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On the Infimum of a Quasiconvex Function over an Intersection. Application to the Distance Function

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We give sufficient conditions for the infimum of a quasiconvex function f over the intersection $\bigcap_{i\in I} R_i$ to agree with the supremum of the infima of f over the R_i's. We apply these results to the distance function in a normed space.

Keywords: Quasiconvex functions, distance to the intersection

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1. Introduction

In this paper we are interested in the following problem. Let S be a set, $(R_i)_{i\in I}$ be a family of subsets of S, $C = \bigcup_{i\in I} R_i$ its union and $R = \bigcap_{i\in I} R_i$ its intersection. Given a map $f : C \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$, is the equality

$$\inf_{x\in R} f(x) = \sup_{i\in I} \inf_{x\in R_i} f(x) \tag{1}$$

true? Of course, the inequality $\geq$ is always satisfied.

We consider several settings. Our main results about equality (1) are obtained in the case when $S = V$ is a linear space, C is a convex subset of V, f is a quasiconvex function and the family $(R_i)_{i\in I}$ satisfies certain conditions, basically $R_i \cap R_j = R$ $(i \neq j)$.

If $S = X$ is a normed space, $C \subset X$ is a convex subset and $x \in X$ is a given point, the distance function $d(x, \cdot) : C \longrightarrow \mathbb{R}$ is convex, hence quasiconvex, and we can apply

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the above results. In this setting the problem can be formulated in the following way: is the equality

$$d\,(x,R) = \sup_{i\in I} d\,(x,R_i) \tag{2}$$

true? Several authors have studied the validity of equality (2). A. Hoffmann [4] obtains an upper estimate of the distance from a point to the intersection of two convex sets. The papers [6, 8] consider the case of a family $(R_i)_{i\in I}$ consisting of two subsets; in [6] it is proved that, in the setting of metric spaces, (2) is true if and only if C is x-boundedly connected.

On the other hand, J.-E. Martínez-Legaz, A. M. Rubinov and I. Singer [7, 9, 10, 11] have proved (2) for families of normal and downward subsets of the finite dimensional space ℓ_∞^d. In this paper, we obtain (2) for any family of downward subsets of ℓ_∞.

G. Chacón, V. Montesinos and A. Octavio [2] consider a nested sequence of subspaces of a Banach space and prove that the space is reflexive if (2) is true for every such sequence. J. M. F. Castillo and P. L. Papini [1] have also studied the validity of equality (2) for a nested sequence of subsets of a Banach space. In this paper we prove a result about nested sequences of subsets in complete metric spaces.

Besides the general results about quasiconvex functions and the distance function, we consider the case $S=\overline{\mathbb{R}}$ and $f=I:C\longrightarrow\overline{\mathbb{R}}$, the identity map; that is, we study when the infimum of the intersection of a family of subsets of $\overline{\mathbb{R}}$ coincides with the supremum of the infimum of the sets of the family.

2. The infimum of a family of subsets of $\overline{\mathbb{R}}$

In this section the situation we study is $S=\overline{\mathbb{R}}$ and $f=I:C\longrightarrow\overline{\mathbb{R}}$, the identity function. Recall that $\inf\emptyset=\infty$ and $\inf A=\inf\overline{A}$, where $\overline{A}$ is the closure of $A\subset\overline{\mathbb{R}}$.

We consider on $\overline{\mathbb{R}}$ the usual distance $d(x,y)=|\arctan x-\arctan y|$. For $\emptyset\neq E\subset\overline{\mathbb{R}}$, we have

$$d(-\infty,E)=\inf_{x\in E}\left|\arctan x+\frac{\pi}{2}\right|=\arctan\inf E+\frac{\pi}{2},$$

hence

$$\inf E=\tan\left(d(-\infty,E)-\frac{\pi}{2}\right);$$

note that this formula is also valid if $E=\emptyset$ and we put $d(-\infty,\emptyset)=\pi$, the diameter of $\overline{\mathbb{R}}$.

The next result relates the validity of the equalities (1) for $f=I$ and (2) for $x=-\infty$.

Proposition 2.1. *Let $(E_i)_{i\in I}$ be a family of subsets of $\overline{\mathbb{R}}$. Consider the following statements:*

(*1*) $\sup_{i\in I}\inf E_i\in\overline{\bigcap_{i\in I}E_i}$
(*2*) $\inf\bigcap_{i\in I}E_i=\sup_{i\in I}\inf E_i$
(*3*) $d(-\infty,\bigcap_{i\in I}E_i)=\sup_{i\in I}d(-\infty,E_i)$
(*4*) $\lambda\in\mathbb{R}$ *and* $(\forall i\in I,[-\infty,\lambda]\cap E_i\neq\emptyset)\Longrightarrow[-\infty,\lambda]\cap\overline{\bigcap_{i\in I}E_i}\neq\emptyset$

Then

$$(1) \Longrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4).$$

Moreover, if $\bigcap_{i\in I} E_i \neq \emptyset$*, then*

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4).$$

Proof. (1) $\Longrightarrow$ (2). The inequality $\inf \bigcap_{i\in I} E_i \geq \sup_{i\in I} \inf E_i$ is clear. Moreover, if $\inf \bigcap_{i\in I} E_i > \sup_{i\in I} \inf E_i$, then there exists $\lambda \in \mathbb{R}$ such that

$$\inf \bigcap_{i\in I} E_i > \lambda > \sup_{i\in I} \inf E_i,$$

hence $[-\infty, \lambda[$ is an open set with $\sup_{i\in I} \inf E_i \in [-\infty, \lambda[$, but $\sup_{i\in I} \inf E_i \notin \overline{\bigcap_{i\in I} E_i}$, since $\inf \bigcap_{i\in I} E_i \notin [-\infty, \lambda]$ and $\inf \bigcap_{i\in I} E_i = \inf \overline{\bigcap_{i\in I} E_i}$.

(2) $\Longleftrightarrow$ (3) We have that

$$\begin{aligned}\inf \bigcap_{i\in I} E_i &= \tan\left(d(-\infty, \bigcap_{i\in I} E_i) - \frac{\pi}{2}\right) \geq \tan\left(\sup_{i\in I} d(-\infty, E_i) - \frac{\pi}{2}\right) \\ &= \tan \sup_{i\in I}\left(d(-\infty, E_i) - \frac{\pi}{2}\right) = \sup_{i\in I} \tan\left(d(-\infty, E_i) - \frac{\pi}{2}\right) = \sup_{i\in I} \inf E_i\,.\end{aligned}$$

Hence the equality

$$\inf \bigcap_{i\in I} E_i = \sup_{i\in I} \inf E_i$$

holds if and only if the equality

$$d(-\infty, \bigcap_{i\in I} E_i) = \sup_{i\in I} d(-\infty, E_i)$$

holds.

(2) $\Longrightarrow$ (4). Let $\lambda \in \mathbb{R}$ be such that, for every $i \in I$, $[-\infty, \lambda] \cap E_i \neq \emptyset$, hence $\sup_{i\in I} \inf E_i \leq \lambda$. By (2), given $n \in \mathbb{N}$, there exists $z_n \in \bigcap_{i\in I} E_i$ with $z_n < \lambda + 1/n$ and $\liminf_{n\to\infty} z_n \in \overline{\bigcap_{i\in I} E_i}$. Therefore $[-\infty, \lambda] \cap \overline{\bigcap_{i\in I} E_i} \neq \emptyset$.

(4) $\Longrightarrow$ (2). If $\sup_{i\in I} \inf E_i < \inf \bigcap_{i\in I} E_i = \inf \overline{\bigcap_{i\in I} E_i}$, then there exists $\lambda \in \mathbb{R}$ such that

$$\sup_{i\in I} \inf E_i < \lambda < \inf \overline{\bigcap_{i\in I} E_i}\,,$$

hence $[-\infty, \lambda] \cap E_i \neq \emptyset$, for every $i \in I$, but $[-\infty, \lambda] \cap \overline{(\bigcap_{i\in I} E_i)} = \emptyset$.

(2) $\Longrightarrow$ (1). If $\bigcap_{i\in I} E_i \neq \emptyset$, then

$$\sup_{i\in I} \inf E_i = \inf \bigcap_{i\in I} E_i \in \overline{\bigcap_{i\in I} E_i}\,.$$

$\square$

Now let S be a nonempty set. Let $(R_i)_{i\in I}$ be a family of subsets of S. We write

$$R = \bigcap_{i\in I} R_i\,, \qquad C = \bigcup_{i\in I} R_i\,.$$

Let $f : S \to \overline{\mathbb{R}}$ be a map. For $A \subset S$ we consider $\inf_{x\in A} f(x) = \inf f(A) \in \overline{\mathbb{R}}$.

We are interested in conditions for equality (1) to hold. We denote $E = f(R)$ and $E_i = f(R_i)$ $(i \in I)$, hence $E \subset \bigcap_{i\in I} E_i$, and consequently, for every $i \in I$,

$$\inf_{x\in R} f(x) = \inf f(R) = \inf E \geq \inf \bigcap_{i\in I} E_i \geq \inf E_i = \inf_{x\in R_i} f(x)\,.$$

Then equality (1) implies

$$\inf E = \inf \bigcap_{i\in I} E_i = \sup_{i\in I} \inf E_i\,. \tag{3}$$

Remark 2.2. It is possible that the three numbers in equation (3) be different. Let $S = \mathbb{R}$, $f(x) = x^2$, $R_1 = [1, \sqrt{2}] \cup [2\sqrt{2}, 3]$, $R_2 = [-4, -2] \cup \{3\}$. Then we obtain $E_1 = f(R_1) = [1, 2] \cup [8, 9]$, $E_2 = f(R_2) = [4, 16]$, $E_1 \cap E_2 = [8, 9]$ and $E = \{9\}$, hence $\inf E = 9$, $\inf(E_1 \cap E_2) = 8$ and $\max\{\inf E_1, \inf E_2\} = 4$.

Remark 2.3. Note that the equality

$$\inf \bigcap_{i\in I} E_i = \sup_{i\in I} \inf E_i \tag{4}$$

does not imply equality (3), as seen in the following example. If we take $S = \mathbb{R}$, $f(x) = x^2$, $R_1 = [1, 3]$, $R_2 = [-4, -2] \cup \{3\}$, then we obtain $E_1 = f(R_1) = [1, 9]$, $E_2 = f(R_2) = [4, 16]$, $E_1 \cap E_2 = [4, 9]$ and $\inf(E_1 \cap E_2) = 4 = \max\{\inf E_1, \inf E_2\}$, but $\inf_{x\in R_1\cap R_2} f(x) = 9 > 4 = \max\{\inf_{x\in R_1} f(x), \inf_{x\in R_2} f(x)\}$. In the case of f injective, then $\inf f(R) = \inf(\bigcap_{i\in I} f(R_i))$ and the equalities (4) and (3) are equivalent.

3. The infimum of a quasiconvex function

In this section we prove several results about equality (1), f being a quasiconvex function defined on a convex subset $C = \bigcup_{i\in I} R_i$ of a vector space V and $R = \bigcap_{i\in I} R_i$.

We basically consider two hypotheses: first, $[r_i, r_j] \cap R_i \cap R_j \neq \emptyset$, for every $r_i \in R_i$, $r_j \in R_j$ $(i \neq j)$; second, the sets R_i are linearly closed in C. In several situations the second hypothesis implies the first one.

Proposition 3.1. *Let V be a vector space, $C \subset V$ a convex set and $f : C \longrightarrow \overline{\mathbb{R}}$ a quasiconvex function. Let $(R_i)_{i\in I}$ be a family of subsets of C satisfying $[r_i, r_j] \cap R_i \cap R_j \neq \emptyset$, for every $r_i \in R_i$ and $r_j \in R_j$ $(i, j \in I$ with $i \neq j)$, and $C = \bigcup_{i\in I} R_i$. Let $R = \bigcap_{i\in I} R_i$ and assume that $R_i \cap R_j = R$, for $i, j \in I$ with $i \neq j$. Then*

$$\exists h \in I, \forall k \in I, k \neq h, \quad \inf_{x\in R} f(x) = \inf_{x\in R_k} f(x), \tag{5}$$

hence equality (1) *holds.*

Proof. Notice that, for every $i \in I$, we have

$$\inf_{x\in C} f(x) \le \inf_{x\in R_i} f(x) \le \sup_{i\in I} \inf_{x\in R_i} f(x) \le \inf_{x\in R} f(x).$$

Hence, if $\inf_{x\in C} f(x) = \inf_{x\in R} f(x)$, then (5) is clear. In the following we assume that $\inf_{x\in C} f(x) < \inf_{x\in R} f(x)$, hence, as $\inf_{x\in C} f(x) = \inf_{i\in I} \inf_{x\in R_i} f(x)$, there exists $h \in I$ such that

$$\inf_{x\in C} f(x) \le \inf_{x\in R_h} f(x) < \inf_{x\in R} f(x).$$

For each $i \in I$, let $\lambda_i > \inf_{x\in R_i} f(x)$ and choose $r_i \in R_i$ such that

$$f(r_i) < \lambda_i\ .$$

For every $k \in I$ with $h \neq k$, we choose $r \in [r_h, r_k] \cap R$ and we obtain, taking into account that f is quasiconvex,

$$\inf_{x\in R} f(x) \le f(r) \le \max\{f(r_h), f(r_k)\} < \max\{\lambda_h, \lambda_k\}\,.$$

Because of the arbitrariness of the λ_i's we have, for every $k \in I$ with $k \neq h$,

$$\inf_{x\in R_h} f(x) < \inf_{x\in R} f(x) \le \max\left\{\inf_{x\in R_h} f(x),\ \inf_{x\in R_k} f(x)\right\};$$

hence, for every $k \in I$ with $k \neq h$, we obtain

$$\max\left\{\inf_{x\in R_h} f(x),\ \inf_{x\in R_k} f(x)\right\} = \inf_{x\in R_k} f(x),$$

so

$$\inf_{x\in R_h} f(x) < \inf_{x\in R} f(x) \le \inf_{x\in R_k} f(x)$$

and, consequently,

$$\inf_{x\in R} f(x) = \inf_{x\in R_k} f(x).$$

From this we obtain the announced result. □

Now we recall a definition:

Definition 3.2. Let V be a vector space and $C \subset V$. The subset $A \subset C$ is said *linearly closed in* C if $A \cap C \cap L$ is closed in L, for every straight line L of V endowed with the natural topology.

In the next result the family $(R_i)_{i\in I}$ is at most countable and the R_i's are linearly closed subsets.

Proposition 3.3. *Let V be a vector space, $C \subset V$ a convex set and $f : C \longrightarrow \overline{\mathbb{R}}$ a quasiconvex function. Let $(R_i)_{i\in\mathbb{N}}$ be a sequence of subsets of C which are linearly closed in C and such that $C = \bigcup_{i\in\mathbb{N}} R_i$. Let $R = \bigcap_{i\in\mathbb{N}} R_i$ and assume that $R_i \cap R_j = R$, for $i, j \in \mathbb{N}$ with $i \neq j$. Then condition* (5) *is satisfied, hence equality* (1) *holds.*

Proof. In view of Prop. 3.1, we only have to prove that if $r_h \in R_h$ and $r_k \in R_k$ with $h, k \in \mathbb{N}$ and $h \neq k$, then the segment $[r_h, r_k]$ satisfies $[r_h, r_k] \cap R \neq \emptyset$. For $i \in \mathbb{N}$ we denote $S_i = R_i \cap [r_h, r_k]$. As $[r_h, r_k] = \bigcup_{i\in\mathbb{N}} S_i$ is a continuum (compact, connected and Hausdorff topological space) and no continuum can be written as the union of countably many disjoint closed sets [12, Problem 28E.2], we have $S_i \cap S_j \neq \emptyset$ for some $i, j \in \mathbb{N}$ with $i \neq j$, hence we obtain $[r_h, r_k] \cap R = S_i \cap S_j \neq \emptyset$. □

Remark 3.4. The above result is not true for an arbitrary infinite family of subsets (R_i). Consider $X = \mathbb{R}$, $C = [-1, 1]$, $R = \{1\}$, $R_i = \{i-1, i, 1\}$ $(0 \leq i < 1)$ and $f(x) = (1 + x)^2$. Then we have $\inf_{x\in R} f(x) = 4$ and $\sup_{0\leq i<1} \inf_{x\in R_i} f(x) = \sup_{0\leq i<1} i^2 = 1$.

In the following proposition the family $(R_i)_{i\in I}$ is finite, but it is only required that $R_h \cap R_j = R$ with $\inf_{x\in R_h} f(x) \leq \inf_{x\in R_j} f(x)$ $(j \neq h)$. Note that the condition $R \neq \emptyset$ is obtained as conclusion.

Proposition 3.5. *Let V be a vector space, $C \subset V$ a convex set and $f : C \longrightarrow \overline{\mathbb{R}}$ a quasiconvex function. Let $R_1, R_2, ..., R_n \subset C$ be nonempty and linearly closed in C and such that $C = \bigcup_{1\leq i\leq n} R_i$, and let $R = \bigcap_{1\leq i\leq n} R_i$. Assume that, for certain $h = 1, ..., n$ and for every $j = 1, ..., n$ with $j \neq h$, $R_h \cap R_j = R$ and*

$$\inf_{x\in R_h} f(x) \leq \inf_{x\in R_j} f(x). \tag{6}$$

Then $R \neq \emptyset$ and

$$\forall j = 1, ..., n; j \neq h, \quad \inf_{x\in R} f(x) = \inf_{x\in R_j} f(x),$$

hence equality (1) *holds.*

Proof. Let $r \in R_h$, $s \in R_j$ $(1 \leq j \leq n,\ j \neq h)$ and, assuming that $r \neq s$, let L be the straight line that contains the segment $[r, s] \subset C$. Define

$$\overline{\lambda} = \sup\{\lambda \in [0, 1] : (1 - \lambda)r + \lambda s \in R_h\}$$

and $t = (1 - \overline{\lambda})r + \overline{\lambda}s$. We have $t \in R_h$. If $\overline{\lambda} = 1$, then $t = s \in [r, s] \cap R_h \cap R_j$. Let $S = (\bigcup_{k\neq h} R_k) \cap [r, s]$. If $\overline{\lambda} < 1$, for every $p = 1, 2, 3...$ there exists $y_p \in S$ such that $y_p = (1 - \lambda_p)r + \lambda_p s$, being $\overline{\lambda} < \lambda_p \leq \overline{\lambda} + 1/p$. As S is the finite union of the sets $R_k \cap [r, s]$, there is a subsequence $(y_{p_m})_{m\geq 1}$ of $(y_p)_{p\geq 1}$ contained in some $R_k \cap [r, s]$. Since $R_k \cap [r, s]$ is closed in $C \cap L$, it is clear that t is the limit of a sequence of points of $R_k \cap L$ and, consequently, $t \in R_k$; hence $t \in R_h \cap R_k = R = R_h \cap R_j$. We have thus proved that $[r, s] \cap R_h \cap R_j \neq \emptyset$, for every $r \in R_h$ and $s \in R_j$. Therefore, by (6), the inclusion $R \subset R_j$, the equality $R = R_h \cap R_j$ and Proposition 3.1, for every $j = 1, ..., n$, $j \neq h$,

$$\inf_{x\in R_j} f(x) \leq \inf_{x\in R} f(x) = \inf_{x\in R_h\cap R_j} f(x) = \max\left\{\inf_{x\in R_h} f(x), \inf_{x\in R_j} f(x)\right\} = \inf_{x\in R_j} f(x).$$

From this we obtain the result. □

Remark 3.6. The above proposition is not true for a countable family of subsets. In fact, let $V = \mathbb{R}$, $C = [0,1]$, $R_0 = \{0,1\}$, $R_n = \left[\frac{1}{n+1}, \frac{1}{n}\right] \cup \{1\}$ $(n \in \mathbb{N})$ and $f(x) = x$; then $R = \{1\} = R_0 \cap R_n$ $(n \in \mathbb{N})$, $\inf_{x\in R} f(x) = 1$, $\inf_{x\in R_0} f(x) = 0$, $\inf_{x\in R_n} f(x) = \frac{1}{n+1}$ $(n \in \mathbb{N})$ and $\sup_{i\geq 1} \inf_{x\in R_i} f(x) = \frac{1}{2}$; hence $\inf_{x\in R} f(x) > \sup_{i\geq 1} \inf_{x\in R_i} f(x)$.

Remark 3.7. The proof of the preceding proposition is still valid if the assumption $C = \bigcup_{1\leq i\leq n} R_i$ is replaced by the weaker condition that $\bigcup_{1\leq i\leq n} R_i$ contains all segments one of whose endpoints is in R_h and the other one is in R_j. This condition holds, in particular, when $\bigcup_{1\leq i\leq n} R_i$ contains the convex hull of $R_h \cup R_j$ for every j.

In the above results we have considered a family $(R_i)_{i\in I}$ of sets linearly closed with intersection R and $R_i \cap R_j = R$ $(i \neq j)$; in Proposition 3.3 we have considered $I = \mathbb{N}$, and in Proposition 3.5 the set I is finite. The next proposition analyzes this type of condition in a topological environment.

Proposition 3.8. *Let C be a locally connected space, $R \subset C$ a closed set and $(A_i)_{i\in I}$ be a family of nonempty connected subsets of C which are pairwise disjoint and such that $R \cap A_i = \emptyset$ for every $i \in I$. Then*

(*1*) *If the components C_i of $C \setminus R$ that contain A_i, respectively, are all different, then there exist a family $(R_i)_{i\in I}$ of closed subsets of C such that $C = \bigcup_{i\in I} R_i$, $R = \bigcap_{i\in I} R_i$, $A_i \subset R_i$ and $R_i \cap R_j = R$ $(i \neq j)$.*

(*2*) *If I is finite and there exists a family $(R_i)_{i\in I}$ of closed subsets of C such that $C = \bigcup_{i\in I} R_i$, $R = \bigcap_{i\in I} R_i$, $A_i \subset R_i$ for every $i \in I$ and $R_i \cap R_j = R$ $(i \neq j)$, then the components C_i of $C \setminus R$ that contain A_i, respectively, are all different.*

Proof. (*1*) Let R be a closed subset of C such that $\bigcup_{i\in I} A_i \subset C \setminus R$ and the components C_i of $C \setminus R$ that contain A_i, respectively, are all different: if $i \neq j$, then $C_i \neq C_j$, hence $C_i \cap C_j = \emptyset$. Let C_0 be the union of all the components of $C \setminus R$ which are different from C_i $(i \in I)$. We take $R_j = R \cup C_j \cup C_0$, for some fix $j \in I$, and, for $i \neq j$, $R_i = R \cup C_i$. As C is locally connected and $C \setminus R$ is open, we have that each component of $C \setminus R$ is open and closed in $C \setminus R$, and open in C. The sets $C \setminus R_i$ $(i \in I)$ are union of components of $C \setminus R$, hence union of subsets of $C \setminus R$ which are open in $C \setminus R$ and in C, so the sets R_i are closed in C and it is clear that they satisfy the conditions of the statement.

(*2*) Suppose that R is the intersection of closed subsets R_i such that $C = \cup_{i\in I} R_i$, the sets $R_i \setminus R$ are pairwise disjoint and $A_i \subset R_i \setminus R_j$ for each i and every $j \neq i$. For each i one has $A_i \subset R_i \setminus R$ and $A_i \cap (R_j \setminus R) = \emptyset$ for every $j \neq i$. As $\bigcup_{j\neq i}(R_j \setminus R)$ is closed in $C \setminus R$ and is disjoint with $R_i \setminus R$, which is also closed in $C \setminus R$, and $\bigcup_{j\in I} (R_j \setminus R) = C \setminus R$, we obtain that the component C_i is contained in $R_i \setminus R$, hence, for $i \neq j$, we have that $C_i \neq C_j$. □

Remark 3.9. The statement (*2*) of the above proposition is not true for countable families, as seen in the next example. Let C be the set of all natural numbers endowed with the cofinite topology (the closed subsets are C and the finite sets). Note that the intersection of two nonempty open sets is nonempty. From this we obtain that every neighborhood of a point is open and connected, hence C is a locally connected space. Consider the sequence of connected subsets $(A_n)_{n\geq 2}$, where $A_n = \{n\}$. The closed set $R = \{1\}$ satisfies $R \cap A_n = \emptyset$, for every $n = 2, 3....$ Moreover, the closed sets $R_n = A_n \cup R$ $(n = 2, 3...)$ satisfy the following properties: $C = \bigcup_{n\geq 2} R_n$, $R = \bigcap_{n\geq 2} R_n$, $R_i \cap R_j = R$ $(i \neq j)$, $A_n \subset R_n$, and $C \setminus R$ is connected, hence it only has one component.

Remark 3.10. Dovgoshey and Martio [3] proved the following result: let R be a subspace of $\mathbb{R}$ with the usual metric; then the set $\mathbb{R} \setminus \overline{R}$ is connected if and only if the radius of every closed ball $B(a, \varepsilon)$ in R is equal to its effective radius

$$r(B(a, \varepsilon)) = \sup\{|a - z| : z \in R, |z - a| \leq \varepsilon\}.$$

Let C be the space $\mathbb{R}$ with the usual metric. Given $x, y \in \mathbb{R}$ with $x < y$, from the above proposition we obtain that a subset R is the intersection of two closed sets R_1 and R_2 such that $\mathbb{R} = R_1 \cup R_2$, $x \in R_1 \setminus R_2$ and $y \in R_2 \setminus R_1$ if and only if R is closed, $x, y \notin R$ and $R\cap]x, y[\neq \emptyset$. In this case, from the result by Dovgoshey and Martio, there are $a \in R$ and $\varepsilon > 0$ such that the $r(B(a, \varepsilon)) < \varepsilon$.

To finish this section, we present a result on the particular case when the family I has only two members.

Proposition 3.11. *Let X be a topological space, $C \subset X$ be a closed set and $f : X \longrightarrow \mathbb{R}$ be an upper semicontinuous function. The following assertions are equivalent:*

(i) *The set $C \cap f^{-1}(]-\infty, \lambda[)$ is connected for every $\lambda \in \mathbb{R}$.*

(ii) *If R and S are nonempty closed subsets of X and $R \cup S = C$ then*

$$\inf_{x\in R\cap S} f(x) = \max\left\{\inf_{x\in R} f(x), \inf_{x\in S} f(x)\right\}. \tag{7}$$

Proof. $(i) \implies (ii)$. Let $\lambda > \max\{\inf_{x\in R} f(x), \inf_{x\in S} f(x)\}$ and set $U = f^{-1}(]-\infty, \lambda[)$. By assumption, $C \cap U$ is connected; hence, as $R \cap U$ and $S \cap U$ are closed in $C \cap U$ and nonempty and $(R \cap U) \cup (S \cap U) = (R \cup S) \cap U = C \cap U$, we have $R \cap S \cap U \neq \emptyset$. Take $y \in R \cap S \cap U$. Given that $\inf_{x\in R\cap S} f(x) \leq f(y) < \lambda$, we conclude that $\inf_{x\in R\cap S} f(x) \leq \max\{\inf_{x\in R} f(x), \inf_{x\in S} f(x)\}$.

$(ii) \implies (i)$. Suppose that $C \cap U$, with $U = f^{-1}(]-\infty, \lambda[)$, is not connected for some $\lambda \in \mathbb{R}$. Then there exist two closed sets $\widetilde{R}, \widetilde{S} \subset X$ such that $\widetilde{R} \cap C \cap U \neq \emptyset$, $\widetilde{S}\cap C\cap U \neq \emptyset$, $(\widetilde{R}\cup\widetilde{S})\cap C\cap U = C\cap U$ and $\widetilde{R}\cap\widetilde{S}\cap C\cap U = \emptyset$. Define $R = (\widetilde{R}\cap C)\cup(C\setminus U)$ and $S = (\widetilde{S}\cap C)\cup(C\setminus U)$. These sets are closed in C; moreover, $R\cap U = \widetilde{R}\cap C\cap U \neq \emptyset$, $S\cap U = \widetilde{S}\cap C\cap U \neq \emptyset$, $R\cup S = ((\widetilde{R}\cup\widetilde{S})\cap C)\cup(C\setminus U) \supset ((\widetilde{R}\cup\widetilde{S})\cap C\cap U)\cup(C\setminus U) = C \cap U \cup (C \setminus U) = C$ and $R \cap S \cap U = \widetilde{R} \cap \widetilde{S} \cap C \cap U = \emptyset$. Take $y \in R \cap U$ and $z \in S \cap U$. We have $\inf_{x\in R} f(x) \leq f(y) < \lambda$ and $\inf_{x\in S} f(x) \leq f(z) < \lambda$; hence $\max\{\inf_{x\in R} f(x), \inf_{x\in S} f(x)\} < \lambda$. On the other hand, since $R \cap S \cap U = \emptyset$, for every $x \in R \cap S$ one has $x \notin U$, that is, $f(x) \geq \lambda$; therefore $\inf_{x\in R\cap S} f(x) \geq \lambda > \max\{\inf_{x\in R} f(x), \inf_{x\in S} f(x)\}$, which contradicts (ii). □

4. The distance to the intersection of a family of sets

Let X be a normed space and fix $x \in X$. Consider the function $f : X \longrightarrow \mathbb{R}$ given by $f(y) = d(x, y) = \|x - y\|$, where $\| \cdot \|$ denotes the norm of X. The results of the above section can be specialized to this environment, since f is convex and $d(x, A) = \inf_{a \in A} f(a)$.

Proposition 4.1. *Let X be a normed space and $C \subset X$ a convex set. Let $(R_i)_{i \in I}$ be a family of subsets of C satisfying $[r_i, r_j] \cap R_i \cap R_j \neq \emptyset$, for every $r_i \in R_i$ and $r_j \in R_j$ ($i, j \in I$ with $i \neq j$) and $C = \bigcup_{i \in I} R_i$. Let $R = \bigcap_{i \in I} R_i$ and assume that $R_i \cap R_j = R$, for $i, j \in I$ with $i \neq j$. Then, for every $x \in X$,*

$$\exists h \in I, \forall k \in I, k \neq h, \quad d(x, R) = d(x, R_k), \tag{8}$$

hence equality (2) *holds.*

Proposition 4.2. *Let X be a normed space and $C \subset X$ a convex set. Let $(R_i)_{i \in \mathbb{N}}$ be a sequence of subsets of C which are linearly closed in C and such that $C = \bigcup_{i \in \mathbb{N}} R_i$. Let $R = \bigcap_{i \in \mathbb{N}} R_i$ and assume that $R_i \cap R_j = R$, for $i, j \in \mathbb{N}$ with $i \neq j$. If $R \neq \emptyset$, then, for every $x \in X$, condition* (8) *is satisfied, hence equality* (2) *holds.*

From Proposition 3.5 we obtain

Proposition 4.3. *Let X be a normed space and $C \subset X$ a convex set. Let $R_1, R_2, ..., R_n \subset C$ be nonempty and closed in C and such that $C = \bigcup_{1 \leq i \leq n} R_i$, and let $R = \bigcap_{1 \leq i \leq n} R_i$. Assume that, for certain $h = 1, ..., n$ and for every $j = 1, ..., n$ with $j \neq h$, $R_h \cap R_j = R$ and*

$$d(x, R_h) \leq d(x, R_j). \tag{9}$$

Then $R \neq \emptyset$ and, for every $x \in X$,

$$\forall j = 1, ..., n; j \neq h, \quad d(x, R) = d(x, R_j)$$

hence equality (2) *holds.*

Example 4.4. Let X, Y be infinite dimensional Banach spaces. We denote by $L(X, Y)$ the class of all bounded linear operators from X into Y. For $T \in L(X, Y)$, $N(T)$ is its null space and $R(T)$ its range. The operator $T \in L(X, Y)$ is *semi-Fredholm* ($T \in SF$) if $R(T)$ is a closed subspace of Y and $N(T)$ or $Y/R(T)$ is finite dimensional; its *index* is defined by

$$\operatorname{ind} T = \dim N(T) - \dim Y/R(T) \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}.$$

For $n \in \overline{\mathbb{Z}}$, we put

$$SF^n = \{T \in SF : \operatorname{ind} T = n\}.$$

Acccording to [5, Proposition 2.c.9], the sets SF^n are open in $L(X, Y)$, so $SF^c = L(X, Y) \setminus SF$ is closed. Denote by $R_n = SF^n \cup SF^c$ ($n \in \overline{\mathbb{Z}}$) and $R = SF^c$. Then $(R_n)_{n \in \overline{\mathbb{Z}}}$ is a countable family of closed subsets of $L(X, Y)$ and, moreover,

$$L(X, Y) = \bigcup_{n \in \overline{\mathbb{Z}}} R_n \ ; \qquad R_n \cap R_m = R \ \ (n \neq m); \qquad R = \bigcap_{n \in \overline{\mathbb{Z}}} R_n \neq \emptyset.$$

Fix $T \in L(X,Y)$ and consider the distance $d(T,H)$ from T to the class $H \subset L(X,Y)$. In view of Proposition 4.2 , there are two possibilities:

(1) For every $n \in \overline{\mathbb{Z}}$,
$$d(T, L(X,Y)) = d(T, R_n) = d(T, R).$$
As $d(T, L(X,Y)) = 0$, we obtain $T \in R$; that is, $T \notin SF$.

(2) There exists $h \in \overline{\mathbb{Z}}$ such that, for every $k \in \overline{\mathbb{Z}}$, with $k \neq h$,
$$d(T, L(X,Y)) = d(T, R_h) < d(T, R_k) = d(T, R).$$
Then $T \in R_h$, but $T \notin R$, hence $T \in SF_h$. Moreover,
$$d(T, R) = d(T, R_k) = d(T, SF^k \cup SF^c) \leq d(T, SF^k),$$
hence $d(T, SF^c) \leq d(T, SF^k)$.

5. The distance to the intersection of a family of downward sets

We now present a result about the real space ℓ_∞ of all bounded sequences of real numbers with the supremum norm: $\|x\|_\infty = \|(x_n)_{n\geq 1}\|_\infty = \sup_{n\geq 1} |x_n|$. This result states the validity of the formula (2) for a certain type of subsets of ℓ_∞. The validity of this formula for the finite dimensional space ℓ_∞^d ($\mathbb{R}^d$ endowed with the norm $\|\cdot\|_\infty$) was established in [7].

In ℓ_∞ we consider the following order: $x = (x_n)_{n\geq 1} \leq y = (y_n)_{n\geq 1}$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$.

Definition 5.1. A subset A of ℓ_∞ is said to be *downward* if $x \in A$, $y \in \ell_\infty$ and $y \leq x$, implies $y \in A$.

Given $A \subset \ell_\infty$ and $x^0 \in \ell_\infty$, we write $P_A(x^0) = \{a \in A : \|a - x^0\|_\infty = d(x^0, A)\}$. The constant sequence $(1,1,1...)$ is denoted by $\mathbf{1}$.

Lemma 5.2. *Let $A \subset \ell_\infty$ be a nonempty closed downward set and $x^0 \in \ell_\infty$. Then*
$$a^0 = \min P_A(x^0),$$
where $a^0 = x^0 - r\mathbf{1}$ and $r = d(x^0, A)$; hence $P_A(x^0) \neq \emptyset$.

Proof. If $r = 0$, then $x^0 \in A$ and, consequently, $x^0 \in P_A(x^0)$. Assume $r > 0$. For each $n = 1,2,3...$, there exists $a_n \in A$ such that $\|a_n - x^0\|_\infty \leq r + \frac{1}{n}$, hence $x^0 - (r + \frac{1}{n})\mathbf{1} \leq a_n$. Then
$$a^0 = \lim_{n\to\infty} \left[x^0 - \left(r + \frac{1}{n} \right) \mathbf{1} \right] \leq \liminf_{n\to\infty} a_n \ .$$
By definition we have that
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k \ ;$$
and as A is a downward set we obtain $\inf_{k\geq n} a_k \in A$. Moreover $\liminf_{n\to\infty} a_n \in A$ since A is closed. Thus $a^0 \in A$. As $\|x^0 - a^0\|_\infty = r$, we obtain $a^0 \in P_A(x^0)$. Moreover, if $a \in P_A(x^0)$ we have that $\|a - x^0\|_\infty \leq r$, hence $a^0 = x^0 - r\mathbf{1} \leq a$. Thus a^0 is the minimum of $P_A(x^0)$. □

Proposition 5.3. *Let $(R_i)_{i\in I}$ be a family of closed downward subsets of ℓ_∞, and $R=\bigcap_{i\in I} R_i$. Then, for every $x\in\ell_\infty$,*

$$d(x,R)=\sup_{i\in I} d(x,R_i).$$

Proof. The proof given in [7, Theorem 4] for the finite dimensional case is based on [7, Corollary 2], which is deduced from [7, Proposition 2]. If we apply the above Lemma, then we obtain that [7, Proposition 2] is true in ℓ_∞ and, consequently, [7, Corollary 2] is also true. □

6. The distance to the intersection of a nested sequence

We end with a simple result on nested sequences of subsets in complete metric spaces. Our proof is based on the Cantor intersection property.

Proposition 6.1. *Let M be a complete metric space with distance d. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of non empty closed subsets whose diameters* $\operatorname{diam} R_n$ *satisfy* $\operatorname{diam} R_n \longrightarrow 0$ *($n\longrightarrow\infty$) and such that $R_n\supset R_{n+1}$ ($n\in\mathbb{N}$). Then, for every $x\in M$,*

$$d(x,\bigcap_{n\in\mathbb{N}} R_n)=\sup_{n\in\mathbb{N}} d(x,R_n)=\lim_{n\longrightarrow\infty} d(x,R_n).$$

Proof. The Cantor intersection theorem assures that there exists $a\in M$ such that $\{a\}=\bigcap_{n\in\mathbb{N}} R_n$. It is obvious that $d(x,a)\geq\sup_{n\in\mathbb{N}} d(x,R_n)$.

Let $\varepsilon>0$. We choose $k\in\mathbb{N}$ such that $\operatorname{diam} R_k<\varepsilon$. For every $y\in R_k$, we have that $d(x,a)\leq d(x,y)+d(y,a)<d(x,y)+\varepsilon$, hence

$$d(x,a)\leq d(x,R_k)+\varepsilon\leq\sup_{n\in\mathbb{N}} d(x,R_n)+\varepsilon\ .$$

As $\varepsilon>0$ is arbitrary, we finally obtain $d(x,a)\leq\sup_{n\in\mathbb{N}} d(x,R_n)$. □

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