corollary 4.6 is violated, this cone is not polyhedral. Notice also that the condition of Theorem 4.3 is not satisfied either, for the intersection of the second-order asymptotic set $R^\nu(A;u)$ with the tangent cone $T(C-u)$ is the set $R^\nu(A;u)$ itself and does contain a nonzero vector orthogonal to u.

5. Closedness of the sum of two closed sets

Let A and B be two closed sets in $\mathbb{R}^k$. We are interested in conditions under which the sum $A+B$ is closed. This important problem of applied analysis has been studied by a number of researchers starting with Dieudonne's work on convex sets (see [4], [6], [7], [9], [13], [21]). The classical result states that if $R'(A)$ and $R'(-B)$ have no nonzero vector in common, then the sum $A+B$ is a closed set. A nice extension has been given in [12] which reduces to the classical case by a quotient space technique. In this section we relax the above condition by using second-order asymptotic directions.

Definition 5.1. Let u be a unit norm first-order asymptotic direction of A. We say that A is strongly closed in direction u if for every sequence $\{a_n\}_{n\geq 1}$ in A with $\lim_{n\to\infty} \frac{a_n}{\|a_n\|} = u$ and every $t_n > 0$ converging to ∞, all cluster points of the sequence $\{a_n - t_n u\}_{n\geq 1}$ belong to A. We say that A is closed in direction u if either u is isolated, or for every sequence $\{a_n\}_{n\geq 1}$ in A with $\lim_{n\to\infty} \frac{a_n}{\|a_n\|} = u$ there exists a sequence $t_n > 0$ converging to ∞, such that some cluster point of the sequence $\{a_n - t_n u\}_{n\geq 1}$ belongs to A.

It is clear that if A is strongly closed in direction u, then it is closed in that direction. The converse is not always true. Moreover, if u is an isolated asymptotic direction of A, then A is strongly closed in this direction.

Theorem 5.2. *Let A and B be nonempty closed sets. Assume that for every unit norm direction $u \in R'(A) \cap R'(-B)$, the following conditions hold*

(i) *Either u is an isolated direction of A or $-B$, or one of the sets A and $-B$ is closed in u and the other is strongly closed in u;*

(ii) $R^\nu(A;u) \cap R^\nu(-B,u) = \{0\}$.

Then the sum $A+B$ is closed.

Proof. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences in A and B such that $\{a_n + b_n\}_{n\geq 1}$ converges to some limit c. We need to show that this limit belongs to $A+B$. Note that the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are both bounded or both unbounded. If they are bounded, then by selecting subsequences instead we may assume that they converge, respectively, to some points $a \in A$ and $b \in B$. In this case $c = a + b$ belongs to $A + B$ and we are done. If those sequences are unbounded, then we may assume that $\{\|a_n\|\}_{n\geq 1}$ and $\{\|b_n\|\}_{n\geq 1}$ converge to ∞. By taking again a subsequence if necessary, we obtain that $\{a_n/\|a_n\|\}_{n\geq 1}$ converges to some unit norm asymptotic vector $u \in R'(A)$ which is also the limit of the sequence $\{-b_n/\|a_n\|\}_{n\geq 1}$ and an asymptotic direction of $-B$. We now show that the sequence $\{a_n - \|a_n\|u\}_{n\geq 1}$ is bounded. If not, we may assume that the sequence $\{\|(a_n - \|a_n\|u)\|\}_{n\geq 1}$ converges to ∞ and the sequence $\{\frac{a_n - \|a_n\|u}{\|(a_n - \|a_n\|u)\|}\}_{n\geq 1}$ converges to some unit norm vector $v \in R^\nu(A; u)$. In particular, $v \perp u$. Note that

$$\lim_{n\to\infty}\left(\frac{a_n - \|a_n\|u}{\|(a_n - \|a_n\|u)\|} + \frac{b_n + \|a_n\|u}{\|(a_n - \|a_n\|u)\|}\right) = 0$$

hence the sequence $\{\frac{-b_n - \|a_n\|u}{\|(a_n - \|a_n\|u)\|}\}_{n\geq 1}$ converges to v. Since

$$\lim_{n\to\infty}\frac{\|a_n\|}{\|(a_n - \|a_n\|u)\|} = \lim_{n\to\infty}\frac{1}{\left\|\frac{a_n}{\|a_n\|} - u\right\|} = \infty$$

and $v \perp u$, we have $v \in R^\nu(-B; u)$. But this contradicts assumption (ii).

Since the sequence $\{a_n - \|a_n\|u\}_{n\geq 1}$ is bounded,$\{b_n + \|a_n\|u\}_{n\geq 1}$ is also bounded. Thus u is an isolated asymptotic direction neither for A nor for $-B$. Now assume that A is strongly closed in u. Since $-B$ is closed in u, we may assume that for some $t_n > 0$ converging to ∞, $\{-b_n - t_n u\}_{n\geq 1}$ converges to some $-b \in -B$. Then from $a_n + b_n \to c$ we get that $\{a_n - t_n u\}_{n\geq 1}$ converges to $a := c - b$. Since A is strongly closed in u, we have $a \in A$, hence $c \in A + B$. The case where $-B$ is strongly closed in u and A is closed in u is similar. The proof is complete. □

It is worthwhile noticing that in the first condition of Theorem 5.2 if none of the sets A and B is strongly closed in direction u, then the conclusion may fail as the next example shows.

Example 5.3. Consider the following sets in $\mathbb{R}^2$:

$$A = \left\{\left(1 - \frac{1}{n}, n - 1\right) \in \mathbb{R}^2 : n = 1, 2, ...\right\} \cup \{(x, 0) : x \geq 0\},$$
$$B = \left\{\left(1 - \frac{1}{n}, 2 - n\right) \in \mathbb{R}^2 : n = 1, 2, ...\right\} \cup \{(x, 0) : x \geq 0\}.$$

It is clear that $R'(A) \cap R'(-B) = \{(0, y) : y \geq 0\}$. For a unit norm vector $u = (0, 1)$ from $R'(A) \cap R'(-B)$ the canonic second-order asymptotic direction sets of A and $-B$ at u are zero, so that condition (ii) on the canonic second-order asymptotic set is satisfied. It can be seen that the sets A and $-B$ are closed in direction u; but none of them is strongly closed in this direction. The sum $A + B$ is not closed because the limit point $(2, 1)$ of the sum $(1 - \frac{1}{n}, n - 1) + (1 - \frac{1}{n}, 2 - n)$ when n tends to ∞ does not belong to $A + B$.

Here is a useful corollary when a set has only isolated asymptotic directions.

Corollary 5.4. *Let A and B be nonempty closed sets. Assume that every unit norm vector u from $R'(A) \cap R'(-B)$ is an isolated direction of A or B, and $R^\nu(A;u) \cap R^\nu(-B,u) = \{0\}$. Then $A+B$ is closed.*

Proof. This is immediate from Theorem 5.2. □

Below we present some examples to illustrate the second-order condition given in Theorem 5.2. The two first examples show that the new condition works well even when the classical first-order condition does not hold. The last example proves the role of condition (ii).

Example 5.5. Let us define

$$A = \left\{ \left(1 - \frac{1}{n}, n-1\right) \in \mathbb{R}^2 : n = 1, 2, ... \right\} \cup \{(1,0)\},$$
$$B = \{(1,t) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Then we have $R'(A) \cap R'(-B) = \{(0,y) : y \geq 0\}$. Hence the classical theorem does not apply. Let us verify the conditions of Theorem 5.2. For the unit norm vector $u = (0,1) \in R'(A) \cap R'(-B)$, the set A is closed in u, and the set B is strongly closed in u. Moreover, both of the sets $R^\nu(A;u)$ and $R^\nu(-B;u)$ are trivial, so that their intersection is the zero vector. By Theorem 5.2 the sum $A+B$ is closed.

Example 5.6. In this example we consider the sets

$$A = \{(n, n^2) \in \mathbb{R}^2 : n = 1, 2, ...\},$$
$$B = \{(n, -n^3) \in \mathbb{R}^2 : n = 1, 2, ...\}.$$

It is clear that $R'(A) \cap R'(-B) = \{(0,y) : y \geq 0\}$. Hence the classical criterion is not applicable. Since the unit norm vector $u = (0,1) \in R'(A) \cap R'(-B)$ is an isolated direction of A and $-B$, and

$$R^\nu(A;u) = \{(x,0) \in \mathbb{R}^2 : x \geq 0\},$$
$$R^\nu(-B;u) = \{(x,0) \in \mathbb{R}^2 : x \leq 0\},$$

we see that the hypothesis of Corollary 5.4 is satisfied, by which the sum $A+B$ is closed.

Example 5.7. The sets A and B are now defined as follows

$$A = \{(n, n^2) \in \mathbb{R}^2 : n = 1, 2, ...\},$$
$$B = \left\{ \left(-n + \frac{1}{n}, -n^2\right) \in \mathbb{R}^2 : n = 1, 2, ... \right\}.$$

Again we have $R'(A) \cap R'(-B) = \{(0,y) : y \geq 0\}$. It is clear that the unit norm vector $u = (0,1) \in R'(A) \cap R'(-B)$ is an isolated direction of A and $-B$. Direct computation shows that $R^\nu(A;u) \cap R^\nu(-B;u)$ consists of all vectors $(x,0) \in \mathbb{R}^2$ with $x \geq 0$. The condition on second-order asymptotic directions of Corollary 5.4 is not fulfilled; and the sum $A+B$ is not closed either.

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