Journal of Convex Analysis
Volume 18 (2011), No. 2, 465–487

Alternative Iterative Methods for Nonexpansive Mappings, Rates of Convergence and Applications

Vittorio Colao*
Dipartimento di Matematica, Universitá della Calabria,
87036 Arcavacata di Rende (CS), Italy
colao@mat.unical.it

Laurenţiu Leuştean†
Department of Mathematics, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany
and: Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
Calea Griviţei 21, 010702 Bucharest, Romania
leustean@mathematik.tu-darmstadt.de

Genaro López‡
Departamento de Análisis Matemático,
Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain
glopez@us.es

Victoria Martín-Márquez§
Departamento de Análisis Matemático,
Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain
victoriam@us.es

Received: May 28, 2009
Revised manuscript received: April 28, 2010

Alternative iterative methods for a nonexpansive mapping in a Banach space are proposed and proved to be convergent to a common solution to a fixed point problem and a variational inequality. Rates of asymptotic regularity for such iterations are given using proof-theoretic techniques. Some applications of the convergence results are presented.

Keywords: Nonexpansive mapping, iterative algorithm, fixed point, viscosity approximation, uniformly smooth Banach space, rates of asymptotic regularity, proof mining, variational inequality, accretive operator

1991 Mathematics Subject Classification: 47H06, 47H09, 47H10, 47J20, 03F60

*Supported by Regione Calabria POR/FSE 2007-2013 and Universitá della Calabria.
†Supported by the Oberwolfach Leibniz Fellow Programme (OWLF) at the Oberwolfach Mathematics Institute and by the German Science Foundation (DFG Project KO 1737/5-1).
‡Supported by DGES, Grant MTM2006-13997-C02-01 and Junta de Andalucía, Grant FQM-127 and partially by the Oberwolfach Mathematics Institute under the OWLF Programme.
§Supported by Junta de Andalucía, Grant FQM-127 and Ministerio de Ciencia e Innovación, Grant AP2005-1018.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

1. Introduction

Many problems arising in different areas of mathematics such as optimization, variational analysis and game theory, can be formulated as the fixed point problem:

$$\text{find } x \in X \text{ such that } x = Tx, \tag{1}$$

where T is a nonexpansive mapping defined on a metric space X, i.e., T satisfies the property $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in X$.

For instance, let $A : C \to H$ be a nonlinear operator where $C \subset H$ is a closed convex subset of a Hilbert space. The variational inequality problem associated to A, $\mathrm{VIP}(A, C)$, is formulated as finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{2}$$

It is well-known that the $\mathrm{VIP}(A, C)$ is equivalent to the problem of finding the fixed point

$$x^* = P_C(x^* - \lambda Ax^*), \tag{3}$$

where $\lambda > 0$ and P_C is the metric projection onto C, which is a nonexpansive mapping in this case. Besides, if $f : C \to \mathbb{R}$ is a differentiable convex function and we denote by A the gradient operator of f, then (2) is the optimality condition for the minimization problem

$$\min_{x \in C} f(x). \tag{4}$$

Bearing in mind that the iterative methods for approximating a fixed point of a nonexpansive mapping can be applied to find a solution to a variational inequality, a zero of an accretive operator and a minimizer of a convex function, in the recent years the study of the convergence of those methods has received a great deal of attention. Basically two types of iterative algorithms have been investigated: Mann algorithm and Halpern algorithm.

In the following, let X be a real Banach space, $C \subset X$ a closed convex subset and $T : C \to C$ a nonexpansive mapping with fixed point set $F = \{x \in C : x = Tx\} \neq \emptyset$.

Mann algorithm generates a sequence according to the iterative scheme

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n \geq 0, \tag{5}$$

where the initial guess $x_0 \in C$ and $\{t_n\}$ is a sequence in $(0, 1)$.

Halpern algorithm generates a sequence via the recursive formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \tag{6}$$

where $x_0, u \in C$ are arbitrary and the sequence $\{\alpha_n\} \subset (0, 1)$.

Whenever a fixed point of the mapping T exists, Halpern algorithm strongly converges, whereas we just get weak convergence for Mann algorithm as was established in [11] thanks to a counterexample. The references [29, 21, 15, 36, 10, 49] can be consulted for convergence results of Mann algorithm. Some modifications have been proposed in [33, 16] to get strong convergence. As for Halpern algorithm, see [14, 25, 45, 39, 48, 6, 43, 27] and references therein for studies dedicated to its convergence.

Another iterative approach to solving the problem (1) which may have multiple solutions, is to replace it by a family of perturbed problems admitting a unique solution, and then to get a particular original solution as the limit of these perturbed solutions as the perturbation vanishes. For example, Browder [2, 3] proved that if the underlying space H is Hilbert, then, given $u \in H$ and $t \in (0, 1)$, the approximating curve $\{x_t\}$ defined by

$$x_t = tu + (1 - t)Tx_t \tag{7}$$

strongly converges, as $t \to 0$, to the fixed point of T closest to u from F. Browder's result has been generalized and extended to a more general class of Banach spaces [37, 41, 34]. Combettes and Hirstoaga [8] introduced a new type of approximating curve for fixed point problems in the setting of a Hilbert space. This curve whose iterative scheme is a more general version of the implicit formula

$$x_t = T(tu + (1 - t)x_t), \tag{8}$$

was proved to converge to the best approximation to u from F. In [51] Xu studied the behavior of $\{x_t\}$ defined by (8) in the setting of a Banach space X and discretized this regularization method studying the strong convergence of the explicit algorithm

$$x_{n+1} = T(\alpha_n u + (1 - \alpha_n)x_n), \tag{9}$$

where $\{\alpha_n\} \subset (0, 1)$. Moreover, he proved that the convergence point is the image of u under the unique sunny nonexpansive retraction Q from X to F (see, for instance, [35, 37]).

On the other hand, Moudafi in [32] introduced the viscosity approximation method for nonexpansive mappings, which generalizes Browder (7) and Halpern (6) iterations, by using a contraction Φ instead of an arbitrary point u. The convergence of the implicit and explicit algorithms has been the subject of many papers because under suitable conditions these iterations strongly converge to the unique solution $q \in F$ to the variational inequality

$$\langle (I - \Phi)q, J(x - q)\rangle \geq 0 \quad \forall x \in F, \tag{10}$$

where J is a duality mapping; that is, q is the unique fixed point of the contraction $Q \circ \Phi$. This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions. See [50, 42, 44] and references therein for convergence results regarding viscosity approximation methods.

In this paper, we analyze the behavior of a new approximating curve in the setting of Banach spaces, which constitutes a hybrid method of the ones presented by Combettes and Hirstoaga (8) and Moudafi. This curve is defined by

$$x_t = T(t\Phi(x_t) + (1 - t)x_t), \tag{11}$$

for some contraction Φ; that is, for any $t \in (0, 1)$ x_t is the unique fixed point of the contraction $T_t = T(t\Phi + (1 - t)I)$. The discretized iteration

$$x_{n+1} = T(\alpha_n \Phi(x_n) + (1 - \alpha_n)x_n) \tag{12}$$

is also considered and studied under suitable conditions on the sequence $\{\alpha_n\} \subset (0,1)$. From this explicit algorithm we obtain the so-called hybrid steepest descent method

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n). \tag{13}$$

This iterative method was suggested by Yamada [46] as an extension of viscosity approximation methods for solving the variational inequality VIP(g, F) (2) in the case when g is strongly monotone and Lipschitz continuous, and F is the fixed point set of a mapping T which belongs to a subclass of the quasi-nonexpansive mappings (also see [47, 28]). We will get the convergence of the algorithm (13) for a nonexpansive mapping T, just requiring $I - \mu g$ to be a contraction for some $\mu > 0$, which it is satisfied in the particular case when g is strongly monotone and Lipschitz continuous.

Asymptotic regularity is a very important concept in metric fixed point theory. It was already implicit in [21, 40, 9], but it was formally introduced by Browder and Petryshyn in [4]. In our setting, the mapping T is called asymptotically regular if for all $x \in C$

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Effective rates of asymptotic regularity for both Mann and Halpern iterations have been obtained (see [17, 18, 19, 22, 23]) by applying methods of proof mining. By "proof mining" we mean the logical analysis, using proof-theoretic tools, of mathematical proofs with the aim of extracting relevant information hidden in the proofs. This new information can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening the premises. Thus, even if one is not particularly interested in the numerical details of the bounds themselves, in many cases such explicit bounds immediately show the independence of the quantity in question from certain input data. A comprehensive reference for proof mining is Kohlenbach's book [20]. One of the aims of this paper is to give effective rates of asymptotic regularity for the algorithm (12) of nonexpansive mappings in the framework of normed spaces.

The organization of the paper is as follows. In Section 2 some preliminary results are presented together with a technical lemma with regards to the behavior of the sequence defined by algorithm (12), which will be useful for the proof of the convergence and the evaluation of the rate of asymptotic regularity in following sections. Section 3 contains the main results about the strong convergence of both implicit (11) and explicit (12) algorithms to the unique solution to the variational inequality (10) in the setting of uniformly smooth Banach spaces, and also in the framework of reflexive Banach spaces with weakly continuous normalized duality mapping. A hybrid method is deduced to converge as well in this section. Section 4 is devoted to the rate of asymptotic regularity for the iteration (12) of nonexpansive mappings in normed spaces. Finally, in Section 5 we give examples of how to apply the main results of Ssection 3 to find a solution to a variational inequality and a zero of an accretive operator.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and dual space X^*. For any $x \in X$ and $x^* \in X^*$ we denote $x^*(x) = \langle x, x^* \rangle$. Given a nonempty closed convex subset $C \subset X$,

$\Phi : C \to C$ will be a ρ-contraction and $T : C \to C$ a nonexpansive self-mapping with nonempty fixed point set $F := \{x \in C : Tx = x\}$.

We include some brief knowledge about geometry of Banach spaces which can be found in more details in [7]. The *normalized duality mapping* $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{14}$$

It is known that

$$J(x) = \partial \left(\frac{1}{2} \| \cdot \|^2 \right) (x),$$

where $\partial(\frac{1}{2}\|\cdot\|^2)(x)$ is the subdifferential of $\frac{1}{2}\|\cdot\|^2$ at x in the sense of convex analysis. Thus, for any $x, y \in X$, we have the subdifferential inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad j(x + y) \in J(x + y). \tag{15}$$

A Banach space X is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{16}$$

exists for each $x, y \in S_X$, where S_X is the unit sphere of X, i.e., $S_X = \{v \in X : \|v\| = 1\}$. When this is the case, the norm of X is said to be *Gâteaux differentiable*. If for each $y \in X$ the limit (16) is uniformly attained for $x \in X$, we say that the norm of X is *uniformly Gâteaux differentiable*, and we say that X is *uniformly smooth* if the limit (16) is attained uniformly for any $x, y \in S_X$.

It is known that a Banach space X is smooth if and only if the duality mapping J is single-valued, and that X is uniformly smooth if and only if the duality mapping J is single-valued and norm-to-norm uniformly continuous on bounded sets of X. Moreover, if the norm of X is uniformly Gâteaux differentiable then J is norm-to-weak* uniformly continuous on bounded sets of X.

Following Browder [3] we say that the duality mapping J is *weakly sequentially continuous* (or simply *weakly continuous*) if J is single-valued and weak-to-weak* sequentially continuous; i.e., if $x_n \rightharpoonup x$ in X, then $J(x_n) \rightharpoonup^* J(x)$ in X^*. A Banach space with weakly continuous duality mapping is known (see [24]) to satisfy *Opial's property* (i.e., whenever $x_n \rightharpoonup x$ and $y \neq x$, we have $\overline{\lim}\|x_n - x\| < \overline{\lim}\|x_n - y\|$), and this fact implies (see [12]) that X satisfies the *Demiclosedness principle*: if C is a closed convex subset of X and T is a nonexpansive self-mapping, then $x_n \rightharpoonup x$ and $(I - T)x_n \to y$ imply that $(I - T)x = y$.

Consider a subset $D \subset C$ and a mapping $Q : C \to D$. We say that Q is a *retraction* provided $Qx = x$ for any $x \in D$. The retraction Q is said to be *sunny* if it satisfies the property: $Q(x + t(x - Qx)) = Qx$ whenever $x + t(x - Qx) \in C$, where $x \in C$ and $t \geq 0$.

Lemma 2.1 ([5, 35, 13]). *Let X be a smooth Banach space and $D \subset C$ be nonempty closed convex subsets of X. Given a retraction $Q : C \to D$, the following three statements are equivalent:*

(a) *Q is sunny and nonexpansive.*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ *for all* $x, y \in C$.
(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in D$.

Consequently, there is at most one sunny nonexpansive retraction from C onto D.

In some circumstances, we can construct the sunny nonexpansive retraction. For the nonexpansive mapping T with fixed point set F, an arbitrary $u \in C$ and $t \in (0, 1)$, let z_t be the unique fixed point of the contraction $z \mapsto tu + (1 - t)Tz$ for $z \in C$; that is, z_t is the unique solution in C to the fixed point equation

$$z_t = tu + (1 - t)Tz_t. \tag{17}$$

It is natural to study the behavior of the net $\{z_t\}$ as $t \to 0^+$. It is unclear if the strong limit of $\{z_t\}$ always exists in a general Banach space. However, the answer is affirmative in some classes of smooth Banach spaces and then the limit defines the sunny nonexpansive retraction from C onto F. Those Banach spaces where the net $\{z_t\}$ strongly converges are said to have *Reich's property* since Reich was the first to show that all uniformly smooth Banach spaces have this property.

Theorem 2.2 ([37, 34]). *Let X be either a reflexive Banach space with a weakly continuous duality mapping or a uniformly smooth Banach space, C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with $F \neq \emptyset$. Then the net $\{z_t\}$ strongly converges as $t \to 0^+$ to a fixed point of T; moreover, the limit*

$$Q(u) := \lim_{t\to 0^+} z_t \tag{18}$$

defines the unique sunny nonexpansive retraction from C onto F.

In [38] Reich proved the following two lemmas which will be needed for the convergence results in Section 3.

Lemma 2.3. *Let $\{x_n\}$ be a bounded sequence contained in a separable subset D of a Banach space X. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\lim_{k\to\infty} \|x_{n_k} - y\|$$

exists for all $y \in D$.

Lemma 2.4. *Let D be a closed convex subset a real Banach space X with a uniformly Gâteaux differentiable norm, and let $\{x_n\}$ be a sequence in D such that*

$$f(y) := \lim_{n\to\infty} \|x_n - y\|$$

exists for all $y \in D$. If the function f attains its minimum over D at u, then

$$\limsup_{n\to\infty} \langle y - u, J(x_n - u)\rangle \leq 0$$

for all $y \in D$.

The following lemma collects some properties of the iteration (12), useful both for proving the convergence of the iteration and for computing the rate of asymptotic regularity.

Lemma 2.5. *Let X be a normed space and $\{x_n\}$ be the sequence defined by the explicit algorithm* (12).

(*1*) *For all $n \geq 0$,*

$$\|\Phi(x_n) - x_n\| \leq (1+\rho)\|x_n - x_0\| + \|\Phi(x_0) - x_0\|, \tag{19}$$
$$\|x_n - Tx_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \|\Phi(x_n) - x_n\|. \tag{20}$$

(*2*) *For all $n \geq 1$,*

$$\begin{aligned} \|x_{n+1} - x_n\| \leq{}& (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ &+ |\alpha_n - \alpha_{n-1}|\, \|\Phi(x_{n-1}) - x_{n-1}\|. \end{aligned} \tag{21}$$

(*3*) *If T has fixed points, then $\{x_n\}$ is bounded for every $x_0 \in C$.*

Proof. (*1*) Let $n \geq 0$.

$$\begin{aligned} \|\Phi(x_n) - x_n\| &\leq \|\Phi(x_n) - \Phi(x_0)\| + \|\Phi(x_0) - x_0\| + \|x_0 - x_n\| \\ &\leq (1+\rho)\|x_n - x_0\| + \|\Phi(x_0) - x_0\|. \end{aligned}$$

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \\ &= \|x_{n+1} - x_n\| + \|T(\alpha_n \Phi(x_n) + (1-\alpha_n)x_n) - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n \Phi(x_n) + (1-\alpha_n)x_n - x_n\| \\ &= \|x_{n+1} - x_n\| + \alpha_n \|\Phi(x_n) - x_n\|. \end{aligned}$$

(*2*) Let $n \geq 1$.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(\alpha_n \Phi(x_n) + (1-\alpha_n)x_n) \\ &\qquad - T(\alpha_{n-1}\Phi(x_{n-1}) + (1-\alpha_{n-1})x_{n-1})\| \\ &\leq \|\alpha_n \Phi(x_n) + (1-\alpha_n)x_n - \alpha_{n-1}\Phi(x_{n-1}) - (1-\alpha_{n-1})x_{n-1}\| \\ &= \|\alpha_n(\Phi(x_n) - \Phi(x_{n-1})) + (1-\alpha_n)(x_n - x_{n-1}) \\ &\qquad + (\alpha_n - \alpha_{n-1})(\Phi(x_{n-1}) - x_{n-1})\| \\ &\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|\, \|\Phi(x_{n-1}) - x_{n-1}\| \\ &= (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|\, \|\Phi(x_{n-1}) - x_{n-1}\|. \end{aligned}$$

(3) Let p be a fixed point of T.

$$\begin{aligned}\|x_{n+1}-p\| &= \|T(\alpha_n\Phi(x_n)+(1-\alpha_n)x_n)-Tp\| \\ &\le \|\alpha_n\Phi(x_n)+(1-\alpha_n)x_n-p\| \\ &= \|\alpha_n(\Phi(x_n)-\Phi(p))+(1-\alpha_n)(x_n-p)+\alpha_n(\Phi(p)-p)\| \\ &\le \alpha_n\rho\|x_n-p\|+(1-\alpha_n)\|x_n-p\|+\alpha_n\|\Phi(p)-p\| \\ &= (1-(1-\rho)\alpha_n)\|x_n-p\|+(1-\rho)\alpha_n\frac{\|\Phi(p)-p\|}{1-\rho} \\ &\le \max\left\{\|x_n-p\|,\frac{\|\Phi(p)-p\|}{1-\rho}\right\}.\end{aligned}$$

By induction, we obtain that for all $n \ge 0$,

$$\|x_n-p\| \le \max\left\{\|x_0-p\|,\frac{\|\Phi(p)-p\|}{1-\rho}\right\},$$

thus $\{x_n\}$ is bounded. □

Let us recall some terminology that is used for expressing the quantitative results in Section 4. We denote by $\mathbb{Z}_+$ the set of nonnegative integers. Let $k \in \mathbb{Z}_+$ and $\{a_n\}_{n\ge k}$ be a sequence of nonnegative real numbers. If $\{a_n\}$ is convergent, then a function $\omega : (0,\infty) \to \{k, k+1, \ldots\}$ is called a *Cauchy modulus* of $\{a_n\}$ if for all $\varepsilon > 0$,

$$|a_{\omega(\varepsilon)+n}-a_{\omega(\varepsilon)}| < \varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{22}$$

If $\lim_{n\to\infty} a_n = a$, then a function $\omega : (0,\infty) \to \{k, k+1, \ldots\}$ is called a *rate of convergence* of $\{a_n\}$ if for any $\varepsilon > 0$

$$|a_n - a| < \varepsilon, \quad \forall n \ge \omega(\varepsilon). \tag{23}$$

If the series $\sum_{n=k}^{\infty} a_n$ is divergent, then a function $\omega : \mathbb{Z}_+ \to \{k, k+1, \ldots\}$ is called a *rate of divergence* of the series if $\sum_{i=k}^{\omega(n)} a_i \ge n$ for all $n \in \mathbb{Z}_+$. If the series $\sum_{n=k}^{\infty} a_n$ converges, then by a *Cauchy modulus* of the series we mean a Cauchy modulus of the sequence of partial sums $\{s_n\}_{n\ge k}$, $s_n = \sum_{i=k}^{n} a_i$.

Lemma 2.6 ([48]). *Assume $\{a_n\}$ is a sequence of nonnegative real number such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n b_n + \epsilon_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ and $\{\epsilon_n\}$ are sequences in $(0,1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty}\gamma_n = \infty$, $\sum_{n=1}^{\infty}\epsilon_n < \infty$ and either $\limsup_{n\to\infty} b_n \le 0$ or $\sum_{n=1}^{\infty}\gamma_n|b_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

The following lemma is a quantitative version of [26, Lemma 2].

Lemma 2.7 ([23, Lemma 9]). *Let $\{\lambda_n\}_{n\ge1}$ be a sequence in $(0,1)$ and $\{a_n\}_{n\ge1}$, $\{b_n\}_{n\ge1}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1-\lambda_{n+1})a_n + b_n \quad \text{for all } n \in \mathbb{N}.$$

Assume that $\sum_{n=1}^{\infty} \lambda_n$ *is divergent,* $\sum_{n=1}^{\infty} b_n$ *is convergent and let* $\delta : \mathbb{Z}_+ \to \mathbb{N}$ *be a rate of divergence of* $\sum_{n=1}^{\infty} \lambda_n$ *and* $\gamma : (0, \infty) \to \mathbb{N}$ *be a Cauchy modulus of* $\sum_{n=1}^{\infty} b_n$*. Then* $\lim_{n\to\infty} a_n = 0$ *and moreover for all* $\varepsilon \in (0, 2)$

$$a_n < \varepsilon, \quad \forall n \geq h(\gamma, \delta, D, \varepsilon), \tag{24}$$

where $D > 0$ *is an upper bound on* $\{a_n\}$ *and*

$$h(\gamma, \delta, D, \varepsilon) = \delta\left(\gamma\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right).$$

3. Convergence of the algorithms

In this section we prove the convergence of the implicit (11), explicit (12) and hybrid steepest descent (13) algorithms in the setting of Banach spaces, which generalize previous results by Combettes and Hirstoaga [8], Xu [51], Yamada [46], and Xu and Kim [52].

Theorem 3.1. *Let* X *be either a reflexive Banach space with weakly continuous normalized duality mapping* J *or a uniformly smooth Banach space,* C *a nonempty closed convex subset of* X*,* $T : C \to C$ *a nonexpansive mapping with fixed point set* $F \neq \emptyset$ *and* $\Phi : C \to C$ *a* ρ*-contraction. Then the approximating curve* $\{x_t\} \subset C$ *defined by*

$$x_t = T(t\Phi(x_t) + (1-t)x_t) \tag{25}$$

strongly converges, as $t \to 0$*, to the unique solution* $q \in F$ *to the inequality*

$$\langle (\Phi - I)q, J(x - q)\rangle \leq 0, \quad \forall x \in F. \tag{26}$$

Proof. First of all, note that if $Q : C \to F$ is the unique sunny nonexpansive retraction whose existence is assured by Theorem 2.2, by the characterization Lemma 2.1, $q \in C$ is the unique fixed point of the contraction $Q \circ \Phi$ if and only if $q \in F$ satisfies inequality (26).

We observe that we may assume that C is separable. To see this, consider the set K defined by

$$\begin{aligned} K_0 &:= \{q\}, \\ K_{n+1} &:= co(K_n \cup T(K_n) \cup \Phi(K_n)), \\ K &:= \overline{\bigcup_n K_n}. \end{aligned}$$

Then $K \subset C$ is nonempty, convex, closed, and separable. Moreover K is invariant under T, Φ and, therefore, $T_t = T(t\Phi + (1-t)I)$. Then $\{x_t\} \subset K$ and we may replace C with K.

We will prove that $\{x_t\}$ converges, as $t \to 0$, to the point $q \in F$ which is the unique solution to the inequality (26).

The sequence $\{x_t\}$ is bounded. Indeed, given $p \in F$,

$$\begin{aligned}\|x_t - p\| &= \|T(t\Phi(x_t) + (1-t)x_t) - Tp\| \\ &\leq \|t(\Phi(x_t) - \Phi(p)) + (1-t)(x_t - p) + t(\Phi(p) - p)\| \\ &\leq (t\rho + (1-t))\|x_t - p\| + t\|\Phi(p) - p\|.\end{aligned}$$

Then, for any $t \in (0,1)$,

$$\|x_t - p\| \leq \frac{1}{1-\rho}\|\Phi(p) - p\|.$$

Take an arbitrary sequence $\{t_n\} \subset (0,1)$ such that $t_n \to 0$, as $n \to 0$, and denote $x_n = x_{t_n}$ for any $n \geq 0$. Let $\Gamma := \limsup_{n\to\infty} \langle \Phi(q) - q, J(x_n - q)\rangle$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle \Phi(q) - q, J(x_{n_k} - q)\rangle = \Gamma.$$

Since $\{x_{n_k}\}$ is bounded, by Lemma 2.3, there exists a subsequence, which also will be denoted by $\{x_{n_k}\}$ for the sake of simplicity, satisfying that

$$f(x) := \lim_{k\to\infty} \|x_{n_k} - x\|$$

exists for all $x \in C$.

We define the set

$$A := \{z \in C : f(z) = \min_{x\in C} f(x)\}$$

and note that A is a nonempty bounded, closed and convex set since f is a continuous convex function and $\lim_{\|x\|\to\infty} f(x) = \infty$. Moreover,

$$\|x_{n_k} - Tz\| \leq t_{n_k}\|\Phi(x_{n_k}) - x_{n_k}\| + \|x_{n_k} - z\|,$$

for any $z \in C$. Since the sequence $\{t_{n_k}\}$ converges to 0 as $k \to \infty$, we deduce that $f(Tz) \leq f(z)$ for any $z \in C$. Then $T(A) \subseteq A$, in other words, T maps A into itself.

Since A is a nonempty bounded, closed and convex subset of either a reflexive Banach space with weakly continuous normalized duality mapping or a uniformly smooth Banach space, it has the fixed point property for nonexpansive mappings (see [12]), that is $F \cap A \neq \emptyset$.

If X is a reflexive Banach space with weakly continuous normalized duality mapping, we can assume that $\{x_{n_k}\}$ has been chosen to be weakly convergent to a point $\tilde{q}$. Since X satisfies Opial's property, we have $A = \{\tilde{q}\}$. Then, since $\tilde{q} \in F$, we obtain that

$$\Gamma = \langle \Phi(q) - q, J(\tilde{q} - q)\rangle \leq 0.$$

If X is uniformly smooth, let $\tilde{q} \in F \cap A$. Then $\tilde{q}$ minimize f over C and, since the norm is uniformly Gâteaux differentiable, by Lemma 2.4,

$$\limsup_{k\to\infty} \langle x - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle \leq 0 \tag{27}$$

holds, for all $x \in C$, and in particular for $x = \Phi(\tilde{q})$.

We shall show that $\{x_{n_k}\}$ strongly converges to $\tilde{q}$. Denote

$$\delta_k := \langle \Phi(\tilde{q}) - \tilde{q}, J(t_{n_k}(\Phi(x_{n_k}) - x_{n_k}) + (x_{n_k} - \tilde{q})) - J(x_{n_k} - \tilde{q})\rangle.$$

Since J is norm-to-weak* uniformly continuous on bounded sets, $\lim_k \delta_k = 0$. Moreover,

$$\begin{aligned}
\|x_{n_k} - \tilde{q}\|^2 &\leq \|t_{n_k}(\Phi(x_{n_k}) - \Phi(\tilde{q})) + (1 - t_{n_k})(x_{n_k} - \tilde{q}) + t_{n_k}(\Phi(\tilde{q}) - \tilde{q})\|^2 \\
&\leq \|t_{n_k}(\Phi(x_{n_k}) - \Phi(\tilde{q})) + (1 - t_{n_k})(x_{n_k} - \tilde{q})\|^2 + 2t_{n_k}\delta_k \\
&\qquad + 2t_{n_k}\langle \Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle \\
&\leq (1 - (1-\rho)t_{n_k})\|x_{n_k} - \tilde{q}\|^2 + 2t_{n_k}\delta_k \\
&\qquad + 2t_{n_k}\langle \Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle.
\end{aligned} \tag{28}$$

From (28) and by (27), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - \tilde{q}\|^2 \leq \limsup_{k\to\infty} \frac{2}{1-\rho}\Big(\delta_k + 2\langle \Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle\Big) \leq 0.$$

That is $\lim_{k\to\infty} x_{n_k} = \tilde{q}$. Since $\tilde{q}$ is a fixed point of T, we also have

$$\Gamma = \lim_{k\to\infty} \langle \Phi(q) - q, J(x_{n_k} - q)\rangle = \langle \Phi(q) - q, J(\tilde{q} - q)\rangle \leq 0.$$

By applying (28) to $\{x_n\}$ and q, since $\Gamma \leq 0$ in both cases, we obtain

$$\lim_{n\to\infty} x_n = q$$

as required. □

Corollary 3.2. *Let H be a Hilbert space, $C \subset H$ a nonempty closed convex subset, $T : C \to C$ a nonexpansive mapping with fixed point set $F \neq \emptyset$ and $\Phi : C \to C$ a ρ-contraction. Then the approximating curve $\{x_t\} \subset C$ defined by (25) strongly converges, as $t \to 0$, to the unique solution $q \in F$ to the inequality*

$$\langle (\Phi - I)q, x - q\rangle \leq 0, \quad \forall x \in F. \tag{29}$$

Theorem 3.3. *Let X be either a reflexive Banach space with weakly continuous normalized duality mapping J or a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $F \neq \emptyset$, $\Phi : C \to C$ a ρ-contraction and $\{\alpha_n\}$ a sequence in $(0,1)$ satisfying*

(*H1*) $\lim_{n\to\infty} \alpha_n = 0$

(*H2*) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(*H3*) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1$.

Then, for any $x_0 \in C$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = T(\alpha_n \Phi(x_n) + (1 - \alpha_n)x_n) \tag{30}$$

strongly converges to the unique solution $q \in F$ to the inequality

$$\langle (\Phi - I)q, J(x - q)\rangle \leq 0, \quad \forall x \in F. \tag{31}$$

Proof. Since T has fixed points, by Lemma 2.5(*3*) we have that $\{x_n\}$ is bounded, and therefore so are $\{T(x_n)\}$ and $\{\Phi(x_n)\}$. The fact that $\{x_n\}$ is asymptotically regular is a consequence of Lemmas 2.5 and 2.6. Indeed, by hypothesis we have that $\sum_{n=1}^{\infty}(1-\rho)\alpha_{n-1}=\infty$ and either $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$ or

$$\limsup_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}=\lim_{n\to\infty}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|=0. \tag{32}$$

Then inequality (21)

$$\|x_{n+1}-x_n\|\le(1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,\|\Phi(x_{n-1})-x_{n-1}\|$$

allows us to use Lemma 2.6 to deduce that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{33}$$

By using inequality (20) and the hypothesis (*H1*) we get

$$\lim_{n\to\infty}\|x_n-Tx_n\|\le\lim_{n\to\infty}\big(\|x_{n+1}-x_n\|+\alpha_n\|\Phi(x_n)-x_n\|\big)=0. \tag{34}$$

Distinguishing both cases according to the underlying space we will see now that

$$\limsup_{n\to\infty}\langle\Phi(q)-q,J(x_n-q)\rangle\le 0. \tag{35}$$

Assume first that X is a reflexive Banach space with weakly continuous normalized duality mapping J. Take a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty}\langle\Phi(q)-q,J(x_n-q)\rangle=\lim_{k\to\infty}\langle\Phi(q)-q,J(x_{n_k}-q)\rangle.$$

Since X is reflexive and $\{x_n\}$ bounded, we may assume that $x_{n_k}\rightharpoonup\bar{x}$. Since X satisfies the Demiclosedness principle and $\{(I-T)x_n\}$ converges to 0 from (34), we deduce that $\bar{x}\in F$. Then by inequality (31) and the weak-to-weak* uniform continuity of J,

$$\begin{aligned}\limsup_{n\to\infty}\langle\Phi(q)-q,J(x_n-q)\rangle&=\lim_{k\to\infty}\langle\Phi(q)-q,J(x_{n_k}-q)\rangle\\&=\langle\Phi(q)-q,J(\bar{x}-q)\rangle\le 0.\end{aligned}$$

If X is uniformly smooth we proceed as follows. Let $\{\beta_k\}$ be a null sequence in $(0,1)$ (i.e., $\{\beta_k\}\to 0$, as $k\to\infty$) and define $\{y_k\}$ by

$$y_k:=T(\beta_k\Phi(y_k)+(1-\beta)y_k).$$

We have proved in Theorem 3.1 that $\{y_k\}$ strongly converges to q. For any $n,k\ge 0$ define

$$\delta_{n,k}:=\|x_n-Tx_n\|^2+2\|x_n-Tx_n\|\|y_k-Tx_n\|$$

and

$$\epsilon_k:=\sup_{n\ge 0}\{\|\Phi(y_k)-x_n\|\|J(\beta_k(\Phi(y_k)-x_n)+(1-\beta_k)(y_k-x_n))-J(y_k-x_n)\|\}.$$

For any fixed $k \in \mathbb{N}$, by (34), $\lim_{n\to\infty} \delta_{n,k} = 0$. Moreover $\lim_{k\to\infty} \epsilon_k = 0$ because of the uniform continuity of J over bounded sets. By using inequality (15) and the nonexpansivity of T we obtain

$$\begin{aligned}
\|y_k - x_n\|^2 &\leq (\|Tx_n - x_n\| + \|y_k - Tx_n\|)^2 \\
&= \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\|\|y_k - Tx_n\| + \|y_k - Tx_n\|^2 \\
&\leq \delta_{n,k} + \|(1-\beta_k)(y_k - x_n) + \beta_k(\Phi(y_k) - x_n)\|^2 \\
&\leq \delta_{n,k} + (1-\beta_k)^2\|y_k - x_n\|^2 \\
&\qquad + 2\beta_k\langle \Phi(y_k) - x_n, J(\beta_k(\Phi(y_k) - x_n) + (1-\beta_k)(y_k - x_n))\rangle \\
&\leq \delta_{n,k} + (1-\beta_k)^2\|y_k - x_n\|^2 + 2\beta_k\langle \Phi(y_k) - x_n, J(y_k - x_n)\rangle + 2\beta_k\epsilon_k \\
&= \delta_{n,k} + (1-\beta_k)^2\|y_k - x_n\|^2 + 2\beta_k\langle y_k - x_n, J(y_k - x_n)\rangle \\
&\qquad + 2\beta_k\langle \Phi(y_k) - y_k, J(y_k - x_n)\rangle) + 2\beta_k\epsilon_k \\
&= \delta_{n,k} + ((1-\beta_k)^2 + 2\beta_k)\|y_k - x_n\|^2 + 2\beta_k\epsilon_k \\
&\qquad + 2\beta_k\langle \Phi(y_k) - y_k, J(y_k - x_n)\rangle
\end{aligned}$$

Then we deduce that

$$\limsup_{n\to\infty} \langle \Phi(y_k) - y_k, J(x_n - y_k)\rangle \leq \frac{\beta_k}{2} \limsup_{n\to\infty} \|y_k - x_n\|^2 + \epsilon_k. \tag{36}$$

On the other hand

$$\begin{aligned}
\langle \Phi(q) - q, J(x_n - q)\rangle = \langle \Phi(q) - q, J(x_n - q) - J(x_n - y_k)\rangle& \\
+ \langle (\Phi(q) - q) - (\Phi(y_k) - y_k), J(x_n - y_k)\rangle& \\
+ \langle \Phi(y_k) - y_k, J(x_n - y_k)\rangle.&
\end{aligned} \tag{37}$$

Note that

$$\lim_{k\to\infty}\left(\sup_{n\to\infty}\{\langle \Phi(q) - q, J(x_n - q) - J(x_n - y_k)\rangle\}\right) = 0 \tag{38}$$

because J is norm to norm uniform continuous on bounded sets. By using (36), (38) and passing first to $\limsup_{n\to\infty}$ and then to $\lim_{k\to\infty}$, from (37) we obtain

$$\limsup_{n\to\infty} \langle \Phi(q) - q, J(x_n - q)\rangle \leq 0.$$

Finally we prove that $\{x_n\}$ strongly converges to q. Set

$$\eta_n := \|J(\alpha_n(\Phi(x_n) - x_n) + (x_n - q)) - J(x_n - q)\|.$$

Then $\eta_n \to 0$, as $n \to 0$. We compute

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|\alpha_n(\Phi(x_n) - q) + (1-\alpha_n)(x_n - q) + \alpha_n(\Phi(q) - q)\|^2 \\
&\leq \|\alpha_n(\Phi(x_n) - q) + (1-\alpha_n)(x_n - q)\|^2 + 2\alpha_n\langle \Phi(q) - q, J(x_n - q)\rangle \\
&\qquad + 2\alpha_n\eta_n\|\Phi(q) - q\| \\
&\leq (1 - (1-\rho)\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle \Phi(q) - q, J(x_n - q)\rangle \\
&\qquad + 2\alpha_n\eta_n\|\Phi(q) - q\|
\end{aligned}$$

and the result follows from (35) and Lemma 2.6. □

Corollary 3.4. *Let X be either a reflexive Banach space with weakly continuous normalized duality mapping J or a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $F \neq \emptyset$ and $g : C \to C$ a mapping such that $I - \mu g$ is a contraction for some $\mu > 0$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying hypotheses (H1)–(H3) in Theorem 3.3. Then the sequence $\{x_n\}$ defined by the iterative scheme*

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n), \tag{39}$$

strongly converges to the unique solution $q \in F$ to the inequality problem

$$\langle g(q), x - q\rangle \geq 0, \quad \forall x \in F. \tag{40}$$

Proof. Consider the sequence $\{z_n\}$ defined by $z_n = Tx_n$, for any $n \geq 0$. Then

$$\begin{aligned} z_{n+1} &= T(Tx_n - \alpha_n g(Tx_n)) \\ &= T(z_n - \frac{\alpha_n}{\mu}\mu g(z_n)) \\ &= T(\alpha'_n(I - \mu g)z_n + (1 - \alpha'_n)z_n), \end{aligned}$$

where $\alpha'_n = \frac{\alpha_n}{\mu}$ for all $n \geq 0$, so the sequence $\{\alpha'_n\}$ satisfies hypotheses (*H1*)–(*H3*). Since $\Phi := I - \mu g$ is a contraction, Theorem 3.3 implies the strong convergence of $\{z_n\}$ to the unique solution $q \in F$ to the inequality problem

$$\langle(\Phi - I)q, x - q\rangle \geq 0, \quad \forall x \in F,$$

which is equivalent to (40). Therefore, from the iteration scheme (39) we deduce that the sequence $\{x_n\}$ strongly converges to q. □

Remark 3.5. It is easily seen that the conclusion of Theorems 3.1, 3.3 and Corollary 3.4 remains true if the uniform smoothness assumption of X is replaced with the following two conditions:

(a) X has a uniformly Gâteaux differentiable norm.
(b) X has Reich's property.

Furthermore, in [31], all previous results were proved to remain true in the more general framework of a reflexive Banach space with a weakly continuous generalized duality mapping J_ϕ associated to a gauge ϕ. Likewise, the relationship between Halpern iteration and algorithms (25) and (30) was studied.

4. Rates of asymptotic regularity

In the following, we apply proof mining techniques to get effective rates of asymptotic regularity for the iteration $\{x_n\}$ defined by (12). The methods we use in this paper are inspired by those used in [22] to obtain effective rates of asymptotic regularity for Halpern iterations. As in the case of Halpern iterations, the main ingredient turns out to be the quantitative Lemma 2.7.

Theorem 4.1. *Let X be a normed space, $C \subseteq X$ a nonempty convex subset and $T : C \to C$ be nonexpansive. Assume that $\Phi : C \to C$ is a ρ-contraction and that $\{\alpha_n\}_{n\geq 0}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n$ is divergent and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$ is convergent. Let $x_0 \in C$ and $\{x_n\}_{n\geq 0}$ be defined by (12). Assume that $\{x_n\}$ is bounded.*

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and moreover for all $\varepsilon \in (0,2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \beta, \theta, \rho, M, D, \varepsilon),$$

where

1. $\varphi : (0,\infty) \to \mathbb{Z}_+$ *is a rate of convergence of* $\{\alpha_n\}$,
2. $\beta : (0,\infty) \to \mathbb{Z}_+$ *is a Cauchy modulus of* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$,
3. $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ *is a rate of divergence of* $\sum_{n=0}^{\infty} \alpha_n$,
4. $M \geq 0$ *is such that* $M \geq \|\Phi(x_0) - x_0\|$,
5. $D > 0$ *satisfies* $D \geq \|x_n - x_m\|$ *for all* $m, n \geq 0$,

and $\Psi(\varphi, \beta, \theta, \rho, M, D, \varepsilon)$ is defined by

$$\Psi := \max\left\{1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil \left(\beta\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil \ln\left(\frac{4D}{\varepsilon}\right)\right\rceil\right)\right),\ 1 + \varphi\left(\frac{\varepsilon}{2P}\right)\right\},$$

with $P = (1+\rho)D + M$.

Proof. Applying (21) and (19), we get that for all $n \geq 1$

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|\,\|\Phi(x_{n-1}) - x_{n-1}\| \\ &\leq (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| + P\,|\alpha_n - \alpha_{n-1}|.\end{aligned}$$

Let us denote for $n \geq 1$

$$a_n := \|x_n - x_{n-1}\|, \qquad b_n := P\,|\alpha_n - \alpha_{n-1}|, \qquad \lambda_n := (1-\rho)\alpha_{n-1}.$$

Then D is an upper bound on $\{a_n\}$ and

$$a_{n+1} \leq (1 - \lambda_{n+1})a_n + b_n \ \text{ for all } n \geq 1.$$

Moreover, $\sum_{n=1}^{\infty} \lambda_n$ is divergent with rate of divergence

$$\delta : \mathbb{Z}_+ \to \mathbb{Z}_+, \qquad \delta(n) = 1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil n\right), \tag{41}$$

since for all $n \in \mathbb{Z}_+$,

$$\sum_{i=1}^{\delta(n)} \lambda_i = (1-\rho) \sum_{i=0}^{\delta(n)-1} \alpha_i = (1-\rho) \sum_{i=0}^{\theta(\lceil 1/1-\rho \rceil\, n)} \alpha_i \geq (1-\rho) \left\lceil \frac{1}{1-\rho} \right\rceil n \geq n.$$

Let $t_n := \sum_{i=0}^{n} |\alpha_{i+1} - \alpha_i|$ and $s_n := \sum_{i=1}^{n} b_i = Pt_{n-1}$ and define

$$\gamma : (0, \infty) \to \mathbb{Z}_+, \qquad \gamma(\varepsilon) := 1 + \beta\left(\frac{\varepsilon}{P}\right), \tag{42}$$

Then for all $n \geq 0$,

$$s_{\gamma(\varepsilon)+n} - s_{\gamma(\varepsilon)} = P\left(t_{\beta(\varepsilon/P)+n} - t_{\beta(\varepsilon/P)}\right) < P\frac{\varepsilon}{P} = \varepsilon.$$

Thus, $\sum_{n=1}^{\infty} b_n$ is convergent with Cauchy modulus γ.

It follows that we can apply Lemma 2.7 to get that for all $\varepsilon \in (0, 2)$ and for all $n \geq h_1(\beta, \theta, \rho, M, D, \varepsilon)$

$$\|x_n - x_{n-1}\| < \frac{\varepsilon}{2}, \tag{43}$$

where

$$h_1(\beta, \theta, \rho, M, D, \varepsilon) := 1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil \left(\beta\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil \ln\left(\frac{4D}{\varepsilon}\right) \right\rceil\right)\right).$$

Using (20) and (19), we get that for all $n \geq 1$,

$$\begin{aligned} \|x_{n-1} - Tx_{n-1}\| &\leq \|x_n - x_{n-1}\| + \alpha_{n-1}\|\Phi(x_{n-1}) - x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + P\alpha_{n-1}. \end{aligned} \tag{44}$$

Let $h_2(\varphi, \rho, M, D, \varepsilon) := 1 + \varphi\left(\frac{\varepsilon}{2P}\right)$. Since φ is a rate of convergence of $\{\alpha_n\}$ towards 0, it follows that

$$P\alpha_{n-1} < \frac{\varepsilon}{2} \quad \text{for all } n \geq h_2(\varphi, \rho, M, D, \varepsilon). \tag{45}$$

As a consequence of (43), (44) and (45), we get that

$$\|x_{n-1} - Tx_{n-1}\| < \varepsilon$$

for all $n \geq \max\{h_1(\beta, \theta, \rho, M, D, \varepsilon),\ h_2(\varphi, \rho, M, D, \varepsilon)\}$, so the conclusion of the theorem follows. □

If C is bounded, then $\{x_n\}$ is bounded for all $x_0 \in C$. Moreover, we can take $M := D := d_C$ in the above theorem, where $d_C := \sup\{\|x - y\| \mid x, y \in C\}$ is the diameter of C.

Corollary 4.2. *In the hypotheses of Theorem* 4.1*, assume moreover that* C *is bounded. Then* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *for all* $x_0 \in C$ *and moreover for all* $\varepsilon \in (0, 2)$*,*

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon),$$

where $\Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon)$ *is defined as in Theorem* 4.1 *by replacing* M *and* D *with* d_C*.*

Thus, for bounded C, we get asymptotic regularity for general $\{\alpha_n\}$ and an explicit rate of asymptotic regularity $\Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon)$ that depends weakly on C (via its diameter d_C) and on the ρ-contraction Φ (via ρ), while it does not depend on the nonexpansive mapping T, the starting point $x_0 \in C$ of the iteration or other data related with C and X.

The rate of asymptotic regularity can be simplified when the sequence $\{\alpha_n\}$ is decreasing.

Corollary 4.3. *Let $X, C, T, \Phi, \{x_n\}$ be as in the hypotheses of Corollary 4.2. Assume that $\{\alpha_n\}$ is a decreasing sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n$ is divergent.*

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x_0 \in C$ and furthermore, for all $\varepsilon \in (0,2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \theta, \rho, d_C, \varepsilon),$$

where $\varphi : (0,\infty) \to \mathbb{Z}_+$ is a rate of convergence of $\{\alpha_n\}$, $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a rate of divergence of $\sum_{n=0}^{\infty} \alpha_n$ and $\Psi(\varphi, \theta, \rho, d_C, \varepsilon)$ is defined by

$$\Psi := \max\left\{1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil \left(\varphi\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil \ln\left(\frac{4d_C}{\varepsilon}\right)\right\rceil\right)\right),\ 1 + \varphi\left(\frac{\varepsilon}{2P}\right)\right\}$$

with $P = (2+\rho)d_C$.

Proof. Remark that $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$ is convergent with Cauchy modulus φ. □

Finally, we get, as in the case of Halpern iterates, an exponential (in $1/\varepsilon$) rate of asymptotic regularity for $\alpha_n = \frac{1}{n+1}$.

Corollary 4.4. *Let $X, C, T, \Phi, \{x_n\}$ be as in the hypotheses of Corollary 4.2. Assume that $\alpha_n = \frac{1}{n+1}$ for all $n \geq 0$.*

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x_0 \in C$ and furthermore, for all $\varepsilon \in (0,2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Theta(\rho, d_C, \varepsilon),$$

where

$$\Theta(\rho, d_C, \varepsilon) = \exp\left(\frac{4}{1-\rho}\left(\frac{16 d_C}{\varepsilon} + 2\right)\right)$$

Proof. We can apply Corollary 4.3 with

$$\varphi : (0,\infty) \to \mathbb{Z}_+, \qquad \varphi(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil - 1.$$

and

$$\theta : \mathbb{Z}_+ \to \mathbb{Z}_+, \qquad \theta(n) = 4^n - 1$$

to conclude that for all $\varepsilon \in (0,2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\rho, d_C, \varepsilon),$$

where $P = (2+\rho)d_C$ and

$$\begin{aligned}\Psi(\rho, d_C, \varepsilon) &= \max\left\{\exp\left(\ln 4 \left\lceil \frac{1}{1-\rho}\right\rceil \left(\left\lceil \frac{4P}{\varepsilon}\right\rceil + 1 + \left\lceil \ln\left(\frac{4d_C}{\varepsilon}\right)\right\rceil\right)\right), \left\lceil \frac{2P}{\varepsilon}\right\rceil\right\} \\ &= \exp\left(\ln 4 \left\lceil \frac{1}{1-\rho}\right\rceil \left(\left\lceil \frac{4P}{\varepsilon}\right\rceil + 1 + \left\lceil \ln\left(\frac{4d_C}{\varepsilon}\right)\right\rceil\right)\right) \\ &< \exp\left(\frac{4}{1-\rho}\left(\frac{16d_C}{\varepsilon} + 2\right)\right) = \Theta(\rho, d_C, \varepsilon).\end{aligned}$$

as $\rho \in (0,1)$, $\lceil a \rceil < a+1$ and $1 + \ln a \le a$ for all $a > 0$. The conclusion follows now immediately. □

5. Applications

As it was pointed out in the introduction, iterative methods for nonexpansive mappings have been applied to solve the VIP(A, C) (2) which, in fact, is equivalent under suitable conditions to the minimization problem of a certain function. On the other hand, the relation between the set of zeros of an accretive operator and the fixed point set of its resolvent allows us to use those iterative techniques for nonexpansive mappings to approximate zeros of such operators. We first apply the explicit iterative method for approximating fixed points, presented in Section 3, to solve a particular variational inequality problem in the setting of Hilbert spaces. Then we focus on the asymptotic behavior of the resolvent of an accretive operator in the framework of Banach spaces.

5.1. A variational inequality problem

Let H be a Hilbert space, $T : H \to H$ be a nonexpansive mapping with fixed point set $F \neq \emptyset$, and $\Phi : H \to H$ be a contraction. Assume that A is a Lipschitzian self-operator on H which is strongly monotone; that is, there exist a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \eta \|x - y\|^2, \quad \forall x, y \in H.$$

It is known that the variational inequality

$$\langle (A - \gamma\Phi)q, q - x\rangle \le 0, \quad \forall x \in F, \tag{46}$$

where $\gamma > 0$, is the optimality condition for the minimization problem

$$\min_{x \in F} f(x) - h(x)$$

where f is a subdifferentiable function with subdifferential $\partial f = A$ and h is a potential function for $\gamma\Phi$ (i.e. $h'(x) = \gamma\Phi(x)$ for $x \in H$). Marino and Xu [30] presented an iterative method to solve the variational inequality (46) for a linear bounded operator. We apply the explicit method (30), in particular algorithm (39), to solve such variational inequality dispensing with the linear condition on the operator A. To this end, we need the following lemma.

Lemma 5.1. *Assume that A is a L-Lipschitzian η-strongly monotone operator, and let Φ be a ρ-contraction. Then, for any $\gamma < \eta/\rho$, $A - \gamma\Phi$ is R-Lipschitzian and δ-strongly monotone with $R = L+\gamma\rho$ and $\delta = \eta-\gamma\rho$. Besides, for any $0 < \mu < 2\delta/R^2$, the mapping $I - \mu(A - \gamma\Phi)$ is a contraction.*

Proof. Since A is L-Lipschitzian and Φ is a ρ-contraction,

$$\|(A - \gamma\Phi)x - (A - \gamma\Phi)y\| \le \|Ax - Ay\| + \gamma\|\Phi x - \Phi y\| \le (L + \gamma\rho)\|x - y\|,$$

that is, $A - \gamma\Phi$ is Lipschitzian with constant $R = L + \gamma\rho$. The strong monotonicity of $A - \gamma\Phi$ is consequence of the strong monotonicity of A as it is showed as follow.

$$\begin{aligned}\langle (A - \gamma\Phi)x - (A - \gamma\Phi)y, x - y\rangle &= \langle Ax - Ay, x - y\rangle - \gamma\langle \Phi x - \Phi y, x - y\rangle \\ &\ge \eta\|x - y\|^2 - \gamma\|\Phi x - \Phi y\|\|x - y\| \\ &\ge (\eta - \gamma\rho)\|x - y\|^2,\end{aligned}$$

where $\delta = \eta - \gamma\rho > 0$. By applying the R-Lipschitz continuity and δ-strong monotonicity of $B := A - \gamma\Phi$ we obtain

$$\begin{aligned}\|(I - \mu B)x - (I - \mu B)y\|^2 &= \|x - y\|^2 + \mu^2\|Bx - By\|^2 - \mu\langle x - y, Bx - By\rangle \\ &\le \|x - y\|^2 + \mu^2 R^2\|x - y\|^2 - 2\mu\delta\|x - y\|^2 \\ &= (1 - \mu(2\delta - \mu R^2))\|x - y\|^2.\end{aligned}$$

Then, for any $0 < \mu < 2\delta/R^2$, the mapping $I - \mu(A - \gamma\Phi)$ is a contraction with constant $\sqrt{1 - \mu(2\delta - \mu R^2)}$. □

Theorem 5.2. *Let T be a nonexpansive mapping with fixed point set F, A a L-Lipschitzian η-strongly monotone operator and Φ a ρ-contraction on a Hilbert space. Then, for any $\gamma < \eta/\rho$, the sequence defined by the iterative scheme*

$$x_{n+1} = Tx_n - \alpha_n(A - \gamma\Phi)Tx_n,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies hypotheses (H1)–(H2) in Theorem 3.3, strongly converges to the unique solution to the variational inequality (46).

Proof. Note that, for any $\gamma < \eta/\rho$, Lemma 5.1 implies that there exists $\mu > 0$ such that $I - \mu g$ is a contraction, where $g = A - \gamma\Phi$. Then, by Theorem 3.4 we obtain the strong convergence of the sequence $\{x_n\}$ to the unique solution to the variational inequality problem (46). □

5.2. Zeros of m-accretive operators

Let X be a real Banach space. A set-valued operator $A : X \to 2^X$ with domain $D(A)$ and range $R(A)$ in X is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \ge 0,$$

where J is the normalized duality mapping. An accretive operator A is *m-accretive* if $R(I+\lambda A)=X$ for all $\lambda>0$. Denote the set of zeros of A by

$$Z := A^{-1}(0) = \{z \in D(A) : 0 \in Az\}.$$

Throughout this subsection it is assumed that A is m-accretive and $A^{-1}(0)\neq\emptyset$. Set $C=\overline{D(A)}$ and assume it is convex. It is known that the *resolvent* of A, defined by

$$J_\lambda = (I+\lambda A)^{-1},$$

for $\lambda>0$, is a single-valued nonexpansive mapping from C into itself (cf. [1]).

If we consider the problem of finding a zero of A, i.e.,

$$\text{find } z \in C \text{ such that } 0 \in Az,$$

it is straightforward to see that the set of zeros $A^{-1}(0)$ coincides with the fixed point set of J_λ, $\mathrm{Fix}(J_\lambda)$, for any $\lambda>0$. Therefore an equivalent problem is to find $z\in\mathrm{Fix}(J_\lambda)$.

As a consequence of the convergence of the implicit iterative scheme (25), we obtain Reich's result (cf. [37]) for approximating zeros of accretive operators in uniformly smooth Banach spaces. Besides, the following theorem in the setting of reflexive Banach spaces with weakly continuous normalized duality mapping constitutes a new approach.

Theorem 5.3. *Let X be either a reflexive Banach spaces with weakly continuous normalized duality mapping J or a uniformly smooth Banach space, and A a m-accretive operator. Then, for each $x\in X$, the sequence $\{J_\lambda(x)\}$ strongly converges, as $\lambda\to\infty$, to the unique zero of A, $q\in A^{-1}(0)$, which satisfies the variational inequality*

$$\langle x-q, J(y-q)\rangle \le 0 \quad \forall y \in A^{-1}(0). \tag{47}$$

Proof. Given $x\in X$ we consider the approximating curve $\{x_t\}$ such that $x_t=J_{1/t}x$, for any $t\in(0,1)$. By definition of the resolvent of A, we obtain the following equivalence:

$$\begin{aligned} x_t = \left(I+\frac{1}{t}A\right)^{-1} x \quad &\Leftrightarrow \quad x \in x_t + \frac{1}{t}Ax_t \\ &\Leftrightarrow \quad t(x-x_t) \in Ax_t \\ &\Leftrightarrow \quad x_t + t(x-x_t) \in (I+A)x_t \\ &\Leftrightarrow \quad x_t = (I+A)^{-1}(x_t+t(x-x_t)) \\ &\Leftrightarrow \quad x_t = T(t\Phi(x_t)+(1-t)x_t), \end{aligned}$$

where $T=(I+A)^{-1}$ is the nonexpansive resolvent with constant 1, and $\Phi=x$ is a constant mapping which is a contraction. Therefore, Theorem 3.1 implies the strong convergence of $\{x_t\}$, as $t\to0$, to the unique solution to the inequality (26); in other words, $\{J_\lambda x\}$ strongly converges, as $\lambda\to\infty$, to the unique solution $q\in A^{-1}(0)$ to the inequality (47). □

Remark 5.4. If we define the mapping $Q : X \to A^{-1}(0)$ such that, for any $x \in X$,

$$Qx = \lim_{\lambda \to \infty} J_\lambda x,$$

then, since Qx satisfies the inequality (47), by Lemma 2.1 we can claim that Q is the unique sunny nonexpansive retraction from X to $A^{-1}(0)$.

References

[1] V. Barbu: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden (1976).

[2] F. E. Browder: Existence and approximation of solutions of nonlinear variational inequalities, Proc. Natl. Acad. Sci. USA 56 (1966) 1080–1086.

[3] F. E. Browder: Convergence of approximants to fixed points of nonlinear maps in Banach spaces, Arch. Ration. Mech. Anal. 24 (1967) 82–90.

[4] F. E. Browder, W. V. Petryshyn: The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72 (1966) 571–575.

[5] R. E. Bruck: Nonexpansive projections on subsets of Banach spaces, Pac. J. Math. 47 (1973) 341–355.

[6] C. E. Chidume, C. O. Chidume: Iterative approximation of fixed points of nonexpansive mappings, J. Math. Anal. Appl. 318 (2006) 288–295.

[7] I. Cioranescu: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht (1990).

[8] P. L. Combettes, S. A. Hirstoaga: Approximating curves for nonexpansive and monotone operators, J. Convex Analysis 13(3&4) (2006) 633–646.

[9] M. Edelstein: A remark on a theorem of M. A. Krasnoselskii, Amer. Math. Monthly 14 (1970) 65–73.

[10] J. Garcia-Falset, W. Kaczor, T. Kuczumov, S. Reich: Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 43 (2001) 377–401.

[11] A. Genel, J. Lindenstrauss: An example concerning fixed points, Israel J. Math. 22(1) (1975) 81–86.

[12] K. Goebel, W. A. Kirk: Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics 28, Cambridge University Press, Cambridge (1990).

[13] K. Goebel, S. Reich: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York (1984).

[14] B. Halpern: Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[15] S. Ishikawa: Fixed points and iteration of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. 59 (1976) 65–71.

[16] T. H. Kim, H. K. Xu: Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51–60.

[17] U. Kohlenbach: A quantitative version of a theorem due to Borwein-Reich-Shafrir, Numer. Funct. Anal. Optimization 22 (2001) 641–656.

[18] U. Kohlenbach: Uniform asymptotic regularity for Mann iterates, J. Math. Anal. Appl. 279 (2003) 531–544.

[19] U. Kohlenbach, L. Leuştean: Mann iterates of directionally nonexpansive mappings in hyperbolic spaces, Abstr. Appl. Anal. 2003 (2003) 449–477.

[20] U. Kohlenbach: Applied Proof Theory: Proof Interpretations and their Use in Mathematics, Springer Monographs in Mathematics, Springer, Berlin (2008).

[21] M. A. Krasnosel'skij: Two remarks on the method of successive approximations, Usp. Mat. Nauk. 10(1(63)) (1955) 123–127 (in Russian).

[22] L. Leuştean: A quadratic rate of asymptotic regularity for CAT(0)-spaces, J. Math. Anal. Appl. 325(1) (2007) 386–399.

[23] L. Leuştean: Rates of asymptotic regularity for Halpern iterations of nonexpansive mappings, J. UCS 13 (2007) 1680–1691.

[24] T. C. Lim, H. K. Xu: Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal. 22 (1994) 1345–1355.

[25] P. Lions: Approximation de points fixes de contractions, C. R. Acad. Sci., Paris, Sér. A 284 (1977) 1357–1359.

[26] L. S. Liu: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194 (1995) 114–125.

[27] G. López, V. Martín-Márquez, H. K. Xu: Halpern's iteration for nonexpansive mappings, in: Nonlinear Analysis and Optimization I: Nonlinear Analysis, A. Leizarowitz et al. (ed.), Contemporary Mathematics 513, AMS Providence (2010) 187–207.

[28] P. E. Maingé: A hybrid extragradient-viscosity method for monotone operators and fixed point problems, SIAM J. Control Optim. 47(3) (2008) 1499–1515.

[29] W. R. Mann: Mean value methods in iteration, Proc. Amer. Math. Soc. 4(3) (1953) 506–510.

[30] G. Marino, H. K. Xu: A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318(1) (2006) 43–52.

[31] V. Martín-Márquez: Fixed Point Approximation Methods for Nonexpansive Mappings: Optimization Problems, PhD Thesis, University of Seville (2010).

[32] A. Moudafi: Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46–55.

[33] K. Nakajo, W. Takahashi: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.

[34] J. G. O'Hara, P. Pillay, H. K. Xu: Iterative approaches to convex feasibility problems in Banach Spaces, Nonlinear Anal. 64 (2006) 2002–2042.

[35] S. Reich: Asymptotic behavior of contractions in Banach spaces, J. Math. Anal. Appl. 44 (1973) 57–70.

[36] S. Reich: Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274–276.

[37] S. Reich: Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292.

[38] S. Reich: Product formulas, nonlinear semigroups, and accretive operators, J. Funct. Anal. 36(2) (1980) 147–168.

[39] S. Reich: Some problems and results in fixed point theory, Contemp. Math. 21 (1983) 179–187.

[40] H. Schaefer: Über die Methode sukzessiver Approximationen, Jahresber. Dtsch. Math.-Ver. 59 (1957) 131–140.

[41] N. Shioji, W. Takahashi: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997) 3641–3645.

[42] Y. Song, R. Chen: Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings, Appl. Math. Comput. 180 (2006) 275–287.

[43] T. Suzuki: A sufficient and necessary condition for Halpern-type strong convergence to fixed point of nonexpansive mappings, Proc. Amer. Math. Soc. 135(1) (2007) 99–106.

[44] T. Suzuki: Moudafi's viscosity approximations with Meir-Keeler contractions, J. Math. Anal. Appl. 325 (2007) 342–352.

[45] R. Wittmann: Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491.

[46] I. Yamada: The hybrid steepest descent method for the variational inequality over the intersection of fixed point sets of nonexpansive mappings, in: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000), D. Butnariu, Y. Censor, S. Reich (eds.), Elsevier, Amsterdam (2001) 473–504.

[47] I. Yamada, N. Ogura: Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, Numer. Funct. Anal. Optimization 25 (2004) 619–655.

[48] H. K. Xu: Iterative algorithms for nonlinear operators, J. Lond. Math. Soc., II. Ser. 66 (2002) 240–256.

[49] H. K. Xu: An iterative approach to quadratic optimization, J. Optimization Theory Appl. 116(3) (2003) 659–678.

[50] H. K. Xu: Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.

[51] H. K. Xu: An alternative regularization method for nonexpansive mappings with applications, in: Nonlinear Analysis and Optimization I: Nonlinear Analysis, A. Leizarowitz et al. (ed.), Contemporary Mathematics 513, AMS, Providence (2010) 239–263.

[52] H. K. Xu, T. H. Kim: Convergence of hybrid steepest descent methods for variational inequalities, J. Optimization Theory Appl. 119 (2003) 185–201.