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About the Existence of an Isotone Retraction onto a Convex Cone

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The existence of continuous isotone retractions onto pointed closed convex cones in Hilbert spaces is studied. The cones admitting such mappings are called isotone retraction cones. In finite dimension, generating, isotone retraction cones are polyhedral. For a closed, pointed, generating cone in a Hilbert space the isotonicity of a retraction and its complement implies that the cone is latticial and the retraction is well defined by the latticial structure. The notion of sharp mapping is introduced. If the cone is generating and normal, it is proved that its latticiality is equivalent to the existence of an isotone retraction onto it, whose complement is sharp. The subdual and autodual latticial cones are also characterized by isotonicity. This is done by attempting to extend Moreau's theorem to retractions.

1. Introduction

G. Isac and A. B. Németh have characterized a cone in the Euclidean space which admits an isotone projection onto it [1], where isotonicity is considered with respect to the order induced by the cone. They called such a cone isotone projection cone. The same authors [2] and S. J. Bernau [3] considered the similar problem for the Hilbert space.

This problem is related to nonlinear complementarity. Both the solvability and the approximation of solutions of nonlinear complementarity problems can be handled by using the metric projection onto the cone defining the problem. The isotonicity of the projection provides new existence results and iterative methods [4, 5, 6, 7].

With an eye on this, bearing also in mind the future possibility of extending the given iterative methods to more general discrete dynamical systems where the projection is replaced by a continuous isotone retraction, S. Z. Németh [8, 9] started to study the extension of the isotonicity problem, when the projection is replaced by a continuous retraction. He called the pointed closed convex cones which admit a continuous isotone retraction onto it, isotone retraction cones. Regardless of its

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immediate applications, this problem is interesting in itself too. It provides a new insight in the convex geometry of cones and relates the ordering structure of a space to its topological and geometrical structure.

The main results proved in the paper are the following:

1. In finite dimension every generating isotone retraction cone is polyhedral.
2. If a generating pointed closed convex cone of a Hilbert space admits a continuous isotone retraction ρ onto it such that $I-\rho$ is also isotone, where I stands for the identity mapping, then K is latticial and $\rho(x)=x^+ := \sup\{0,x\}$ for all x.
3. A normal generating pointed closed convex cone K of a Hilbert space is latticial if and only if it admits an isotone retraction ρ onto it whose complement is sharp (that is, for which $\operatorname{im}(I-\rho)\cap\operatorname{im}(\rho-I)=\{0\}$), where I stands for the identity mapping and im for the image of a mapping. In the case of retractions the later implication is equivalent to $\rho(x)+\rho(y)=x+y \implies x\in K$.

By attempting to extend Moreau's theorem to retractions, we also give two important characterizations of subdual and autodual latticial cones respectively, through isotonicity properties. The paper also contains several other properties of the isotone retractions and it is structured as follows: First we introduce the preliminary notions and the new notations used throughout the paper. Then, we provide the main results: the investigation of the facial structure of the isotone projection cones in Euclidean spaces, and the relation between the existence of an isotone retraction and the latticiality of the cone. Since there are fundamental questions which remain open and the construction of examples is difficult, we considered important to allocate a separate section to examples. We conclude the paper by providing some conclusions, which both give an overview of the paper and raises several questions for the future.

2. Preliminaries

In all what follows $(H,\langle\cdot,\cdot\rangle)$ stands for a *separable Hilbert space* over the reals, denoted simply with H. Let be $\|.\|$ the norm defined by $\langle\cdot,\cdot\rangle$.

The nonempty set $K\subset H$ is called a *convex cone* if

(i) $\lambda x\in K$, for all $x\in K$ and $\lambda\geq 0$ and if
(ii) $x+y\in K$, for all $x,y\in K$.

A convex cone K is called *pointed* if $K\cap(-K)=\{0\}$.

A convex cone is called *generating* if $K-K=H$.

The *relation $\leq$ defined by the pointed convex cone* K is given by $x\leq y$ if and only if $y-x\in K$. Particularly, we have $K=\{x\in H: 0\leq x\}$. The relation $\leq$ is an *order relation*, that is, it is reflexive, transitive and antisymmetric; it is *translation invariant*, that is, $x+z\leq y+z$, $\forall x,y,z\in H$ with $x\leq y$; and it is *scale invariant*, that is, $\lambda x\leq\lambda y$, $\forall x,y\in H$ with $x\leq y$ and $\lambda>0$.

Conversely, for every $\leq$ scale invariant, translation invariant and antisymmetric order relation in H there is a pointed convex cone K, defined by $K=\{x\in H: 0\leq x\}$, such that $x\leq y$ if and only if $y-x\in K$. The cone K is called the *positive cone of* H and $(H,\leq)$ (or (H,K)) is called an *ordered Hilbert space*.

The cone $K_0 \subset K$ is called a *face of* K if from $x \in K$, $y \in K_0$ and $x \leq y$, it follows that $x \in K_0$. If $x \in K$, then the set $\{\lambda y : \ 0 \leq y \leq x, \ \lambda \geq 0\}$ is a face of K called the *face of* x and is denoted $\operatorname{fce} x$. If $K_0 \neq K$, then it is called a *proper face of* K. If L denotes a supporting hyperplane of K and $K_0 = K \cap L$, then K_0 is said an *exposed face of* K.

The ordered Hilbert space $(H, \leq)$ is called *latticially ordered* if for every $x, y \in H$ there exists $x \vee y := \sup\{x, y\}$. In this case the positive cone K is called a *latticial cone*. Denote $x^+ = 0 \vee x$ and $x^- = 0 \vee (-x)$. Then, $x = x^+ - x^-$, x^+ is called the *positive part* of x and x^- is called the *negative part* of x. The mapping $x \mapsto x^+$ is called the *positive part mapping*. The continuity of the positive part mapping is equivalent to the closedness of K.

Let $K \subseteq H$ be a closed convex cone. Recall that

$$K^* = \{x \in H : \langle x, y\rangle \geq 0, \, \forall y \in K\}$$

is called the *dual cone* of K. K^* is a closed convex cone and if K is generating, then K^* is pointed (this is the case for example if K is latticial).

The closed convex cone K is called *subdual* if $K \subseteq K^*$.

The closed convex cone $K^\circ = -K^*$ is called the polar of K. We have $K^\circ = \{x \in H : \langle x, y\rangle \leq 0, \, \forall y \in K\}$ and if K is closed $(K^\circ)^\circ = K$ (Farkas lemma). Therefore, the closed convex cones K and L are called *mutually polar* if $K = L^\circ$ (or equivalently $K^\circ = L$).

Let $P_K : H \to H$ be the *projection mapping* onto K defined by $P_K(x) \in K$ and $\|x - P_K(x)\| = \min\{\|x - y\| : y \in K\}$. The following theorem is proved in [10].

Theorem 2.1 (Moreau). *Let H be a Hilbert space, $K, L \subseteq H$ two mutually polar closed convex cones in H. Then, the following statements are equivalent:*

(i) $\quad z = x + y, \ x \in K, \ y \in L$ *and* $\langle x, y\rangle = 0,$

(ii) $\quad x = P_K(z)$ *and* $y = P_L(z)$

The closed, pointed cone $K \subset H$ is called *normal* if from $x_m \in K$, $x_m \to 0$ and $0 \leq y_m \leq x_m$ it follows $y_m \to 0$. The cone K is called *regular* if every decreasing sequence (and hence each increasing and order bounded sequence) in it is convergent. McArthur in [11] shows that every closed convex normal cone in a Banach space is regular if this space does not contain any subspace isomorphic to the Banach space c_0 of all sequences of real numbers convergent to zero equipped with the supremum norm. In particular, this is true for a Hilbert space. Therefore, any closed convex normal cone in a Hilbert space is regular.

3. New definitions

By the *image* $\operatorname{im} \varphi$ of a mapping φ we mean the set of points $\varphi(x)$, where x goes through all elements of the domain of definition of φ. In the next definition only the notions of *sharp mapping* and *isotone retraction cone* are new. We combined them together with slightly adapted, already known notions for the sake of compactness only.

Definition 3.1. Let H be a Hilbert space, $K \subseteq H$ and $\zeta : H \to H$.

1. The mapping ζ is called a *retraction*, if $\zeta \circ \zeta = \zeta$. If $\operatorname{im} \zeta = K$ this is equivalent to
   (a) $\zeta(x) \in K$, for all $x \in H$,
   (b) $\zeta(u) = u$, for all $u \in K$.
2. The mapping ζ is called *sharp* if $\zeta(0) = 0$ and $\operatorname{im} \zeta \cap \operatorname{im}(-\zeta) = \{0\}$. If ζ is sharp, then $t\zeta$ is also sharp, for all $t \in \mathbb{R}$. If ζ is a retraction and $\operatorname{im} \zeta = K$, then $I - \zeta$ is sharp if and only if $\zeta(x) + \zeta(y) = x + y$ implies $x, y \in K$.
3. Suppose that K is a pointed closed convex cone. The mapping $\zeta : H \to H$ is called K-*isotone* (or simply isotone if there is no ambiguity) if $y - x \in K$ implies $\zeta(y) - \zeta(x) \in K$. This can be written in the equivalent form: $x \leq y$ implies $\zeta(x) \leq \zeta(y)$. If there is a continuous K-isotone retraction $\zeta : H \to H$ with $\operatorname{im} \zeta = K$, then K is called an *isotone retraction cone.*

We remark that the set of isotone retractions onto a given pointed closed convex cone is convex.

The next remark follows easily from item 2 of Definition 3.1.

Remark 3.2. Let $K \subseteq H$ with $K \cap (-K) = \{0\}$ (in particular, K may be a pointed convex cone). Let $\zeta : H \to H$ be a mapping onto K such that $\zeta(0) = 0$. Then, $t\zeta$ is sharp for all $t \in \mathbb{R}$. In particular, if ζ is a retraction onto K, then $t\zeta$ is sharp for all $t \in \mathbb{R}$. If in addition the relation $K + K \subseteq K$ holds (in particular K may be a convex cone) and $\rho : H \to H$ is another mapping onto K with $\rho(0) = 0$, then $\zeta + \rho$ is sharp too.

More generally, it is easy to show that the property of sharpness is independent of the multiplication with an arbitrary scalar. Although we will not cite this result explicitly, we will implicitly assume it.

4. On the facial structure of an isotone retraction cone in $\mathbb{R}^n$

Let $K \subseteq \mathbb{R}^n$ be a pointed closed convex generating cone in $\mathbb{R}^n$. Since K is generating, $\operatorname{int} K \neq \emptyset$, and $K + \operatorname{int} K = \operatorname{int} K$. Each proper face of a generating cone in $\mathbb{R}^n$ is on its boundary $\operatorname{bdr} K$. If $x \in \operatorname{bdr} K$, then always exists an exposed face of K containing it.

The pointed closed convex cone $K \subset \mathbb{R}^n$ is called *polyhedral* if it is the intersection of a finite number of closed half-spaces. Among the defining half-spaces there is a minimal number of defining half-spaces. The intersection with the cone of the hyperplanes defining these half-spaces are exactly its maximal proper faces.

The next lemma exhibits some simple, but fundamental properties of an isotone retraction. These will be used to show that every isotone retraction cone in $\mathbb{R}^n$ is polyhedral.

Lemma 4.1. *Let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ be an isotone retraction onto K. Then,*

1. *For $x \in \mathbb{R}^n \setminus K$ we have $\rho(x) \in \operatorname{bdr} K$ and $\rho(x - K) \subseteq \operatorname{fce} \rho(x)$. If $x \in \operatorname{bdr} K$, then $\rho(x - K) \subseteq \operatorname{fce} x$.*

2. *If K_0 is a face of dimension $n-1$ of K and L is the subspace generated by K_0, if L^- is the closed half-space determined by L which contains K, and $L^+ = -L^-$, then $\rho(L^+) \subseteq K_0$.*

Proof. *1.* Since K is generating, there exist two elements $u, v \in K$ such that $x = u - v$, and hence $x \leq u$. Then, there exists some $t \in [0, 1)$ such that $w = tx + (1-t)u \in \operatorname{bdr} K$. Obviously, $x \leq w$ (since $(1-t)x \leq (1-t)u$). Hence,

$$0 \leq \rho(x) \leq \rho(w) = w.$$

Consequently, $\rho(x) \in \operatorname{fce} w \subseteq \operatorname{bdr} K$.

For every $y \in x - K$ we have $0 \leq \rho(y) \leq \rho(x) \in \operatorname{bdr} K$. Hence,

$$\rho(x - K) \subseteq \operatorname{fce} \rho(x).$$

If $x \in \operatorname{bdr} K$ the conclusion is immediate.

2. Let $x \in L$. Since K_0 generates L, there exists $u \in K_0$ such that $u - x \in K_0$. Hence, $x \leq u$ and then $\rho(x) \leq u$ and since $u \in K_0$, it follows that $\rho(x) \in K_0$.

Let $x \in L^+$. Take $u \in K$ with $x \leq u$. Since L separates L^+ and K, there exists $t \in [0, 1)$ such that $w = tx + (1-t)u \in L$. Then, according to our above proof $\rho(w) \in K_0$. Since $x \leq w$, it follows that $0 \leq \rho(x) \leq \rho(w)$. In conclusion, $\rho(x) \in K_0$. □

The next theorem shows that every isotone retraction cone in $\mathbb{R}^n$ is polyhedral. The construction of isotone retractions on non-latticial polyhedral cones is rather difficult and will be done elsewhere. However, it is not clear yet whether every polyhedral cone is or not an isotone retraction one. This is the main open question which remains to be clarified in the future.

Theorem 4.2. *If K is an isotone retraction cone in $\mathbb{R}^n$, then it is polyhedral.*

Proof. (a) Denote by B_ε the open ball with the center 0 and radius $\varepsilon > 0$, that is, the set

$$B_\varepsilon = \{x \in \mathbb{R}^n : \ \|x\| < \varepsilon\}.$$

We shall show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall\, v \in B_\delta \ \exists x \in K \cap (v + K) \ \text{ with } x \in B_\varepsilon.$$

To this end, take a nonempty open set $U \subseteq \operatorname{int} K \cap B_\varepsilon$, and let $x \in U$. Then, $x - U$ is a neighborhood of 0. Hence, there exists $\delta > 0$ with $B_\delta \subseteq x - U$. If $v \in B_\delta$, then $v \in x - U$ and hence $x \in v + U \subseteq v + K$, and we are done.

(b) From the assertion in (a), it follows that for a sequence (v_n) with $v_n \to 0$, there exists a sequence (x_n) with $x_n \in K \cap (v_n + K)$ and $x_n \to 0$.

(c) Let L be a hyperplane meeting K in 0. Then, for $k \in K \setminus \{0\}$ the set $B = (k + L) \cap K$ will be a bounded closed convex subset of $k + L$, which is a so called *base of* K, which means that each element of K is the positive multiple of an element in $k + L$.

Consider the set B as a subset of $\mathbb{R}^{n-1}$. It is well known that B is the convex hull of its extremal points, and the set of exposed points of B is dense in the set of its extremal points ([12]). (The point $z \in B$ is its exposed point if there exists a hyperplane in $\mathbb{R}^{n-1}$ which meets B in the single point z.)

The exposed points of B have the particularity that they generate exposed rays of K, that is, rays on the boundary of K along which K is "supported" by some hyperplane, in the sense that this hyperplane meets K in that ray. Such a ray is a face of K which can meet any other proper face of it only at 0.

(d) The assertion that K is polyhedral is equivalent with the fact that B possesses a finite set of extremal points in $k + L$.

Assume that K is not polyhedral. Then B has an infinity of extremal points. Let x_0 be an accumulation point of these extremal points. Then, according to the observation in (c), there is a sequence (x_n) of exposed points of B such that $x_n \to x_0$.

From (b), we have a sequence (y_n) with $y_n \in (x_0 - K) \cap (x_n - K)$ and $y_n \to x_0$.

(e) Suppose that $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is an isotone retraction onto K. Then, by Lemma 4.1, $\rho(y_n)$ is on the face of x_0 and on the face of x_n. Since the face of x_n is an exposed ray, we must have by (c) that $\rho(y_n) = 0$ for all n. On the other hand, from $y_n \to x_0$ and the continuity of ρ, it follows that $\rho(y_n) \to \rho(x_0) = x_0 \neq 0$. The obtained contradiction shows that K must be polyhedral. □

5. Isotone retractions and latticiality

As in Section 2, the space H is a (real) separable Hilbert space. The next lemma exhibits some more simple, but useful properties of isotone retractions onto cones. They will be used to prove items *1*, *2* and *3(a)* of Theorem 5.2.

Lemma 5.1. *Let $K \subseteq H$ be a pointed closed convex cone and $\rho : H \to H$ an isotone mapping onto K. We have the following:*

1. *If $\rho(0) = 0$, then $\rho(-K) = \{0\}$ and $-K \subseteq \operatorname{im}(I - \rho)$. In particular, this is the case if ρ is a retraction onto K.*
2. *Suppose that ρ is a retraction onto K.*
   (a) *If there is an $x \in K$ such that $x \leq \rho(x)$, then there exists $x^+ = 0 \vee x$ and $\rho(x) = x^+$.*
   (b) *If $K \subseteq H$ is a latticial cone, then $\rho(x) \leq x^+$, for all $x \in H$.*

Proof. *1.* Let $v \in -K$ arbitrary, or equivalently $v \leq 0$. By the isotonicity of ρ, we have $0 \leq \rho(v) \leq \rho(0) = 0$. Hence, $\rho(v) = 0$, or equivalently $\rho(-K) = \{0\}$. We also have

$$v = \rho(v) + (I - \rho)(v) = (I - \rho)(v) \in \operatorname{im}(I - \rho).$$

Hence, $-K \subseteq \operatorname{im}(I - \rho)$.

2. (a) Suppose that there is an $x \in H$ such that $x \leq \rho(x)$. Then $\rho(x)$ is obviously an upper bound of the set $\{0, x\}$. Let u be an arbitrary upper bound of the set $\{0, x\}$. Then, $0 \leq u$ and $x \leq u$. Since ρ is a retraction onto K, by the isotonicity of ρ, we have $\rho(x) \leq \rho(u) = u$. Hence, $\rho(x) = 0 \vee x = x^+$.

(*b*) Suppose that K is latticial. Since $x \leq x^+$, by the isotonicity of ρ, we have $\rho(x) \leq \rho(x^+) = x^+$, for all $x \in H$. $\square$

Although we couldn't completely clarify the structure of an isotone retraction cone, we can answer the question "when is an isotone retraction cone latticial?" Item *3*(*b*) of the next theorem shows that the existence of an isotone retraction with rather mild property implies its latticiality. For this we have to require an extra condition of sharpness for the complement of the isotone retraction onto the cone.

Theorem 5.2. *Let $K \subseteq H$ be a pointed closed convex cone and $\leq$ the order defined by K. We have the following:*

1. *If K is latticial and $\rho : H \to H$ is defined by $\rho(x) = x^+$, then ρ is a continuous isotone retraction onto K such that $x \leq \rho(x)$, for all $x \in H$, and $I - \rho$ is isotone and sharp.*
2. *If there is a continuous isotone retraction $\rho : H \to H$ onto K such that $x \leq \rho(x)$, for all $x \in H$, then K is latticial and $\rho(x) = x^+$, for all $x \in K$.*
3. *Suppose that K is generating.*
   (*a*) *If there is a continuous retraction $\rho : H \to H$ onto K such that ρ and $I - \rho$ are isotone, then K is latticial and $\rho(x) = x^+$, for all $x \in K$.*
   (*b*) *If K is normal and there is a continuous isotone retraction $\rho : H \to H$ onto K such that the complement $I - \rho$ of ρ is sharp, then K is latticial.*

Proof. *1.* Suppose that K is latticial. Let $\rho : H \to H$ be defined by $\rho(x) = x^+$. Obviously, ρ is a continuous isotone retraction onto K, $x \leq \rho(x)$. Moreover, since

$$(I - \rho)(x) = x - x^+ = -x^- = -(-x)^+,$$

$I - \rho$ is isotone and sharp.

2. Suppose that there is a continuous isotone retraction $\rho : H \to H$ onto K such that $x \leq \rho(x)$, for all $x \in H$. By item *2*(*a*) of Lemma 5.1 it follows that there exists $x^+ = 0 \vee x$ and $\rho(x) = x^+$, for all $x \in H$. By using standard arguments it can be easily shown that $x \vee y = (x - y)^+ + y$, for all $x, y \in H$. Thus, K is latticial.

3. Suppose that K is generating.

(*a*) Suppose that $\rho : H \to H$ is a continuous retraction onto K such that ρ and $I - \rho$ are isotone. By item *2*, it is sufficient to show that $x \leq \rho(x)$, for all $x \in H$. Let $x \in H$ be arbitrary. Since K is generating, it follows that there are $u, v \in K$ such that $x = u - v$. Hence, $x \leq u$. Since $I - \rho$ is isotone and ρ is a retraction onto K, we have $x - \rho(x) \leq u - \rho(u) = 0$. Hence, $x \leq \rho(x)$.

(*b*) Suppose that $\rho : H \to H$ is a continuous isotone retraction onto K. For all $y \in H$ define $\rho_y : H \to H$ by $\rho_y(x) = y + \rho(x - y)$. Since ρ is continuous and isotone, ρ_y is also continuous and isotone. Moreover, $y \leq \rho_y(x)$. Let u and v be arbitrary elements in H. We have to show that there exists $u \vee v$. If u and v are comparable the statement is trivial. Suppose that they are not comparable. First we remark that the set $\{u, v\}$ has an upper bound. Indeed, since K is generating, there exist $u_1, u_2, v_1, v_2 \in K$ such that $u = u_1 - u_2$ and $v = v_1 - v_2$. Hence, $u_1 + v_1$ is an upper bound of the set $\{u, v\}$. Let w be an arbitrary upper bound of the set $\{u, v\}$, that is, an arbitrary element of $(u+K)\cap(v+K)$. The mappings ρ_u and ρ_v are isotone. Moreover, $\rho_u(w)=u+\rho(w-u)=u+(w-u)=w$ and similarly $\rho_v(w)=w$. Consider the operators $\sigma=\rho_u\circ\rho_v$ and $\tau=\rho_v\circ\rho_u$. They are isotone because ρ_u and ρ_v are. Moreover, $\sigma(w)=\tau(w)=w$. Put $v_n=\tau^n(v)$, $u_1=\rho_u(v)$ and $u_n=\sigma^{n-1}(u_1)$. We have $u\le\rho_u(v)=u_1$. Also, $u\le\rho_u(v)$ implies $v\le\rho_v(u)\le\rho_v\circ\rho_u(v)=v_1$ and therefore $u_1=\rho_u(v)\le\rho_u(v_1)$, or equivalently $u_1\le\rho_u\circ\rho_v\circ\rho_u(v)=\sigma(u_1)=u_2$. Bearing in mind that σ, τ are isotone $\sigma(w)=\tau(w)=w$, the relations

$$v\le v_1\le\cdots\le v_n\le\cdots\le w$$

and

$$u\le u_1\le u_2\le\cdots\le u_n\le\cdots\le w$$

can be verified by using mathematical induction. We further have

$$\begin{aligned} v_n=\tau^n(v)=(\rho_v\circ\rho_u)^n(v)&=\rho_v\circ(\rho_u\circ\rho_v)^{n-1}\circ\rho_u(v)\\ &=\rho_v\circ\sigma^{n-1}(u_1)=\rho_v(u_n) \end{aligned}\tag{1}$$

and

$$u_{n+1}=\sigma(u_n)=\rho_u\circ\rho_v(u_n)=\rho_u(v_n).\tag{2}$$

Since H is a Hilbert space and $K\subseteq H$ is normal, K is also regular (see [11]). Since K is regular and closed, the sequences $\{u_n\}$ and $\{v_n\}$ are increasing and bounded above by w, and $u\le u_n\le w$ and $v\le v_n\le w$, there exists the limit

$$u^*=\lim_{n\to\infty}u_n \quad\text{and}\quad v^*=\lim_{n\to\infty}v_n\tag{3}$$

such that

$$u\le u^*\le w \quad\text{and}\quad v\le v^*\le w.\tag{4}$$

From the continuity of the mappings ρ_u and ρ_v and the relations (1), (2) and (3) it follows that $v^*=\rho_v(u^*)$ and $u^*=\rho_u(v^*)$. Hence, we have $v^*-u^*+u^*-v=\rho(u^*-v)$ and $u^*-v^*+v^*-u=\rho(v^*-u)$, or equivalently $(I-\rho)(a)=c$ and $(I-\rho)(b)=-c$, where $a=u^*-v$, $b=v^*-u$ and $c=u^*-v^*$. Thus, we have $c\in\operatorname{im}(I-\rho)\cap\operatorname{im}(\rho-I)$. Since $I-\rho$ is sharp, we have $c=0$, or equivalently $u^*=v^*$. Hence, from relation (4), it follows that $u^*=v^*=u\vee v$. □

Theorem 5.2 also shows that in the case of a continuous isotone retraction ρ onto the cone, the requirement for $I-\rho$ to be isotone completely determines ρ. It is still an open question what would imply the isotonicity of $I-\rho$ only.

Lemma 5.3. *Let $K,L\subseteq H$ be normal generating pointed closed convex cones. If $\rho:H\to H$ is a continuous K-isotone retraction onto K such that* $\operatorname{im}(I-\rho)\subseteq L$, *then K is latticial and $-K\subseteq L$.*

Proof. The inclusion $\operatorname{im}(I-\rho)\subseteq L$ implies that $I-\rho$ is sharp. Hence, from item *3(b)* of Theorem 5.2 we get that K is latticial. By item *1* of Lemma 5.1 we also get $-K\subseteq L$. □

Since the projection is a particular retraction, it is an interesting question whether it is possible to give extensions of Moreau's theorem for retractions or not. The next two theorems provide a partial answer to this question in the case of particular latticial cones only.

Theorem 5.4. *Let $K, L \subseteq H$ be mutually polar normal generating pointed closed convex cones. Then, K is a subdual latticial cone if and only if there are a continuous K-isotone retraction $\rho_K : H \to H$ onto K and a mapping $\rho_L : H \to H$ onto L such that $x = \rho_K(x) + \rho_L(x)$ for all $x \in H$.*

Proof. 1. Suppose that K is a subdual latticial cone. Choose $\rho_K : H \to H$ defined by $\rho_K(x) = x^+$ and $\rho_L : H \to H$ defined by $\rho_L(x) = -x^-$ (with respect to the latticial structure of K). It is easy to see that ρ_K is K-isotone. Since K is subdual, it follows that ρ_L is well defined.

2. Conversely, suppose that there are a continuous K-isotone retraction $\rho_K : H \to H$ onto K and a mapping $\rho_L : H \to H$ onto L such that $x = \rho_K(x) + \rho_L(x)$ for all $x \in H$. Then $\operatorname{im}(I - \rho_K) \subseteq L$. Hence, Lemma 5.3 implies that K is latticial and subdual. □

Theorem 5.5. *Let $K, L \subseteq H$ be mutually polar normal generating pointed closed convex cones. Then, K is an autodual latticial cone if and only if there are a continuous K-isotone retraction $\rho_K : H \to H$ onto K and a continuous L-isotone retraction $\rho_L : H \to H$ onto L such that $x = \rho_K(x) + \rho_L(x)$ for all $x \in H$. In this case $\rho_K(x) = x^+$ (with respect to the latticial structure of K).*

Proof. 1. Suppose that K is an autodual latticial cone. Choose $\rho_K : H \to H$ defined by $\rho_K(x) = x^+$ and $\rho_L : H \to H$ defined by $\rho_L(x) = -x^-$ (with respect to the latticial structure of K). It is easy to see that ρ_K is K-isotone. Since K is autodual, it follows that ρ_L is well defined. Moreover, since $\rho_L(x) = -(-x)^+$ it follows that ρ_L is K-isotone. But, since K is autodual, it follows that $-K = L$. Hence, ρ_L is L-isotone too.

2. Conversely, suppose that there are a continuous K-isotone retraction $\rho_K : H \to H$ onto K and a continuous L-isotone retraction $\rho_L : H \to H$ onto L such that $x = \rho_K(x) + \rho_L(x)$ for all $x \in H$. By Theorem 5.4 we have that both K and L are latticial subdual cones. Hence, K is an autodual latticial cone. Since $I - \rho_K = \rho_L$ is L-isotone and $K = -L$ it follows that $I - \rho_K$ is K-isotone too. Hence, from item *3(a)* of Theorem 5.2 it follows that $\rho_K(x) = x^+$ (with respect to the latticial structure of K). □

6. Miscellaneous examples and counterexamples

We start by stating a well known lemma. We include its proof for the sake of completeness only.

Lemma 6.1. *Let $K \subseteq H$ be a subdual pointed closed convex cone, $\leq$ the order relation defined by K and $u, v \in H$ such that $0 \leq u \leq v$. Then $\|u\| \leq \|v\|$.*

Proof. If $u = 0$ the statement of the lemma is trivial. Let us suppose that $u \neq 0$. Then, $u \in K$ and $v - u \in K \subseteq K^*$ imply $\langle u, v-u \rangle \geq 0$. Thus, by using the Cauchy inequality we get $\|u\|^2 \leq \langle v, u \rangle \leq \|v\|\|u\|$, or equivalently $\|u\| \leq \|v\|$. □

The next example shows that the isotonicity of a retraction ρ onto a cone does not imply that the complement $I - \rho$ of ρ is sharp. According to Theorem 4.2 and item *3(a)* of Theorem 5.2 this gives a chance for the existence of an isotone retraction cone which is polyhedral but not latticial. (Special non-latticial cones with this property will be constructed in a separate note.)

Example 6.2. Let $K \subseteq H$ be a subdual latticial cone. Then, the mapping $\rho : H \to H$ defined by

$$\rho(x) = \begin{cases} \left(1 - \dfrac{\|x^-\|}{\|x^+\|}\right) x^+ & \text{if} \|x^+\| > \|x^-\|, \\ 0 & \text{if} \|x^+\| \leq \|x^-\|. \end{cases}$$

is a continuous isotone retraction onto K, but in general $I - \rho$ is not sharp.

Proof. Obviously $\rho(x) \in K$, for all $x \in H$. Hence, ρ is well defined. If $x \in K \setminus \{0\}$, then $x^+ = x$ and $x^- = 0$. Hence, $\|x^+\| > \|x^-\|$. By the definition of ρ we have $\rho(x) = x$. If $x = 0$, then $x^+ = x^- = 0$. Hence, $\|x^+\| \leq \|x^-\|$. By the definition of ρ we have $\rho(x) = 0 = x$. Thus, $\rho(x) = x$, for all $x \in K$. It follows that ρ is a retraction onto K. Obviously, ρ is continuous. Next, we prove that ρ is isotone. Suppose that $x, y \in H$ such that $x \leq y$. We have to show that $\rho(x) \leq \rho(y)$. Since

$$0 \leq x^+ \leq y^+ \tag{5}$$

and $0 \leq y^- \leq x^-$, we have

$$\|x^+\| \leq \|y^+\| \tag{6}$$

and

$$\|y^-\| \leq \|x^-\|. \tag{7}$$

We consider three different cases:

1. $\|y^+\| > \|y^-\|$ and $\|x^+\| > \|x^-\|$. Then, by the definition of ρ, (5), (6) and (7) we have $\rho(x) \leq \rho(y)$.
2. $\|y^+\| > \|y^-\|$ and $\|x^+\| \leq \|x^-\|$. Then, by the definition of ρ we have $\rho(x) = 0 \leq \rho(y)$.
3. $\|y^+\| \leq \|y^-\|$. Then, by (6) and (7) we have $\|x^+\| \leq \|y^+\| \leq \|y^-\| \leq \|x^-\|$. Hence, by the definition of ρ we have $\rho(x) = 0 \leq 0 = \rho(y)$.

In general $I - \rho$ is not sharp. Indeed, if $K = \mathbb{R}^2_+$, $x = (1, -1) \notin K$, $y = (-1, 1) \notin K$, then by the definition of ρ we have $x + y = 0 = 0 + 0 = \rho(x) + \rho(y)$. Hence, $0 \neq (I - \rho)(x) = -(I - \rho)(y)$, which shows that $\operatorname{im}(I - \rho) \cap \operatorname{im}(-(I - \rho)) \neq \{0\}$. Therefore, $I - \rho$ is not sharp. □

The next example shows the surprising fact that there are isotone retractions ρ onto cones whose complement $I - \rho$ is not sharp, but for an arbitrarily small "homotopic deformation" of ρ we get an isotone retraction whose complement is sharp.

Example 6.3. Consider the previous example with $H = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $t \in [0, 1[$ and $\rho_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\rho_t(x) = t\rho(x) + (1-t)x^+$. Then, ρ_t is a continuous isotone retraction onto K such that $I - \rho_t$ is sharp.

Proof. If $t = 0$ the assertion is obvious. Therefore, we can suppose that $t \in]0, 1[$. By the previous example ρ is a continuous isotone retraction onto K. The mapping $x \mapsto x^+$ bears the same properties. Hence, ρ_t bears the same properties too. It remains to prove that $I - \rho$ is sharp. First, note that

$$x^+ - \rho(x) = \frac{\|x^-\|}{\|x^+\|}x^+$$

implies that

$$\langle x^+ - \rho(x), x^- \rangle = 0 \tag{8}$$

and

$$\|x^+ - \rho(x)\| = \|x^-\| \tag{9}$$

By using (8) and (9) we get

$$\begin{aligned}
&\langle x - \rho_t(x), x^+ - \rho_t(x) \rangle \\
&= \langle tx + (1-t)x - t\rho(x) - (1-t)x^+, tx^+ + (1-t)x^+ - t\rho(x) - (1-t)x^+ \rangle \\
&= \langle t(x - \rho(x)) - (1-t)x^-, t(x^+ - \rho(x)) \rangle = t^2 \langle x - \rho(x), x^+ - \rho(x) \rangle \\
&= t^2 \langle x^+ - \rho(x) - x^-, x^+ - \rho(x) \rangle = t^2 \|x^+ - \rho(x)\|^2 = t^2 \|x^-\|^2.
\end{aligned}$$

By tidying up we have

$$\langle x - \rho_t(x), x^+ - \rho_t(x) \rangle = t^2 \|x^-\|^2. \tag{10}$$

By reductio ad absurdum suppose that $I - \rho_t$ is not sharp. Then, there exists $x, y \notin K$ such that

$$x + y = \rho_t(x) + \rho_t(y) \tag{11}$$

We can suppose that $x, y \notin -K$. Indeed suppose that $x \in -K$ (the argument is similar if we suppose that $y \in -K$). Then, by item *1* of Lemma 5.1 we have $\rho_t(x) = 0$. Thus, (11) implies that $y \in K$, which is a contradiction. Hence $x^+, y^+ \neq 0$. Since ρ_t is an isotone retraction onto K, item *2(b)* of Lemma 5.1 implies $0 \leq \rho_t(x) \leq x^+$. Hence, $0 \leq x^+ - \rho_t(x) \leq x^+$. Then, $\rho_t(x), x^+ - \rho_t(x) \in \operatorname{fce} x^+$. Similarly $\rho_t(y), y^+ - \rho_t(y) \in \operatorname{fce} y^+$. We consider two cases.

1. $\operatorname{fce} x^+ = \operatorname{fce} y^+$. Then, $x^+ - \rho_t(x) = r(y^+ - \rho_t(y))$ for some $r > 0$. By using the latter relation and (10), it follows that

$$\langle x - \rho_t(x), y^+ - \rho_t(y) \rangle > 0 \tag{12}$$

and

$$\langle y - \rho_t(y), y^+ - \rho_t(y) \rangle > 0 \tag{13}$$

By summing up (12) and (13) we get

$$0 = \langle x + y - (\rho_t(x) + \rho_t(y)), y^+ - \rho_t(y) > 0$$

which is a contradiction.

2. $\operatorname{fce} x^+ \perp \operatorname{fce} y^+$. Then, $y^+ - y = r(x^+ - \rho_t(x))$ and $x^+ - x = s(y^+ - \rho_t(y))$, for some $r, s > 0$. From (11) and the last two relations we get

$$\rho_t(x) + \rho_t(y) = x + y = (1-r)x^+ + (1-s)y^+ + r\rho_t(x) + s\rho_t(y),$$

or equivalently $(1-r)(x^+ - \rho_t(x)) = (1-s)(\rho_t(y) - y^+)$. The left hand side of the latter equality is in $K \setminus \{0\}$ and the right hand is in $-K \setminus \{0\}$. But, this contradicts the pointedness of K.

The contradictions obtained in both of the cases show that $I - \rho_t$ is sharp. □

Next, we present an even simpler example than Example 6.2 for an isotone retraction ρ onto a cone whose complement $I - \rho$ in general is not sharp.

Example 6.4. Let $K \subseteq H$ be a subdual latticial cone. Then, the mapping $\rho : H \to H$ defined by

$$\rho(x) = \frac{x^+}{1 + \|x^-\|}$$

is a continuous isotone retraction onto K, but in general $I - \rho$ is not sharp.

Proof. Obviously, ρ is a continuous retraction onto K. From (5), (6) and (7) it follows that ρ is isotone. In general $I - \rho$ is not sharp. Indeed, if $K = \mathbb{R}^2_+$, $x = (3, -2) \notin K$ and $y = (-2, 3) \notin K$, then by the definition of ρ we have

$$\rho(x) + \rho(y) = \frac{(3,0)}{1+2} + \frac{(0,3)}{1+2} = (1,1) = x + y.$$

Similarly to the proof of the previous example, we obtain that $I - \rho$ is not sharp. □

Lemma 6.5. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing mapping such that $h(0) = 0$, H a Hilbert space and $K \subseteq H$ a subdual generating pointed closed convex cone. If $\psi : H \to H$ is an isotone mapping such that $\psi(x) = 0$, for all $x \in K$, then the function $r : H \to \mathbb{R}$ defined by*

$$r(x) = \frac{1}{1 + h(\|\psi(x)\|)}$$

is isotone.

Proof. Let $x, y \in H$ such that $x \leq y$. Since K is generating there is an $u \in K$ such that $x \leq u$. Then, by the isotonicity of ψ we get $0 = -\psi(u) \leq -\psi(y) \leq -\psi(x)$. Hence, by Lemma 6.1 we have $\|\psi(y)\| \leq \|\psi(x)\|$. Therefore, the monotonicity of h implies $r(x) \leq r(y)$. □

The next proposition is an immediate consequence of Lemma 6.5 and extends the result of Example 6.4. It shows that if there is an isotone retraction onto a cone, then there are quite many. Therefore, although it is difficult to construct an isotone retraction, once an isotone retraction is found, it is much easier to generate a whole family of such mappings.

Proposition 6.6. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing mapping such that $h(0) = 0$, H a Hilbert space, $K \subseteq H$ a subdual generating pointed closed convex cone, $\rho : H \to H$ an isotone retraction onto K and $\psi : H \to H$ an isotone mapping such that $\psi(x) = 0$, for all $x \in K$. Then, the mapping $\zeta : H \to H$ defined by*

$$\zeta(x) = \frac{\rho(x)}{1 + h(\|\psi(x)\|)}$$

is an isotone retraction onto K.

We remark that the mapping defined in Example 6.4 is a particular case of the mapping defined in the previous proposition with $h(t) = t$, $\rho(x) = x^+$ and $\psi(x) = -x^-$.

The next theorem shows a fundamental property of isotone retraction, namely, the invariance of an isotone retraction ζ under a coordinate transformation. If the complement $I - \zeta$ of ζ is sharp the complement of the transformed mapping is also sharp. In this way we can generate many examples for new isotone retractions and new mappings whose complements are sharp.

Theorem 6.7. *Let A be an $n \times n$ nonsingular real matrix. Consider a latticial cone K in $\mathbb{R}^n$, the latticial cone L defined by*

$$L = \{x \in \mathbb{R}^n : Ax \in K\},$$

a continuous K-isotone retraction $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ onto K and the mapping $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(x) = A^{-1}\zeta(Ax).$$

Then, ρ is a continuous L-isotone retraction onto L. If $I - \zeta$ is sharp, then $I - \rho$ is sharp too.

Proof. Let $x \in \mathbb{R}^n$ be arbitrary. We have $A\rho(x) = \zeta(Ax) \in K$, which means $\rho(x) \in L$. Hence, ρ is well defined. Let $u \in L$ be arbitrary. Then, $Au \in K$. Thus, $\zeta(Au) = Au$ and therefore $\rho(u) = u$. It follows that ρ is a continuous retraction onto L. Now, let $x, y \in \mathbb{R}^n$ such that $y - x \in L$. This means $A(y - x) \in K$, or equivalently $Ay - Ax \in K$. Hence, by the K-isotonicity of ζ we have $\zeta(Ay) - \zeta(Ax) \in K$, or equivalently $A(\rho(y) - \rho(x)) \in K$. Hence, $\rho(y) - \rho(x) \in L$. Therefore, ρ is L-isotone. Suppose that $I - \zeta$ is sharp and let $x, y \in H$ such that $\rho(x) + \rho(y) = x + y$. Then, $\zeta(Ax) + \zeta(Ay) = Ax + Ay$. Since, $I - \zeta$ is sharp it follows that $Ax, Ay \in K$, or equivalently $x, y \in L$. Hence, $I - \rho$ is sharp. □

7. Conclusions

In this paper we have considered the problem of the existence and construction of continuous isotone retractions onto pointed closed convex cones in Hilbert spaces. In the case of adding an extra sharpness condition to the isotone retraction, these mappings characterize the latticial cones. We presented several other characterizations of latticial cones and particular latticial cones. However, it is still an open question to characterize cones which admit continuous isotone retractions onto them and not satisfying the extra condition. In finite dimension we could show that the generating pointed closed convex cones which admit a continuous isotone retraction onto them must be polyhedral. The construction of non-latticial isotone retraction cones in $\mathbb{R}^n$ is a rather difficult problem and will be done by us in a next note. But we cannot answer yet the question if a general polyhedral cone in $\mathbb{R}^n$ is or not an isotone retraction one. This is the main question which should be answered in the future. Although construction of isotone retractions onto latticial cones is not easy neither, we could give several examples for such mappings.

The results presented in this paper provide a connection between the order structure and the topological-geometrical structure of a space. The remaining challenging issues will hopefully provide further insight in the order-topological-geometrical structure of a space. Apart from its theoretical importance, we plan to use the results to extend the existence and iterative methods of [4, 5, 6, 7] to more general equilibrium problems or/and more general discrete dynamical systems where the projection is replaced by a continuous isotone retraction.

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