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Asplund Sets and Metrizability for the Polynomial Topology*

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The theme of this paper is the study of the separability of subspaces of holomorphic functions respect to the convergence over a given set and its connection with the metrizability of the polynomial topology. A notion closely related to this matter is that of Asplund set. Our discussion includes an affirmative answer to a question of Globevnik about interpolating sequences. We also consider the interplay between polynomials and Asplund sets and derive some consequences of it. Among them we obtain a characterization of Radon-Nikodým composition operators on algebras of bounded analytic functions.

Keywords: Algebras of analytic functions, Asplund set, composition operator, interpolation, polynomial topology, Radon-Nikodým property

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1. Introduction and preliminaries

In this article we deal firstly with the natural matter of the separability of some algebras of bounded analytic functions defined on complex Banach spaces E with open unit ball B_E which may be seen as the natural extensions of the disc algebra. We will prove that the algebra

$$A_\infty(B_E) := \{f : \overline{B}_E \to \mathbb{C} : f \text{ continuous, bounded and analytic on } B_E\}$$

endowed with the sup norm is separable if, and only if, E is finite-dimensional. This is achieved by answering affirmatively the question, raised by Globevnik [15], of the existence in B_E of interpolating sequences for $A_\infty(B_E)$ if E is a non-reflexive Banach space.

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The separability of the subalgebra $A_u(B_E) \subset A_\infty(B_E)$ of analytic and uniformly continuous functions on B_E is also discussed and we give a simple characterization of it in terms of the metrizability of its spectrum for the polynomial topology, that is, the topology $\tau(P(E))$ of the convergence against the continuous polynomials on E. Recall that the spectrum M_A of a Banach algebra A is the set of all non-null scalar-valued homomorphisms of algebras defined on A.

There are a number of equivalent formulations for the concept of Asplund set in a Banach space (see [5] Thm. 5.2.11). The most convenient for our purposes is the following given by Stegall and actually shown equivalent to the original one by Fitzpatrick. For a Banach space E and $A \subset E$, recall that $\|\cdot\|_A$ denotes the seminorm on E^* given by $\|x^*\|_A = \sup_{x\in A} |x^*(x)|$ for any $x^* \in E^*$.

Definition 1.1. Let E be a Banach space. A bounded set $D \subset E$ is said to have the *Asplund property* (AP) or to be an *Asplund set* if for any countable subset $A \subset D$ the seminormed space $(E^*, \|\cdot\|_A)$ is separable.

The connection between the Asplund property and the weak metrizability is recalled in Theorem 1.2 below. We focus on such kind of connection in a context in which the dual space E^* is replaced by the space $P(E)$ of continuous polynomials on E. Along this lines, we will study some properties related to separability in $A_u(B_E)$ and will connect this to the study of the metrizability of some subsets of B_E with respect to the polynomial topology, and, in consequence, to the Asplund property. For instance, we prove that a subset $D \subset E$ has the Asplund property if any separable $A \subset \overline{aco(D)}$ is metrizable for the polynomial topology $\tau(P(E))$, where $aco(D) = \{\sum t_n x_n : x_n \in A, \sum |t_n| \leq 1, t_n \in \mathbb{C}\}$ denotes the absolutely convex hull of D.

It is known that linear bounded operators map Asplund sets into Asplund sets. In the third section, we discuss the behaviour of Asplund sets against polynomials instead of linear operators, and we show that whenever the finite type polynomials are uniformly dense, then polynomials map Asplund sets into Asplund sets. As a consequence, we recover a condition for the Banach space of continuous k-homogeneous polynomials $P(^kE)$ to be a Radon-Nikodým space earlier proved by Aron and Dineen [2].

The study of composition operators is a nowadays vast field of research. We contribute to it in §4 with an application of results in §3 that produces a characterization of Radon-Nikodým composition operators on $H^\infty(B_E)$, the Banach space of bounded analytic functions on B_E endowed with the uniform norm.

Theorem 1.2 (Fitzpatrick, [5] Thm. 5.4.1). *Let E be a Banach space and D a bounded set of E. Then the following statements are equivalent:*

a) *D has the Asplund property,*
b) *If L is a separable subset of $\overline{aco}(D)$, then L is weakly metrizable.*

A warning here: the closed absolutely convex hull of a weakly metrizable set need not be a weakly metrizable set; think for instance of the set L of the unit vectors in ℓ_1, which is weakly discrete and whose closed absolutely convex hull, the unit ball of ℓ_1, cannot be weakly metrizable, since otherwise, by Schur's lemma, the weak topology would coincide there with the norm topology.

Definition 1.3. Let E and F be Banach spaces. The space E is said to be an *Asplund space* if its unit ball B_E has the AP or, equivalently, if any closed separable subspace X of E has a separable dual X^*. A linear operator $T : E \longrightarrow F$ is said to be an *Asplund operator* if $T(B_E)$ has the AP.

Recall (see [5] Proposition 5.2.2 d)) that D has the AP if and only if $\overline{aco}(D)$ also has the AP. It is obvious that a bounded set D has the AP if and only if for any $A \subset D$ norm-separable, the seminormed space $(X^*, \|\cdot\|_A)$ is separable.

All reflexive spaces are Asplund spaces while ℓ_1 is not. We refer to Chapter 5 of [5] for extensive background regarding Asplund sets. Every weakly compact set in a Banach space is an Asplund set (see [16] Cor. 287). A Banach space E is an Asplund space if, and only if, E^* has the Radon-Nikodým property (see [9]).

We briefly recall some notions about holomorphic mappings. Throughout this paper, E and F will denote complex Banach spaces. A mapping is holomorphic (or analytic) if it is Fréchet differentiable on its domain. A k-homogeneous polynomial is a mapping $P : E \to F$ of the form $P(x) = A(x, \ldots, x),\ x \in E$, where A is a continuous F-valued k-linear map on E. We put $P \in P(^kE; F)$, and if $F = \mathbb{C}$, we simply put $P \in P(^kE)$. A polynomial is a finite sum of homogeneous polynomials. The space of all polynomials is denoted by $P(E; F)$ ($P(E)$, respectively) and it is endowed with the topology of the uniform convergence on B_E. The subalgebra of $P(E)$ generated by E^* is denoted by $P_f(E)$ and its elements are referred as to the finite type polynomials. Further background for analytic functions defined on Banach spaces can be found in [10].

Recall (see [20]) that a predual of $H^\infty(B_E)$ is given by $G^\infty(B_E)$, the closed subspace of $H^\infty(B_E)^*$ given by the functionals $u \in H^\infty(B_E)^*$ whose restriction to the closed unit ball of $H^\infty(B_E)$ is continuous for the compact-open topology, that is, the topology of uniform convergence on compact sets of B_E. We have that the closed absolutely convex hull of the set of evaluations $\delta_x : H^\infty(B_E) \longrightarrow \mathbb{C}$, $x \in B_E$, coincides with the unit ball of $G^\infty(B_E)$. Analogously, (see [20] as well) the set $Q(^kE) \subset P(^kE)^*$ of all $u \in P(^kE)^*$ such that u restricted to the closed unit ball of $P(^kE)$ is continuous for the compact-open topology is a closed subspace of $P(^kE)^*$ and $Q(^kE)$ is a predual of $P(^kE)$.

2. Separability and metrizability

2.1. Separability in $A_\infty(B_E)$

J. Globevnik studied in [15] the existence of interpolating sequences for $A_\infty(B_E)$, that is, the existence of sequences $(x_n) \subset B_E$ such that for every bounded sequence $(\alpha_n) \subset \mathbb{C}$, there is $f \in A_\infty(B_E)$ such that $f(x_n) = \alpha_n\ \forall n$. He proved such existence for a class of infinite dimensional Banach spaces including the reflexive spaces and he asked whether this result holds for all infinite dimensional Banach spaces. We answer this question affirmatively by proving the existence of interpolating sequences for $A_\infty(B_E)$ for non-reflexive Banach spaces E.

Denote by S_E the unit sphere of a Banach space E. A point $x \in S_E$ is said to be strongly exposed if there exists $L \in E^*$ such that $L(x) = \|L\| = 1$ and for any sequence (x_n) in E with $\lim_n L(x_n) = 1$, then $\lim_n x_n = x$ in E.

Theorem 2.1 ([15]). *Let E be an infinite dimensional complex Banach space whose unit sphere S_E contains a sequence (x_n) of strongly exposed points of $\bar{B}_E$ with no cluster points. Then (x_n) is an interpolating sequence for $A_\infty(B_E)$.*

As it is pointed out in [15], the class of Banach spaces that satisfies the conditions of Theorem 2.1 contains the infinite dimensional reflexive spaces.

Recall the following theorem of James,

Theorem 2.2 ([18]). *A Banach space E is reflexive if and only if every functional $L \in E^*$ attains its norm on $\bar{B}_E$.*

Next, we extend the result given by Globevnik to all infinite dimensional complex Banach spaces,

Theorem 2.3. *Let E be an infinite dimensional complex Banach space. Then there exist interpolating sequences for $A_\infty(B_E)$.*

Proof. As just said, we only have to deal with the non-reflexive case. Thus, we assume that E is not reflexive. Then by the James Theorem 2.2, there exists a functional $L \in E^*$ such that $\|L\| = 1$ which does not attain its norm on $\bar{B}_E$. Moreover, there exists $(x_n) \subset S_E$ such that $|L(x_n)| \to 1$. We deduce (see Corollary in [17], p. 204) the existence of a subsequence $(L(x_{n_k}))_k$ which is interpolating for H^∞. To check that (x_{n_k}) is interpolating for $A_\infty(B_E)$, set $(\alpha_k)_k \in \ell_\infty$. Then there exists $h \in H^\infty$ such that $h(L(x_{n_k})) = \alpha_k$. Consider the function $g : \bar{B}_E \longrightarrow \mathbb{C}$ defined by

$$g(x) = h \circ L(x) \quad \text{for any } x \in \bar{B}_E.$$

We claim that g belongs to $A_\infty(B_E)$. Indeed, for $x \in \bar{B}_E$, since L does not attain its norm, there exists $\delta_x > 0$ such that $|L(x)| < 1 - \delta_x$. Then,

$$|L(y)| < 1 \quad \text{for any } y \in B(x, \delta_x)$$

so g extends to $\tilde{g}$ on $B(x, \delta_x)$ for any $x \in \bar{B}_E$ and $\tilde{g}$ is analytic there. Hence, $\tilde{g}$ is analytic and bounded on the open neighbourhood $\bigcup_{x \in \bar{B}_E} B(x, \delta_x)$ of $\bar{B}_E$. In particular, $g \in A_\infty(B_E)$ since it is the restriction of $\tilde{g}$ to $\bar{B}_E$. Thus (x_{n_k}) is interpolating for $A_\infty(B_E)$. □

From this result we obtain the following characterization of the separability of $A_\infty(B_E)$ in terms of the finite dimension of E,

Corollary 2.4. *Let E be a complex Banach space. The following statements are equivalent:*

i) $A_\infty(B_E)$ *is non-separable.*
ii) E *is infinite dimensional.*
iii) *There exist interpolating sequences for* $A_\infty(B_E)$.

Proof. If E is finite-dimensional, then $A_\infty(B_E) = A_u(B_E)$ is separable. If E is infinite dimensional, by Theorem 2.3 we have that there exist interpolating sequences for $A_\infty(B_E)$. If there is an interpolating sequence, then the algebra cannot be separable. □

2.2. Separability in $A_u(B_E)$

It is clear that $A_u(B_E)$ is separable if E^* is separable and $P_f(E)$ is dense in $A_u(B_E)$. This condition is satisfied by c_0, the Tsirelson space T^*, the Tsirelson-James space introduced in [2], and $d_*(w)$, the predual of the Lorentz space considered in [8].

On the other hand, it is known that if $\ell_1 \subset E$, then there exists an interpolating sequence for $A_u(B_E)$ and, therefore, the algebra $A_u(B_E)$ is not separable. The converse is false. Indeed, given $1 < p < \infty$, we have that for the vectors (e_n) in the canonical basis the sequence $(\frac{e_n}{2})$ is interpolating for the algebra $A_u(B_{\ell_p})$. This follows easily by choosing $m \in \mathbb{N}$, $m \geq p$, and considering for $\alpha = (\alpha_n) \in \ell_\infty$ the polynomial $P_\alpha \in P(^m\ell_p)$ given by

$$P_\alpha((x_n)) = 2^m \sum_{n=1}^{\infty} \alpha_n x_n^m$$

for which $P_\alpha(\frac{e_k}{2}) = \alpha_k$. However, $\ell_1 \not\subseteq \ell_p$ since ℓ_p is reflexive. Notice that this example also shows that in order to assure that $A_u(B_E)$ is separable it is not sufficient that E does not contain ℓ_1.

It is well-known that a uniform algebra A is separable if, and only if, its spectrum is metrizable for the Gelfand topology, that is, the topology of pointwise convergence against the elements in A. Indeed, it is sufficient to notice that M_A is a subset of $\overline{B}_{A^*}$ and that the Gelfand topology in M_A is the restriction of the $w(A^*, A)$-topology to M_A. Next we give another simple characterization for $A_u(B_E)$ to be separable.

Proposition 2.5. *Let E be a complex Banach space and $A = A_u(B_E)$. Then, A is separable if and only if M_A is $\tau(P(E))$-metrizable.*

Proof. As we have mentioned above, A is separable if and only if M_A is $w(A^*, A)$-metrizable. Since $P(E)$ is a dense set in $A_u(B_E)$, the Hausdorff topology $\tau(P(E))$ coincides on M_A with the finer compact topology $w(A^*, A)$. □

Remark 2.6. *If E is infinite dimensional, then the polynomial topology is strictly finer than the weak topology.* It suffices to consider a linearly independent sequence $(L_n) \subset B_{E^*}$, and the 2-homogeneous polynomial $P(x) = \sum \frac{L_n^2(x)}{2^n}$; it is obviously $\tau(E, P(E))$-continuous and not weakly continuous since it is not of finite type ([1] 4.1).

Proposition 2.5 leads us to study the metrizability for the polynomial topology of bounded sets of M_A, in particular those of B_E, in a similar way as the weak topology is considered in Theorem 1.2.

Let us denote by $\tau(E, P(E))$ the polynomial topology on E. If we replace M_A by a smaller set L in Proposition 2.5, the $\tau(E, P(E))-$metrizability of $L \subset B_E$ may not lead to the separability of $A_u(B_E)$ for the seminorm $\|\cdot\|_L$. Indeed, consider $E = \ell_2$

and $L = \left\{ \frac{e_n}{2} : n \in \mathbb{N} \right\}$. It is clear that L is a closed bounded separable $\tau(\ell_2, P(\ell_2))$-metrizable subset of B_{ℓ_2} since it is $\tau(\ell_2, P(\ell_2))$-discrete. However, $(P(^2\ell_2), \|\cdot\|_L)$ is not separable since L is an interpolating sequence for $P(^2\ell_2)$ as we have seen above.

We have the following strengthening of the necessary condition in 2.5.

Proposition 2.7. *Let $B \subset E$ be a separable bounded set. If $(P(^kE), \|\ \|_B)$ is separable for all $k \in \mathbb{N}$, then B is $\tau(E, P(E))$-metrizable.*

Proof. Since for L, the absolutely convex closed hull of B in $Q(^kE)$, we have that $\|P\|_L = \|P\|_B \ \forall P \in P(^kE)$, our assumption shows that $(P(^kE), \|\ \|_L)$ is separable. Consequently, $L \subset Q(^kE)$ is an Asplund set. Hence Theorem 1.2 shows that the separable subset $B \subset L$ is $w(Q(^kE), P(^kE))$-metrizable. To conclude, observe that a net is convergent in $(B, \tau(E, P(E)))$ if, and only if, it is convergent in $(B, w(Q(^kE), P(^kE))$ for all $k \in \mathbb{N}$. So $(B, \tau(E, P(E)))$ is metrizable as the supremum of a countable family of metric topologies. $\square$

Proposition 2.8. *Let E be a Banach space and let L be a bounded separable absolutely convex set in E which is $\tau(E, P(E))$-metrizable. Then, $(E^*, \|\cdot\|_L)$ is separable.*

Proof. Since L is a bounded set, we can also assume without loss of generality that $L \subset B_E$. Since L is $\tau(E, P(E))$-metrizable, there exists a countable basis of $\tau(E, P(E))$-neighbourhoods (V_n) of 0 in L. Therefore, there exists a sequence (F_n) of finite subsets of $P(E)$ such that the sequence (V_n) is given by

$$V_n = \{x \in L : |P(x) - P(0)| \le 1 \text{ for any } P \in F_n\}.$$

Set $G_n = \{dP_x : x \in L,\ P \in F_n\}$. Since the mapping $x \in E \mapsto dP_x \in E^*$ is continuous, G_n is a norm separable set in E^* for all $n \in N$. Let G be the norm closure in E^* of $\operatorname{span}(\cup_{n=1}^{\infty} G_n)$. It is clear that G is a separable subspace of E^* and G is $\|\cdot\|_L$-separable as well.

We want to prove that $E^* = G + L^\circ$. For this, let $g \in E^*$ and consider U the $\tau(E, P(E))$-neighbourhood of 0 defined by

$$U = \{x \in L : |g(x)| \le 1\}.$$

Then, we find $m \in \mathbb{N}$ such that $V_m \subset U$. We claim that $G_m^\circ \cap L \subset \{g\}^\circ$. Fix $z \in G_m^\circ \cap L$. For each of the polynomials $P \in F_m$, we apply the mean value theorem to the function $\lambda \in [0,1] \mapsto \overline{(P(z) - P(0))} P(\lambda z)$. Then it turns out that for some $\mu \in [0,1]$, $|P(z) - P(0)| \le \sup_{|t|<1} |dP_{\mu z}(tz)| \le 1$ since $\mu z \in L$. Therefore, $|P(z) - P(0)| \le 1$ for all $P \in F_m$, that is, $z \in V_m$. Hence, $|g(z)| \le 1$ and, therefore, $z \in \{g\}^\circ$. Thus the claim is proved.

Further, since $\overline{aco(G_m)}^{w*}$, the $w(E^*, E)$-closure of the absolutely convex hull of G_m, is also a $w(E^*, E)$-compact set and L° is a $w(E^*, E)$-closed set, their sum is an absolutely convex $w(E^*, E)$-closed set to which g belongs to, since, otherwise, if $g \notin \overline{aco(G_m)}^{w*} + L^\circ$, we may appeal to the Hahn-Banach theorem to get some $x \in E$ such that $|g(x)| > 1$ and $|\varphi(x)| < 1$ for all $\varphi \in \overline{aco(G_m)}^{w*} + L^\circ$, that is $x \in L^{\circ\circ} = L$

since L is a closed absolutely convex set, and $x \in G_n^\circ$, contradicting the relation $G_m^\circ \cap L \subset \{g\}^\circ$. Hence, $g \in G + L^\circ$.

Finally, for any $p \in \mathbb{N}$, we have that $pg \in G + L^\circ$, so there are $\alpha \in G$ and $\beta \in L^\circ$ such that $pg = \alpha + \beta$, so

$$\left\| g - \frac{\alpha}{p} \right\|_L = \left\| \frac{\beta}{p} \right\|_L \leq \frac{1}{p}.$$

Therefore, G is $\| \cdot \|_L$-dense in E^*, thus $(E^*, \| \cdot \|_L)$ is a separable space. □

The next result is to be compared to Theorem 1.2.

Theorem 2.9. *Let $D \subset E$ be a bounded set. If any separable subset $L \subset \overline{aco(D)}$ is $\tau(E, P(E))$-metrizable, then D has the Asplund property.*

Proof. For any countable set $A \subset D$, $L := \overline{aco(A)}$ is $\tau(E, P(E))$-metrizable by assumption, so $(E^*, \| \cdot \|_L)$ is separable by Proposition 2.8 and, clearly, $(E^*, \| \cdot \|_A)$ is separable as well. □

Next we show that the converse to Theorem 2.9 does not hold.

Example 2.10. *The unit ball of any separable reflexive Banach space E which is also a Λ-space is an example of an Asplund set, that is, a weakly metrizable set, which is not metrizable for the polynomial topology.* By the very definition of Λ-space (see [6]), the convergent sequences in B_E for the polynomial topology are norm convergent. If the polynomial topology were metrizable on B_E, it would coincide with the norm topology on B_E. However this is not possible since the unit sphere is dense for the polynomial topology in B_E([1] Thm. 4.3). Recall that the spaces ℓ_p, $p > 1$, ([6]) and T, the dual of the (original) Tsirelson space ([19] or [7]), fall within the assumptions of this example.

3. Polynomials and Asplund property

We begin with an easy example showing that, contrary to the case of linear operators, polynomials do not necessarily preserve the Asplund property.

Example 3.1. *Consider the polynomial $P \in P(^2\ell_2, \ell_1)$ given by*

$$P((x_n)_{n=1}^\infty) = (x_n^2)_{n=1}^\infty. \tag{1}$$

Then P does not transform Asplund sets into Asplund sets.

To show this, consider the set $D = \{e_n : n \in \mathbb{N}\}$, which is an Asplund set in ℓ_2. However, $P(D) = D$ is not an Asplund set in ℓ_1 because $\ell_1^* = \ell_\infty$ and the space $(\ell_\infty, \| \cdot \|_D)$ is not separable since $\| \cdot \|_D$ is the usual norm for ℓ_∞. □

Notice that if F^* is separable and U is a subset of E, then every mapping $f : U \longrightarrow F$ transforms any subset of U into an Asplund set.

Next we aim to find sufficient conditions for analytic mappings to transform Asplund sets into Asplund sets.

Recall the following lemma (see [5] Lemmas 5.3.3. and 5.3.4.)

Lemma 3.2. *Let E be a Banach space. Then,*

a) *The sum and the union of a finite number of Asplund sets in E is an Asplund set.*

b) *Let (D_n) be a sequence of Asplund sets in E and let (t_n) be a sequence of positive numbers such that $\lim_n t_n = 0$. Then, the set*

$$D = \bigcap_{n=1}^{\infty} (D_n + t_n B_E)$$

is an Asplund set.

Let U be an open subset of E. A subset $D \subset U$ is said to be U-bounded if D is bounded and $d(D, E \setminus U) > 0$. A function $f : U \longrightarrow F$ is said to be of bounded type if it maps U-bounded sets into bounded sets.

Recall that the space of approximable polynomials from E into F is $P_A(E, F) := \overline{P_f(E) \otimes F} \subset P(E, F)$. It is well-known that $P_A(E, F)$ equals to the set of polynomials which are weakly continuous on bounded sets, that is, $P_A(E, F) = P_w(E, F)$, if, and only if, E^* has the approximation property. If $F = \mathbb{C}$, we will denote these spaces by $P_A(E)$ and $P_w(E)$ respectively.

We have the following,

Proposition 3.3. *Let E and F be Banach spaces.*

a) *Let $k \in \mathbb{N}$ and $P \in P(^kE, F)$. Suppose that $x^* \circ P \in P_A(^kE)$ for all $x^* \in F^*$. Then P maps Asplund sets into Asplund sets. In particular, if $P \in P_A(^kE, F)$, then P maps Asplund sets into Asplund sets.*

b) *Let $P \in P(E, F)$. Suppose that $x^* \circ P \in P_A(E)$ for all $x^* \in F^*$. Then P maps Asplund sets into Asplund sets. In particular, if $P \in P_A(E, F)$, then P maps Asplund sets into Asplund sets.*

c) *If $P(^kE) = \overline{P_f(^kE)}$ for some $k \in \mathbb{N}$ and $P \in P(^kE, F)$, then P maps Asplund sets into Asplund sets.*

d) *If $P(E) = P_A(E)$ and $f : U \subset E \longrightarrow F$ is an analytic function of bounded type, then f maps U-bounded Asplund sets into Asplund sets.*

Proof. Consider D a bounded set of E with the AP. We can suppose, without loss of generality, that $D \subset B_E$, since the class of Asplund sets is stable under translations and homotheties.

a) Let $A \subset P(D)$ be a countable set, that is, $A = P(C)$ for some countable set $C \subset D$. Since D is Asplund, we have that $(E^*, \|\cdot\|_C)$ is separable, so there exists a countable set S which is dense in $(E^*, \|\cdot\|_C)$. Consider the adjoint mapping $P^* : F^* \longrightarrow P_A(^kE)$ given by

$$P^*(x^*) = x^* \circ P.$$

Clearly, the mapping $P^* : (F^*, \|\cdot\|_A) \longrightarrow (P_A(^kE), \|\cdot\|_C)$ is a linear isometry. Therefore, $(F^*, \|\cdot\|_A)$ is separable if $P^*(F^*) \subset (P_A(^kE), \|\cdot\|_C)$ is separable and this will be a consequence of the separability of the seminormed space $(P_A({}^kE), \|\cdot\|_C)$. This space is separable since the algebra generated by S is dense in $(P_f({}^kE), \|\cdot\|_C)$, which in turn is dense in $(P_A({}^kE), \|\cdot\|_C)$ since by definition, $P_A({}^kE) = \overline{P_f({}^kE)}$ and the norm topology is finer than the $\|\cdot\|_C$ - topology.

b) It follows from *a*) and 3.2.*a*.

c) It follows from *a*) and *b*).

d) Let D be an Asplund set. By *c*) and 3.2.*a*, any polynomial defined on E maps D into an Asplund set. Set $n \in \mathbb{N}$. Since f is of bounded type, there exists a polynomial P_n such that $\|f(x) - P_n(x)\| \leq 1/n$ for any $x \in D$. Therefore,

$$f(D) \subset P_n(D) + \frac{1}{n} B_F,$$

and, hence, $f(D)$ is an Asplund set by 3.2.*b*. □

The following application of 3.3 yields a result due to Aron and Dineen [2] Cor. 6.

Corollary 3.4. *If $P({}^kE)$ has the Radon-Nikodým property, then E is an Asplund space. Conversely, if E is an Asplund space and further $P_f({}^kE)$ is dense in $P({}^kE)$, then $P({}^kE)$ has the Radon-Nikodým property.*

Proof. For the first statement, recall that according to [3] E^* is a complemented subspace of $P({}^kE)$. Hence E^* has also the Radon-Nikodým property. For the other statement, recall that the mapping $x \in E \mapsto \delta_x \in Q({}^kE)$ is a k-homogeneous polynomical, hence the above Proposition 3.3 shows that $\delta(B_E)$ is an Asplund set in $Q({}^kE)$ and by [5] 5.2.2.d, also its closed absolutely convex hull is such a set. Moreover [20] 2.5, this hull is the unit ball of $Q({}^kE)$, thus $Q({}^kE)$ is an Asplund space and we conclude that its dual $P({}^kE)$ has the Radon-Nikodým property. □

The following example shows that the $G^\infty(B_E)$ is not an Asplund space regardless of the Banach space E.

Example 3.5. *The space $G^\infty(B_E)$ is not an Asplund space.* Indeed, it is well-known that $c_0 \subset H^\infty(B_E)$ since $H^\infty(B_E)$ is an infinite dimensional uniform algebra. Then, $\ell_1 \subset G^\infty(B_E)$ by Theorem 4 in [4]. Therefore, $G^\infty(B_E)$ is not an Asplund space since $\ell_1^* = \ell_\infty$ is not separable.

This remark leads us to point out that Proposition 3.3.*d* does not hold, in general, for non U-bounded sets. To show this, consider the predual G^∞ of H^∞ and the bounded analytic map $\delta : \mathbf{D} \to G^\infty$, given by $\delta(x) = \delta_x$, the evaluation at x. Then $\delta(\mathbf{D}) = \{\delta_x : x \in \mathbf{D}\}$ is not an Asplund set since otherwise the closure of its absolutely convex hull B_{G^∞} would have the AP, but as we have just proved, it has not.

The following result gives a sufficient condition for all functions in the algebra $A_u(B_E, F)$ to transform Asplund sets into Asplund sets. This result can be extended to other algebras of analytic functions on B_E.

Proposition 3.6. *Let E and F be Banach spaces and suppose that A_E is a separable space of scalar functions on B_E. Let $A_{E,F}$ a space of functions from B_E into F such that for any $x^* \in F^*$ and $f \in A_{E,F}$, the function $x^* \circ f$ belongs to A_E. Then, any $f \in A_{E,F}$ maps Asplund sets into Asplund sets.*

Proof. We denote by A the algebra $A_{E,F}$. Pick $f \in A$ and consider the restriction $f^* := C_f|_{F^*} : F^* \longrightarrow A_E$ given by $f^*(x^*) = x^* \circ f$. Let $D \subset B_E$ be an Asplund set. To show that $f(D) \subset F$ is also an Asplund set, consider the space $(F^*, \| \cdot \|_{f(D)})$ and $B \subset f(D)$ a countable set. There exists a countable set $C \subset D$ such that $f(C) = B$. In consequence, $(A_E, \| \cdot \|_C)$ is separable since $\|g-h\|_C \leq \|g-h\|_\infty$ for any $g, h \in A_E$. Since the mapping $f^* : (F^*, \| \cdot \|_{f(C)}) \to (A_E, \| \cdot \|_C)$ is a linear isometry, we have that $(F^*, \| \cdot \|_{f(C)})$ is separable and, therefore, $f(D)$ is an Asplund set. □

Corollary 3.7. *If $A_u(B_E)$ is separable, then any $f \in A_u(B_E, F)$ transforms Asplund sets into Asplund sets.*

Remark 3.8. (i) There exist Banach spaces E such that E^* is separable, so E is Asplund, but there exist functions $f \in A_u(B_E, F)$ which do not transform Asplund sets into Asplund sets. To show this, it is sufficient to consider the function given in Example 3.1.

(ii) There exist Banach spaces E whose dual spaces E^* are not separable but any $f \in A_u(B_E, F)$ transforms Asplund sets into Asplund sets for any Banach space F. Consider, for instance, $E = c_0(\Gamma)$ for an uncountable set Γ, whose dual space $\ell_1(\Gamma)$ is not separable. However, any function $f \in A_u(B_E, F)$ transforms Asplund sets into Asplund sets by Proposition 3.3.*d*.

4. Radon-Nikodým Composition Operators

Every holomorphic map $\varphi : B_E \to B_F$ gives raise to the composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ defined according to $C_\varphi(f) = f \circ \varphi$.

The range of $C_\varphi^*|_{G^\infty(B_E)}$ is contained in $G^\infty(B_F)$ since for any $u \in G^\infty(B_E)$, the composition $u \circ C_\varphi$ is still continuous for the compact-open topology on $B_{G^\infty(B_F)}$. We denote by C^φ the restriction $C_\varphi{}^*|_{G^\infty(B_E)}$.

In this section, we will study when C^φ is an Asplund operator, which will lead us to characterize Radon-Nikodým composition operators C_φ because of the duality between the Asplund property and the Radon-Nikodým property.

In order to introduce the class of Radon-Nikodým operators, we give some background on the Radon-Nikodým property for Banach spaces. For further results on this property or concepts on vector measures see [9].

Definition 4.1. A closed, bounded, convex subset C of a Banach space E is called a *Radon-Nikodým set* if, for each probability space (Ω, Σ, μ) and any vector measure $m : \Sigma \longrightarrow E$ which is absolutely continuous with respect to μ and whose average range is in C (i.e. $m(A)/\mu(A)$ is contained in C for every $A \in \Sigma$, $\mu(A) > 0$), there exists a Bochner integrable function $f : \Omega \longrightarrow E$ such that

$$m(A) = \int_A f d\mu \text{ for all } A \in \Sigma.$$

The concept of Radon-Nikodým operator is the following [14],

Definition 4.2. The linear operator $T : E \longrightarrow F$ is said to be a *Radon-Nikodým operator* if for every finite measure space (Ω, Σ, μ) and for every vector measure $m : \Sigma \longrightarrow E$ such that $\|m(A)\| \leq \mu(A)$ for all $A \in \Sigma$, there exists a Bochner integrable function $f : \Omega \longrightarrow F$ such that

$$T \circ m(A) = \int_A f d\mu \quad \text{for all } A \in \Sigma.$$

The set of Radon-Nikodým operators is a closed ideal of operators [21]. In connection with Radon-Nikodým operators, Ghoussoub and Saab introduced the concept of strong Radon-Nikodým operator, i.e., an operator $T : E \longrightarrow F$ such that $\overline{T(B_E)}$ is a Radon-Nikodým set. Edgar [11] proved that strong Radon-Nikodým operators are Radon-Nikodým operators. The converse is not true in general since any quotient Q from ℓ_1 onto c_0 is a Radon-Nikodým operator but fails to be a strong Radon-Nikodým operator [14]. However, as pointed out in [14], these concepts become equivalent when we deal with adjoint maps, i.e., for an operator $T : E \longrightarrow F$, $T^* : F^* \longrightarrow E^*$ is a Radon-Nikodým operator if and only if $T^* : F^* \longrightarrow E^*$ is strong Radon-Nikodým. Thus we conclude from Theorem 2.11 in [22],

Theorem 4.3. *Let $T : E \longrightarrow F$ be a linear operator. Then,*

T is Asplund if and only if T^* is Radon-Nikodým.

And bearing in mind that $(C^\varphi)^* = C_\varphi$, the following is clear,

Corollary 4.4. *A composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým if and only if $C^\varphi : G^\infty(B_E) \longrightarrow G^\infty(B_F)$ is Asplund.*

Therefore, we aim to find conditions to ensure that C^φ is Asplund. We begin with the following result,

Proposition 4.5. *The operator C^φ is an Asplund operator if and only if the set $\left\{\delta_{\varphi(x)} : x \in B_E\right\}$ is Asplund in $G^\infty(B_F)$.*

Proof. Set $B = B_{G^\infty(B_E)}$. The set $C^\varphi(B) = \{\mu \circ C_\varphi : \mu \in B\}$ is Asplund if and only if $\overline{aco}(\{\delta_x \circ C_\varphi : x \in B_E\})$ is Asplund, which is equivalent to $\{\delta_x \circ C_\varphi : x \in B_E\}$ to be Asplund and this is satisfied if and only if $\left\{\delta_{\varphi(x)} : x \in B_E\right\}$ is Asplund. □

Notice that Example 3.5 shows that $\left\{\delta_{\varphi(x)} : x \in B_E\right\}$ being an Asplund set is not equivalent to $\varphi(B_E)$ being an Asplund set.

We need the following lemma, whose proof follows the same pattern as Proposition 2 in [12].

Lemma 4.6. *Let $\varphi : B_E \longrightarrow B_F$ be an analytic map and suppose that $\varphi(B_E) \nsubseteq rB_F$ for all $0 < r < 1$. Then, there exist linear operators $T : H^\infty(B_E) \longrightarrow \ell^\infty$ and $S : \ell^\infty \longrightarrow H^\infty(B_F)$ such that*

$$T \circ C_\varphi \circ S = Id_{\ell^\infty}.$$

Proof. We may find, by hypothesis, a sequence $(x_n) \subset B_E$ such that $\lim_{n\to\infty} \|\varphi(x_n)\| = 1$. We can consider, passing to a subsequence if necessary, that $(\|\varphi(x_n)\|)$ converges fast enough in order to satisfy the Hayman-Newman condition, that is,

$$\frac{1 - \|\varphi(x_{n+1})\|}{1 - \|\varphi(x_n)\|} < c \quad \text{for some } 0 < c < 1.$$

By Corollary 8 in [13], the sequence $(\varphi(x_n))$ is interpolating for $H^\infty(B_F)$. In particular, there exist a sequence $(f_k) \subset H^\infty(B_F)$ and $M > 0$ such that

$$f_k(\varphi(x_n)) = \delta_{kn} \quad \text{for all } n, k \in \mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} |f_n(x)| \leq M \quad \text{for all } x \in B_F.$$

Define the operators $S : \ell^\infty \longrightarrow H^\infty(B_F)$ by $S((\alpha_k)) = \sum_{k=1}^\infty \alpha_k f_k$ and $T : H^\infty(B_E) \longrightarrow \ell^\infty$ by $T(f) = (f(x_k))_{k=1}^\infty$. Both are well-defined, linear and continuous. Finally, it is clear that $T \circ C_\varphi \circ S = Id_{\ell^\infty}$. $\square$

Proposition 4.7. *Let $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ be a Radon-Nikodým operator. Then there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$.*

Proof. If such an $0 < r < 1$ does not exist, we apply Lemma 4.6 and find linear operators $S : \ell_\infty \longrightarrow H^\infty(B_F)$ and $T : H^\infty(B_E) \longrightarrow \ell_\infty$ such that $T \circ C_\varphi \circ S = Id_{\ell_\infty}$. This is not possible since the class of Radon-Nikodým operators is an ideal of operators and Id_{ℓ_∞} is not a Radon-Nikodým operator. $\square$

These results allow us to give the following characterization of Radon-Nikodým composition operators,

Theorem 4.8. *The composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým if and only if there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$ and $(P(F), \|\cdot\|_A)$ is separable for any countable set $A \subset \varphi(B_E)$.*

Proof. To prove the sufficient condition we show firstly that $\delta_{\varphi(B_E)} = \{\delta_x : x \in \varphi(B_E)\}$ is an Asplund set in $G^\infty(B_F)$. Let $A \subset \varphi(B_E)$ be a countable set. This set can be described as $A = \varphi(C)$ for some countable set $C \subset B_E$. Recall that the Taylor series of $f \in H^\infty(B_F)$ converges uniformly to f on rB_F and, therefore, f is uniformly approximable by polynomials on $\varphi(C)$. Hence, $P(F)$ is dense in $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$. Since $(P(F), \|\cdot\|_{\varphi(C)})$ is separable and $G^\infty(B_F)^* = H^\infty(B_F)$, it follows that $(G^\infty(B_F)^*, \|\cdot\|_{\varphi(C)})$ is separable. Thus, $\delta_{\varphi(B_E)}$ is an Asplund set in $G^\infty(B_F)$. Hence C^φ is an Asplund operator by Proposition 4.5 and, according to Corollary 4.4, C_φ is a Radon-Nikodým operator.

Now we prove the necessary conditions. Let C_φ be a Radon-Nikodým operator. By Proposition 4.7, there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$. To show the remaining condition, let $A \subset \varphi(B_E)$ be a countable set, that is, $A = \varphi(C)$ for some countable set $C \subset B_E$. Since $C^\varphi(B_{G^\infty(B_E)})$ is an Asplund set, the space $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$ is separable. Therefore, since $P(F) \subset H^\infty(B_F)$, it follows that $(P(F), \|\cdot\|_{\varphi(C)})$ is a subspace of the seminormed space $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$. Thus, $(P(F), \|\cdot\|_A)$ is separable. $\square$

Corollary 4.9. *Let $\varphi : B_E \longrightarrow B_F$ be an analytic map.*

a) *Suppose that $P(^kF) = \overline{P_f(^kF)}$ for any $k \in \mathbb{N}$. If $\varphi(B_E)$ is an Asplund set and there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$, then the composition operator $C_\varphi :\ H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým.*

b) *Suppose that the algebra $A_u(B_F)$ is separable. If $\varphi(B_E)$ is an Asplund set and there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$, then the composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým.*

Proof. *a)* We show in Proposition 3.3.*a* that for any countable set $C \subset \varphi(B_E)$, the space $(P(^kF), \|\cdot\|_C)$ is separable. Then,

$$(P(F), \|\cdot\|_C) = \bigcup_{k\in\mathbb{N}} (P(^kF), \|\cdot\|_C)$$

is also separable and, therefore, we can use the previous theorem to get the result.

b) Since $\varphi(B_E) \subset rB_F$ is an Asplund set and the function $\delta|_{rB_F}$ belongs to $A_u(rB_F, G^\infty(B_F))$, we have that the set $\delta_{\varphi(B_E)}$ is also an Asplund set by Proposition 3.6 and, therefore, the composition operator C_φ is Radon-Nikodým. □

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