Journal of Convex Analysis
Volume 18 (2011), No. 3, 721–736

On Evenly Convex Functions

Margarita M. L. Rodríguez*
Department of Statistics and Operations Research,
University of Alicante, 03080 Alicante, Spain
marga.rodriguez@ua.es

José Vicente-Pérez†
Department of Statistics and Operations Research,
University of Alicante, 03080 Alicante, Spain
jose.vicente@ua.es

Received: July 17, 2009
Revised manuscript received: July 14, 2010

A subset of $\mathbb{R}^n$ is said to be evenly convex if it is the intersection of some family (possibly empty) of open halfspaces. This class of convex sets was introduced by Fenchel in 1952 in order to extend the polarity theory to nonclosed convex sets. This paper deals with functions with evenly convex epigraphs, the so-called evenly convex functions. We study the main properties of this class of convex functions that contains the important class of lower semicontinuous convex functions. In particular, a characterization of even convexity in terms of lower semicontinuity is given. We also show that the class of evenly convex functions is closed under the main operations.

Keywords: Evenly convex set, evenly convex function, lsc convex function

2010 Mathematics Subject Classification: 26B25, 52A41, 90C25

Introduction

Frequently, the study of different classes of functions, considered within convex and quasiconvex analysis, can be reduced to the study of the basic sets considered in this field. So, for example, properties of convex sets are often used to study convex and quasiconvex functions because these classes of functions are characterized by the convexity of their epigraphs and sublevel sets, respectively.

In this sense, in the eighties, Martínez-Legaz [11] and Passy and Prisman [13], independently, started to use evenly convex sets (i.e., intersections of open halfspaces) in quasiconvex programming, defining the evenly quasiconvex functions as those having evenly convex sublevel sets. In this paper, we consider extended real-valued functions defined on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, with evenly convex epigraphs, the so-called evenly convex functions.

The paper is organized as follows. Section 1 contains the necessary notation and some basic results on evenly convex sets to be used later. In Section 2, we introduce

*Research supported by MEC of Spain and FEDER of EU, Grant MTM2008-06695-C03-01.
†Research supported by FPI Program of MEC of Spain, Grant BES-2006-14041.

ISSN 0944-6532 / $ 2.50

the new concept of evenly convex function and study the main properties of this class of convex functions that contains the important class of lower semicontinuous convex functions. In particular, a characterization of even convexity in terms of lower semicontinuity is given. Finally, in Section 3, we show that the class of evenly convex functions is closed under the main operations.

1. Preliminaries: Evenly convex sets

All the vectors in $\mathbb{R}^n$ will be interpreted as column vectors. The inner product of two vectors u and x will be denoted by $\langle u, x\rangle$. Moreover, we shall denote by $\|\cdot\|$, 0_n and $B(x; r)$ the Euclidean norm, the zero vector and the open ball with radius $r > 0$ and center x in $\mathbb{R}^n$. Given a set $X \subset \mathbb{R}^n$, $\operatorname{conv} X$, $\operatorname{cl} X$, $\operatorname{rint} X$ and $\operatorname{rbd} X$ denote the convex hull, closure, relative interior and relative boundary of X, respectively. If $X \neq \emptyset$ is convex, the recession cone of X is given by $O^+X := \{v \in \mathbb{R}^n \mid x + \mu v \in X, \text{ for all } x \in X \text{ and for all } \mu \geq 0\}$. Given $X, Y \subset \mathbb{R}^n$, the cartesian product of X and Y is denoted by $X \times Y$.

A set is said to be *evenly convex* (in [2]) if it is the intersection of a family (possibly empty) of open halfspaces. From the definition, it is easy to see that the intersection of a family of evenly convex sets is also evenly convex. We shall use the following characterizations of evenly convex sets.

Proposition 1.1 ([4, Proposition 3.1]). *Given $C \subset \mathbb{R}^n$ such that $\emptyset \neq C \neq \mathbb{R}^n$, the following conditions are equivalent to each other:*

(*i*) *C is evenly convex;*
(*ii*) *C is a convex set and for each $x \in \mathbb{R}^n \backslash C$ there exists a hyperplane H such that $x \in H$ and $H \cap C = \emptyset$;*
(*iii*) *C is the result of eliminating from a closed convex set the union of a certain family of its exposed faces;*
(*iv*) *C is a convex set and for any convex set K contained in $(\operatorname{cl} C)\backslash C$, there exists a hyperplane containing K and not intersecting C.*

Observe that the second part in condition (*ii*) always holds for any $x \in \mathbb{R}^n \backslash (\operatorname{cl} C)$, since $\operatorname{cl} C$ is a closed convex set and it can always be strongly separated from any outside point by a hyperplane. So, we only have to prove this condition for each $x \in (\operatorname{cl} C)\backslash C$. According to this, it is obvious that any closed convex set is evenly convex. On the other hand, as a consequence of Proposition 1.1 and [15, Theorem 11.2], any relatively open convex set is also evenly convex. In particular, it is obvious that, for any convex set in $\mathbb{R}$, condition (*ii*) holds and, therefore, it is evenly convex.

The *evenly convex hull* of X [2], denoted by $\operatorname{eco} X$, is the smallest evenly convex set which contains X (i.e., it is the intersection of all the open halfspaces which contain X). Obviously, X is evenly convex if and only if $\operatorname{eco} X = X$. It is known that $\operatorname{eco} X$ is obtained by eliminating from $\operatorname{cl}(\operatorname{conv} X)$ those exposed faces which do not contain points of X [6, Proposition 2.1]. From the definition, given $\bar{x} \in \mathbb{R}^n$, $\bar{x} \notin \operatorname{eco} X$ if and only if there exists $z \in \mathbb{R}^n$ such that $\langle z, x - \bar{x}\rangle > 0$ for all $x \in X$.

The following result will be frequently used in this paper.

Proposition 1.2. *Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be two non-empty sets. Then, $C \times D$ is evenly convex if and only if C and D are evenly convex.*

Proof. Suppose that $C \times D$ is evenly convex. According to [6, Proposition 2.3], we have

$$(\operatorname{eco} C) \times (\operatorname{eco} D) = \operatorname{eco}(C \times D) = C \times D$$

and we can deduce that C and D are necessarily evenly convex. We can see the converse in [4, Proposition 3.6]. □

Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, its *effective domain*, *graph*, *epigraph* and *sublevel set* for $r \in \mathbb{R}$, are

$$\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\},$$

$$\operatorname{gph} f = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \mid x \in \mathbb{R}^n, f(x) \in \mathbb{R} \right\},$$

$$\operatorname{epi} f = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f, f(x) \leq \lambda \right\}$$

and

$$L(f, r) = \{x \in \mathbb{R}^n \mid f(x) \leq r\},$$

respectively. The function f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$ and $\operatorname{epi} f$ does not contain vertical lines, i.e., f does not take on the value $-\infty$. In other case, we say that f is *improper*.

We shall say that f is *lower semicontinuous*, lsc in breaf, (*upper semicontinuous*, usc in breaf) at $\bar{x} \in \mathbb{R}^n$ if for any $\lambda \in \mathbb{R}$, $\lambda < f(\bar{x})$ (resp. $\lambda > f(\bar{x})$), there exists a neigbourhood of $\bar{x}$, $V_{\bar{x}}$, such that $\lambda < f(x)$ (resp. $\lambda > f(x)$) for all $x \in V_{\bar{x}}$. Any convex function f is always lsc except perhaps at relative boundary points of $\operatorname{dom} f$ (see [15, Theorems 7.2 and 7.4]). Moreover, it is well-known that a function is lsc at any point of $\mathbb{R}^n$ if and only if its epigraph is closed.

Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we define the *lsc hull function* of f, denoted by $\bar{f}$, as the greatest lsc function majorized by f. It is well-known that $\operatorname{epi} \bar{f} = \operatorname{cl}(\operatorname{epi} f)$ and f is lsc at $\bar{x}$ if and only if $f(\bar{x}) = \bar{f}(\bar{x})$. Moreover, if f is a convex function, then $\bar{f}$ is a convex function too.

The next proposition is a geometrical interpretation of the subdifferentiability of a proper convex function on the relative interior of its effective domain [15, Theorem 23.4].

Proposition 1.3. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function and $\bar{x} \in \operatorname{rint}(\operatorname{dom} f)$. Then, there exists $u \in \mathbb{R}^n$ such that*

$$\langle u, x - \bar{x} \rangle + (x_{n+1} - f(\bar{x})) \geq 0$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$.

2. Evenly convex functions

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *evenly convex* if its epigraph is an evenly convex set. Obviously, any evenly convex function is convex and, since any closed convex set is evenly convex, lsc convex functions constitute a subclass of evenly convex functions. In particular, any finite-valued convex function is evenly convex. The next two examples show that not every convex function is evenly convex and not every evenly convex function is a lsc convex function, respectively.

Example 2.1. Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 1, \\ 3 & \text{if } x = 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Obviously, its epigraph is a convex set, but it is not evenly convex. In fact, considering $\bar{x} = \binom{1}{2} \in \mathbb{R}^2 \setminus (\operatorname{epi} f)$, we have that, for any hyperplane H containing $\bar{x}$, $H \cap (\operatorname{epi} f) \neq \emptyset$.

Example 2.2. Consider the function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > -1, \\ +\infty & \text{if } x \leq -1. \end{cases}$$

It is easy to see that the epigraph of f is evenly convex but it is not closed, so f is an evenly convex function but it is not lsc.

It is well-known that sublevel sets are convex for convex functions, although a function whose sublevel sets are all convex need not be convex. On the other hand, closedness of sublevel sets characterizes the class of lsc functions. Next we show that all the sublevel sets of an evenly convex function are evenly convex.

Proposition 2.3. *If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is an evenly convex function, then f is evenly quasiconvex.*

Proof. It is well-known that, for any $r \in \mathbb{R}$, $L(f, r) \times \{r\} = (\operatorname{epi} f) \cap (\mathbb{R}^n \times \{r\})$. Since $\operatorname{epi} f$ is evenly convex and $\mathbb{R}^n \times \{r\}$ is a closed convex set (evenly convex too), we obtain that the cartesian product $L(f, r) \times \{r\}$ is an evenly convex set and, by Proposition 1.2, $L(f, r)$ is evenly convex, for all $r \in \mathbb{R}$. $\square$

The converse of Proposition 2.3 is not true in general, even though f is convex (recall Example 2.1).

Concerning the effective domain, we know that it is a convex set when f is a convex function, since it is the projection on $\mathbb{R}^n$ of the convex set $\operatorname{epi} f$. Nevertheless, the projection of an evenly convex set is not, in general, evenly convex (see [9, Example 3.1]). The following example shows an evenly convex function with non-evenly convex domain.

Example 2.4. Let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} x_1 \ln \frac{x_1}{x_2} & \text{if } x \in E, \\ 0 & \text{if } x = 0_2, \\ +\infty & \text{otherwise,} \end{cases}$$

where $E = \{x \in \mathbb{R}^2 \mid 0 < x_1 \leq 1,\ 0 < x_2 \leq x_1\}$. Observe that $\operatorname{dom} f = E \cup \{0_2\}$ is not evenly convex, although, as we shall see, f is a lsc convex function and, therefore, it is evenly convex.

Consider the function $g : \mathbb{R}^2_{++} \to \mathbb{R}$ defined by $g(x) = x_1 \ln \frac{x_1}{x_2}$, where $\mathbb{R}^2_{++} = (]0, +\infty[)^2$. Since g is a twice continuously differentiable real-valued function on the open convex set $\mathbb{R}^2_{++}$ and its Hessian matrix is positive semi-definite for every $x \in \mathbb{R}^2_{++}$, according to [15, Theorem 4.5], we have that g is a convex function on $\mathbb{R}^2_{++}$. On the other hand, both functions f and g coincide on the convex subset E of $\mathbb{R}^2_{++}$ and, hence, f is convex on E. The convexity of f on $\operatorname{dom} f = E \cup \{0_2\}$ can be easily proved showing that $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $x \in E$, $y = 0_2$ and $0 < \lambda < 1$. So, f is a convex function.

Since every convex function is always lsc except perhaps at relative boundary points of its domain, we only have to prove that f is lsc on $\operatorname{rbd}(\operatorname{dom} f)$.

Since g is a finite-valued convex function on $\mathbb{R}^2_{++}$, g is lsc at any $\overline{x} \in \mathbb{R}^2_{++}$, i.e., for all $\lambda < g(\bar{x})$, there exists a neighbourhood of $\bar{x}$, $V_{\bar{x}}$, in $\mathbb{R}^2_{++}$ such that $\lambda < g(x)$ for all $x \in V_{\bar{x}}$. Then, given $\bar{x} \in \operatorname{rbd}(\operatorname{dom} f) \cap E \subset \mathbb{R}^2_{++}$, we have that, for all $\lambda < f(\bar{x}) = g(\bar{x})$, there exists a neighbourhood of $\bar{x}$, $V_{\bar{x}}$, in $\mathbb{R}^2_{++} \subset \mathbb{R}^2$ such that $\lambda < g(x) = f(x)$ for all $x \in V_{\bar{x}} \cap E$. Obviously, if $x \in V_{\bar{x}} \cap (\mathbb{R}^2 \backslash E) \subset \mathbb{R}^2 \backslash (\operatorname{dom} f)$, then $f(x) = +\infty > \lambda$, so that f is lsc at $\bar{x}$.

On the other hand, lower semicontinuity of f at 0_2 is a direct consequence of $f(x) \geq 0 = f(0_2)$ for all $x \in \mathbb{R}^2$.

Finally, for $\bar{x} \in]0, 1] \times \{0\}$, we have $f(\bar{x}) = +\infty > \lambda$ for all $\lambda \in \mathbb{R}$. Moreover, since

$$\lim_{\substack{x \to \bar{x} \\ x \in E}} f(x) = \lim_{\substack{x \to \bar{x} \\ x \in E}} g(x) = +\infty,$$

we have that, given any $\lambda \in \mathbb{R}$, there exists a neighbourhood of $\bar{x}$, $V_{\bar{x}}$, in $\mathbb{R}^2 \backslash \{0_2\}$ such that $f(x) > \lambda$, for all $x \in V_{\bar{x}} \cap E$, and $f(x) = +\infty > \lambda$, for all $x \in V_{\bar{x}} \cap (\mathbb{R}^2 \backslash E)$. Then, f is also lsc on $]0, 1] \times \{0\}$.

The function f in Example 2.4 is a proper evenly convex function. Now, we shall consider improper functions with non-empty domain (a function with empty domain has empty epigraph and it is trivially evenly convex) and we shall obtain necessary and sufficient conditions for even convexity. One of these conditions will be even convexity of the effective domain.

Lemma 2.5. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an evenly convex function such that $f(x_0) = -\infty$ for some $x_0 \in \mathbb{R}^n$. Then $f(x) = -\infty$ for all $x \in \operatorname{dom} f$.*

Proof. Since f is an improper convex function, according to [15, Theorem 7.2], $f(x) = -\infty$ for all $x \in \operatorname{rint}(\operatorname{dom} f)$. Suppose that there exists

$$\bar{x} \in (\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f) \subset \operatorname{rbd}(\operatorname{dom} f)$$

such that $f(\bar{x}) \in \mathbb{R}$. Then $\binom{\bar{x}}{f(\bar{x})-1} \notin \operatorname{epi} f$ and, by the even convexity of $\operatorname{epi} f$, there exists $\binom{z}{\gamma} \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{0_{n+1}\}$ such that

$$\left\langle \begin{pmatrix} z \\ \gamma \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ x_{n+1} - f(\bar{x}) + 1 \end{pmatrix} \right\rangle > 0, \tag{2}$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$.

On the other hand, since $\bar{x} \in \operatorname{rbd}(\operatorname{dom} f)$, we can consider $\{x^k\} \subset \operatorname{rint}(\operatorname{dom} f)$ such that $\lim_{k\to\infty} x^k = \bar{x}$. Then, for any $k \in \mathbb{N}$, $f(x^k) = -\infty$ and $\binom{x^k}{\lambda} \in \operatorname{epi} f$, for all $\lambda \in \mathbb{R}$. In particular, taking $\lambda = f(\bar{x}) - 2$, we obtain

$$\left\langle \begin{pmatrix} z \\ \gamma \end{pmatrix}, \begin{pmatrix} x^k - \bar{x} \\ -1 \end{pmatrix} \right\rangle = \langle z, x^k - \bar{x} \rangle - \gamma > 0, \tag{3}$$

for all $k \in \mathbb{N}$. Similarly, taking $\lambda = f(\bar{x})$, we have

$$\left\langle \begin{pmatrix} z \\ \gamma \end{pmatrix}, \begin{pmatrix} x^k - \bar{x} \\ 1 \end{pmatrix} \right\rangle = \langle z, x^k - \bar{x} \rangle + \gamma > 0, \tag{4}$$

for all $k \in \mathbb{N}$. Taking limits as $k \to \infty$ in (3) and (4), we can conclude that $\gamma = 0$. Hence, (2) becomes to

$$\langle z, x - \bar{x} \rangle > 0,$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$ and this contradicts $\binom{\bar{x}}{f(\bar{x})} \in \operatorname{epi} f$. Therefore, $f(x) = -\infty$ for all $x \in \operatorname{dom} f$. □

Theorem 2.6. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an improper function such that $f(x_0) = -\infty$ for some $x_0 \in \mathbb{R}^n$. Then, f is evenly convex if and only if* $\operatorname{dom} f$ *is evenly convex and $f(x) = -\infty$ for all $x \in \operatorname{dom} f$.*

Proof. If f is an evenly convex function, then, by Lemma 2.5, $f(x) = -\infty$ for all $x \in \operatorname{dom} f$, so that $\operatorname{epi} f = \operatorname{dom} f \times \mathbb{R}$ and, since $\operatorname{epi} f$ is evenly convex, $\operatorname{dom} f$ is also an evenly convex set, according to Proposition 1.2.

Conversely, if $f(x) = -\infty$ for all $x \in \operatorname{dom} f$, we have that $\operatorname{epi} f = \operatorname{dom} f \times \mathbb{R}$ and $\operatorname{epi} f$ is evenly convex as a consequence of the even convexity of $\operatorname{dom} f$ and $\mathbb{R}$. □

So, any improper evenly convex function has evenly convex domain (if $f \equiv +\infty$ then $\operatorname{dom} f = \emptyset$ that is trivially evenly convex). Nevertheless, this property is not exclusive for improper functions.

Proposition 2.7. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper evenly convex function. If f is bounded from above on* $\operatorname{dom} f$*, then* $\operatorname{dom} f$ *is an evenly convex set.*

Proof. If f is a bounded function on its domain, then there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \operatorname{dom} f$, so that

$$\operatorname{dom} f \subset L(f, M) \subset \operatorname{dom} f.$$

Therefore, $\operatorname{dom} f = L(f, M)$ and this set is evenly convex as a consequence of Proposition 2.3. □

According to Proposition 2.7, even convexity of the domain is a necessary condition for even convexity of proper functions which are bounded from above. Next, we shall see that it is not a sufficient condition.

Example 2.8. Consider the proper convex function f defined in Example 2.1. f is bounded on $\operatorname{dom} f = [-1, 1]$ that is a closed convex set (and, therefore, evenly convex), but f is not an evenly convex function.

Up to now, we know that the class of evenly convex functions is between the class of lsc convex functions and the class of convex functions (that are always lsc except perhaps at relative boundary points of their domains). Now, we shall show a characterization of proper evenly convex functions in terms of lower semicontinuity.

Theorem 2.9. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Then, f is evenly convex if and only if f is a convex function and f is lsc on* $\operatorname{eco}(\operatorname{dom} f)$.

Proof. Assume that f is a proper evenly convex function. Since f is a proper convex function, $\operatorname{dom} f$ is a non-empty convex set and f is lsc on $\operatorname{rint}(\operatorname{dom} f)$. So, we only have to prove lower semicontinuity on $\operatorname{eco}(\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f) \subset \operatorname{rbd}(\operatorname{dom} f)$.

Suppose that f is not lsc at $\bar{x} \in \operatorname{eco}(\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f)$. Then, there exist $\lambda \in \mathbb{R}$, $\lambda < f(\bar{x})$, and $x^k \in B_k := B(\bar{x}; 1/k)$, for any $k \in \mathbb{N}$, such that $f\left(x^k\right) \leq \lambda$. Observe that $\lim_{k\to\infty} x^k = \bar{x}$. Taking $\mu \in \mathbb{R}$ such that $\lambda < \mu < f(\bar{x})$, we have that $\binom{\bar{x}}{\mu} \notin \operatorname{epi} f$ and, since $\operatorname{epi} f$ is an evenly convex set, there exits $\binom{z}{\gamma} \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{0_{n+1}\}$ such that

$$\left\langle \begin{pmatrix} z \\ \gamma \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ x_{n+1} - \mu \end{pmatrix} \right\rangle > 0, \tag{5}$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$. In particular, $\binom{x^k}{\mu+1} \in \operatorname{epi} f$, for any $k \in \mathbb{N}$, so that

$$\left\langle z, x^k - \bar{x} \right\rangle + \gamma > 0, \tag{6}$$

for all $k \in \mathbb{N}$. In the same way, we obtain

$$\left\langle z, x^k - \bar{x} \right\rangle + \gamma \left(\frac{\lambda - \mu}{2} \right) > 0, \tag{7}$$

for all $k \in \mathbb{N}$, replacing $\binom{x}{x_{n+1}}$ with $\binom{x^k}{(\lambda+\mu)/2} \in \operatorname{epi} f$ in (5). Taking limits as $k \to \infty$ in (6) and (7), we conclude that $\gamma = 0$ and (5) becomes to

$$\langle z, x - \bar{x} \rangle > 0,$$

for all $x \in \operatorname{dom} f$, and this entails $\bar{x} \notin \operatorname{eco}(\operatorname{dom} f)$ that is a contradiction. So, f is lsc at any point of $\operatorname{eco}(\operatorname{dom} f)$.

Conversely, assume that f is a proper convex function and it is lsc on $\operatorname{eco}(\operatorname{dom} f)$. We shall prove that $\operatorname{epi} f$ is evenly convex by Proposition 1.1(ii).

First, $\operatorname{epi} f$ is a convex set because f is a convex function. Now, let us consider $\binom{\bar{x}}{\bar{\lambda}} \notin \operatorname{epi} f$.

If $\bar{x} \in \operatorname{rint}(\operatorname{dom} f)$, then, by Proposition 1.3, there exists $u \in \mathbb{R}^n$ such that

$$\langle u, x - \bar{x} \rangle + (x_{n+1} - f(\bar{x})) \geq 0,$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$. Since $\binom{\bar{x}}{\bar{\lambda}} \notin \operatorname{epi} f$, we have $f(\bar{x}) > \bar{\lambda}$ and, therefore,

$$\langle u, x - \bar{x} \rangle + \left(x_{n+1} - \bar{\lambda}\right) > \langle u, x - \bar{x} \rangle + (x_{n+1} - f(\bar{x})) \geq 0,$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$, so that we can conclude that

$$\left\langle \begin{pmatrix} u \\ 1 \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ x_{n+1} - \bar{\lambda} \end{pmatrix} \right\rangle > 0,$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$.

If $\bar{x} \notin \operatorname{eco}(\operatorname{dom} f)$, then we can consider $z \in \mathbb{R}^n \setminus \{0_n\}$ such that $\langle z, x - \bar{x} \rangle > 0$, for all $x \in \operatorname{dom} f$. So, we can again conclude that

$$\left\langle \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ x_{n+1} - \bar{\lambda} \end{pmatrix} \right\rangle > 0,$$

for all $\binom{x}{x_{n+1}} \in \operatorname{epi} f$.

Finally, if $\bar{x} \in \operatorname{eco}(\operatorname{dom} f) \setminus \operatorname{rint}(\operatorname{dom} f)$, given $\beta \in \mathbb{R}$ such that $\bar{\lambda} < \beta < f(\bar{x})$, since f is lsc at $\bar{x}$, there exists $\delta > 0$ such that $f(x) > \beta$ for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}\| < \delta$. Let us consider $B := B\left(\binom{\bar{x}}{\bar{\lambda}}; \varepsilon\right)$ with $\varepsilon := \min\left\{\delta, \beta - \bar{\lambda}\right\}$. We shall prove that $B \cap \operatorname{epi} f = \emptyset$ that entails $\binom{\bar{x}}{\bar{\lambda}} \notin \operatorname{cl}(\operatorname{epi} f)$ and, since $\operatorname{cl}(\operatorname{epi} f)$ is evenly convex, we can again conclude that there exists a hyperplane $H \subset \mathbb{R}^{n+1}$ such that $\binom{\bar{x}}{\bar{\lambda}} \in H$ and $H \cap \operatorname{epi} f \subset H \cap \operatorname{cl}(\operatorname{epi} f) = \emptyset$.

Let us consider $\binom{x}{\lambda} \in B \cap \operatorname{epi} f$. Taking into account that

$$\|x - \bar{x}\| \leq \left\| \begin{pmatrix} x \\ \lambda \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix} \right\| < \varepsilon \leq \delta,$$

we get $\lambda \geq f(x) > \beta$. On the other hand, we can obtain the opposite inequality from

$$\left|\lambda - \bar{\lambda}\right| \leq \left\| \begin{pmatrix} x \\ \lambda \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix} \right\| < \varepsilon \leq \beta - \bar{\lambda}.$$

$\square$

Corollary 2.10. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an evenly convex function with closed convex domain. Then, f is a convex lsc function.*

Proof. Obviously, the result is true if f is identically $+\infty$, since $\operatorname{epi} f = \emptyset$. Otherwise, if f is improper, by Lemma 2.5, f is identically $-\infty$ on $\operatorname{dom} f$ and $\operatorname{epi} f = \operatorname{dom} f \times \mathbb{R}$ that is a closed convex set. Therefore, f is a lsc convex function.

Finally, if f is a proper evenly convex function, by Theorem 2.9, f is a convex function and it is lsc on $\operatorname{dom} f = \operatorname{cl}(\operatorname{dom} f)$. Since every proper convex function is lsc on $\mathbb{R}^n \setminus \operatorname{cl}(\operatorname{dom} f)$, we can conclude that f is a lsc convex function. $\square$

It is well-known that any convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is continuous relative to $\operatorname{rint}(\operatorname{dom} f)$ (see [15, Theorem 10.1]). Moreover, if f is a proper evenly convex function, then f is lsc on the greater set $\operatorname{eco}(\operatorname{dom} f)$. When we ask whether any proper convex function f can be assumed usc on $\operatorname{rbd}(\operatorname{dom} f) \cap \operatorname{dom} f$ relative to $\operatorname{dom} f$, the answer is no in general (see [15, page 83]). However, it is easy to prove that this property holds for univariate functions. As a consequence of this, we consider the concept of upper semicontinuity along lines (as in [10]).

Given a non-empty convex set $A \subset \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *upper* (resp. *lower) semicontinuous along lines* on A if, for every $\bar{x}, \bar{y} \in A$, the function $\varphi : [0,1] \to \overline{\mathbb{R}}$, given by $\varphi(t) = f(\bar{x} + t(\bar{y} - \bar{x}))$, is usc (resp. lsc) at t relative to $[0,1]$, for any $t \in [0,1]$. Moreover, f is said to be *continuous along lines* on A, if f is usc and lsc along lines on A.

For any proper convex function f, $\operatorname{dom} f$ is a non-empty convex set and, for every $\bar{x}, \bar{y} \in \operatorname{dom} f$, φ, defined from $[0,1]$ to $\mathbb{R}$, is a univariate convex function and, therefore, it is usc relative to $[0,1]$. As a consequence of this, any proper convex function is usc along lines on its domain.

Furthermore, it is easy to prove that any proper convex function f that is lsc on a non-empty convex set $A \subset \operatorname{dom} f$, is also lsc along lines on A.

Proposition 2.11. *Any proper evenly convex function is continuous along lines on its domain.*

Proof. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper evenly convex function. As a consequence of Theorem 2.9, f is lsc convex on the non-empty convex set $\operatorname{dom} f$ and, therefore, f is lsc along lines on its domain. Since any proper convex function is usc along lines on its domain, we obtain the result. $\square$

Proposition 2.12. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper evenly convex function. Then, the image of f, given by*

$$\operatorname{Im} f := \{f(x) \mid x \in \operatorname{dom} f\},$$

is a convex set.

Proof. Let us consider $a, b \in \operatorname{Im} f$ and $x, y \in \operatorname{dom} f$ such that $a = f(x)$ and $b = f(y)$. Since, by Proposition 2.11, $\varphi(t) := f(x + t(y - x))$ is continuous relative to $[0,1]$, applying the Intermediate Value Theorem to the function φ, we obtain that $[a,b] \subset \operatorname{Im} f$. $\square$

Given a non-empty set $C \subset \mathbb{R}^{n+1}$, we can consider the function whose graph is the lower boundary of C, that is, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $f(x) = \inf\left\{a \in \mathbb{R} \mid \binom{x}{a} \in C\right\}$. By [15, Theorem 5.3], we know that if C is a convex set, the function f is also convex. Moreover, if C is a closed convex set, then f is a lsc convex function. However, this result is not true, in general, when we replace lsc convexity by even convexity (consider the evenly convex set in [9, Example 3.1]). In the next proposition, we shall see that the condition $\binom{0_n}{1} \in O^+C$ is sufficient for this result to become true.

Proposition 2.13. *Let $C \subset \mathbb{R}^{n+1}$ be a non-empty evenly convex set such that $\binom{0_n}{1} \in O^+C$. Then the function $f(x) = \inf\left\{a \in \mathbb{R} \mid \binom{x}{a} \in C\right\}$ is evenly convex and* $\operatorname{epi} f = C \cup \operatorname{gph} f$.

Proof. Obviously, $C \cup \operatorname{gph} f \subset \operatorname{epi} f$. Conversely, given $\binom{x}{\lambda} \in \operatorname{epi} f$, if $f(x) = \lambda$, then $\binom{x}{\lambda} \in \operatorname{gph} f$. Otherwise, there exists $a \in \mathbb{R}$, $f(x) \le a < \lambda$, such that $\binom{x}{a} \in C$ and, since $\binom{0_n}{1} \in O^+C$, we have

$$\begin{pmatrix} x \\ a \end{pmatrix} + (\lambda - a)\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in C.$$

Hence, $\operatorname{epi} f = C \cup A$, where $A := \operatorname{gph} f$. Now, we shall see that $C \cup A$ is an evenly convex set and, therefore, f is evenly convex.

As C is evenly convex, by Proposition 1.1(*iii*), we know that

$$C = \operatorname{cl} C \backslash \left(\bigcup_{t \in U} C_t \right),$$

where C_t is an exposed face of $\operatorname{cl} C$, for each $t \in U$. Moreover, it is clear that $\operatorname{cl} C = \operatorname{cl}(C \cup A)$, so proving that

$$C \cup A = \operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right), \tag{8}$$

where $V = \{t \in U \mid C_t \cap A = \emptyset\}$, and applying Proposition 1.1(*iii*) again, we will obtain that $C \cup A$ is evenly convex. Let us then prove (8).

First, since $V \subset U$, it is obvious that

$$C \subset \operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right). \tag{9}$$

On the other hand, as $A \subset (C \cup A) \subset \operatorname{cl}(C \cup A) = \operatorname{cl} C$ and $C_t \cap A = \emptyset$ for all $t \in V$, we have

$$A \subset \operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right). \tag{10}$$

From (9) and (10), we obtain the inclusion

$$C \cup A \subset \operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right). \tag{11}$$

Conversely, assume that $\binom{\bar{x}}{\bar{\lambda}} \in \operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right)$ and $\binom{\bar{x}}{\bar{\lambda}} \notin C \cup A$. As C is a non-empty evenly convex set, by [4, Proposition 3.3], we have $O^+C = O^+(\operatorname{cl} C)$ and, therefore, $\binom{0_n}{1} \in O^+(\operatorname{cl} C)$ and $\binom{\bar{x}}{\bar{\lambda}+\varepsilon} \in \operatorname{cl} C$, for all $\varepsilon \geq 0$.

Now, consider the non-empty convex set

$$X := \left\{ \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix} + \varepsilon \begin{pmatrix} 0_n \\ 1 \end{pmatrix}, \ \varepsilon \in \mathbb{R} \right\} \cap \operatorname{cl} C.$$

This set satisfies that $X \cap (C \cup A) = \emptyset$. In fact, for any $\varepsilon \leq 0$, $\binom{\bar{x}}{\bar{\lambda}+\varepsilon} \in C \cup A$ implies that $f(\bar{x}) \leq \bar{\lambda} + \varepsilon \leq \bar{\lambda}$ and, therefore, $\binom{\bar{x}}{\bar{\lambda}} \in C \cup A$, that is a contradiction.

On the other hand, for any $\varepsilon > 0$, if $\binom{\bar{x}}{\bar{\lambda}+\varepsilon} \in C \cup A$, we can take $a := \bar{\lambda} + \varepsilon$, if $\binom{\bar{x}}{\bar{\lambda}+\varepsilon} \in C$ or $a > \bar{\lambda} + \varepsilon = f(\bar{x})$ with $\binom{\bar{x}}{a} \in C$, otherwise. Taking into account that C is evenly convex, $\binom{\bar{x}}{\bar{\lambda}} \in \operatorname{cl} C$ and $\binom{\bar{x}}{a} \in C$, by [2, statement 3.5], we obtain that

$$(1-t) \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix} + t \begin{pmatrix} \bar{x} \\ a \end{pmatrix} \in C,$$

for all $t \in]0,1]$. Therefore, $f(\bar{x}) \leq \bar{\lambda}$ and $\binom{\bar{x}}{\bar{\lambda}} \in C \cup A$ again.

By Proposition 1.1(iv), there exists a hyperplane H such that $X \subset H$ and $H \cap C = \emptyset$. Since X is a vertical semiline, H is a vertical hyperplane, and since $H \cap C = \emptyset$, then $H \cap A = \emptyset$. Consider the set $H \cap \operatorname{cl} C$. It satisfies $X \subset (H \cap \operatorname{cl} C)$ and $(H \cap \operatorname{cl} C) \cap C = \emptyset$. Hence $H \cap \operatorname{cl} C$ is an exposed face of $\operatorname{cl} C$ that does not intersect C. So, there exists $t \in U$ such that $H \cap \operatorname{cl} C = C_t$ and $C_t \cap A = (H \cap \operatorname{cl} C) \cap A = \emptyset$. Then, $t \in V$ and $(\bar{x}, \bar{\lambda})' \in X \subset C_t \subset \bigcup_{t \in V} C_t$, which is a contradiction. So, we can conclude that

$$\operatorname{cl} C \backslash \left(\bigcup_{t \in V} C_t \right) \subset C \cup A. \tag{12}$$

From (11) and (12), we obtain (8). □

Corollary 2.14. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function. If its strict epigraph*

$$\operatorname{epi}_S f := \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f, f(x) < \lambda \right\}$$

is an evenly convex set, then f is an evenly convex function.

Proof. It is a consequence of Proposition 2.13 and the expression

$$f(x) = \inf \left\{ a \in \mathbb{R} \mid \begin{pmatrix} x \\ a \end{pmatrix} \in \operatorname{epi}_S f \right\}.$$

□

It is easy to see that not every evenly convex function has evenly convex strict epigraph. We can consider, for example, the evenly convex function $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f(x) = \begin{cases} -\sqrt{1-x^2} & \text{if } -1 \leq x \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

3. Functional operations preserving even convexity

It is well-known that convexity and lower semicontinuity are preserved by the most important functional operations. In this section, we shall see that the same thing happens to even convexity.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an evenly convex function and let $\alpha > 0$. Then αf is an evenly convex function.*

Proof. Since f is a convex function, we know that αf is also a convex function. Assume that $\binom{\bar{x}}{\bar{a}} \notin \operatorname{epi}(\alpha f)$. Then, since $\binom{\bar{x}}{\frac{\bar{a}}{\alpha}} \notin \operatorname{epi} f$ and $\operatorname{epi} f$ is evenly convex, there exists $\binom{z}{t} \neq 0_{n+1}$ such that

$$\left\langle \begin{pmatrix} z \\ t \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ a - \frac{\bar{a}}{\alpha} \end{pmatrix} \right\rangle > 0, \tag{13}$$

for all $\binom{x}{a} \in \operatorname{epi} f$. From (13) we obtain

$$\left\langle \begin{pmatrix} \alpha z \\ t \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ b - \bar{a} \end{pmatrix} \right\rangle > 0,$$

for all $\binom{x}{b} \in \operatorname{epi}(\alpha f)$ and, by Proposition 1.1, αf is an evenly convex function. $\square$

Proposition 3.2. *Let $\{f_i : \mathbb{R}^n \to \overline{\mathbb{R}},\ i \in I\}$ be a family of evenly convex functions and let $f := \sup_{i \in I} f_i$. Then, f is evenly convex.*

Proof. It is a consequence of the fact that the intersection of a collection of evenly convex sets is evenly convex and the epigraph of f is the intersection of those of the functions f_i. $\square$

Now, we shall consider the sum of two evenly convex functions. It is well-known that, given two extended real-valued functions f and g, $\operatorname{dom}(f+g) = (\operatorname{dom} f) \cap (\operatorname{dom} g)$. In the following, we shall assume, without loss of generality, that this set is non-empty.

Proposition 3.3. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two proper evenly convex functions. Then, $f + g$ is also an evenly convex function.*

Proof. According to Theorem 2.9, f and g are convex and lsc on $\operatorname{eco}(\operatorname{dom} f)$ and $\operatorname{eco}(\operatorname{dom} g)$, respectively. Since the class of convex functions is closed for the sum, we have that $f + g$ is convex. On the other hand, the sum of proper lsc functions at a point is also lsc at this point, so $f+g$ is lsc on $\operatorname{eco}(\operatorname{dom} f) \cap \operatorname{eco}(\operatorname{dom} g)$. Due to the expression,

$$\operatorname{eco}(\operatorname{dom}(f+g)) = \operatorname{eco}((\operatorname{dom} f) \cap (\operatorname{dom} g)) \subset \operatorname{eco}(\operatorname{dom} f) \cap \operatorname{eco}(\operatorname{dom} g),$$

we have that $f+g$ is lsc on $\operatorname{eco}(\operatorname{dom}(f+g))$ and, by Theorem 2.9 again, $f+g$ is an evenly convex function. □

In general, the last result is not true when one of the functions is not proper.

Example 3.4. Let $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$, where f is the proper evenly convex function defined in Example 2.4 and g is the improper evenly convex function defined by

$$g(x) = \begin{cases} -\infty & \text{if } x \in [0,1]^2, \\ +\infty & \text{otherwise.} \end{cases}$$

Obviously, $\operatorname{dom}(f+g) = (\operatorname{dom} f) \cap (\operatorname{dom} g) = \operatorname{dom} f$, that is not an evenly convex set, and, in particular,

$$(f+g)(x) = \begin{cases} -\infty & \text{if } x \in \operatorname{dom} f, \\ +\infty & \text{otherwise,} \end{cases}$$

so $f+g$ is not an evenly convex function by Theorem 2.6.

Proposition 3.5. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two evenly convex functions and assume that f is not proper. Then $f+g$ is an evenly convex function if and only if* $\operatorname{dom}(f+g)$ *is an evenly convex set.*

Proof. As we are assuming that $(\operatorname{dom} f) \cap (\operatorname{dom} g) \neq \emptyset$, then $\operatorname{dom} f \neq \emptyset$ and, therefore, there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) = -\infty$. Since f is evenly convex, according to Lemma 2.5, we have that $f \equiv -\infty$ on $\operatorname{dom}(f+g) \subset \operatorname{dom} f$. On the other hand, $g(x) < +\infty$ for all $x \in \operatorname{dom}(f+g) \subset \operatorname{dom} g$, so $f+g \equiv -\infty$ on its domain and

$$\operatorname{epi}(f+g) = \operatorname{dom}(f+g) \times \mathbb{R}. \tag{14}$$

Finally, we obtain the result applying Proposition 1.2 on (14). □

As a consequence of the fact that the intersection of evenly convex sets is an evenly convex set and the equality $\operatorname{dom}(f+g) = (\operatorname{dom} f) \cap (\operatorname{dom} g)$, we obtain the following results, applying Proposition 3.5.

Corollary 3.6. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two improper evenly convex functions. Then $f+g$ is evenly convex.*

Corollary 3.7. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two evenly convex functions. If f is improper and g is proper with evenly convex domain, then $f+g$ is evenly convex.*

Corollary 3.8. *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be two evenly convex functions such that f is improper and g is proper and bounded on its domain. Then $f+g$ is evenly convex.*

Given a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$, we define its *evenly convex hull*, denoted by f^{ec}, as the largest evenly convex function minorizing f. Obviously, since the pointwise supremum of a family of evenly convex functions is evenly convex, f^{ec} coincides with the pointwise supremum of all the evenly convex functions minorizing f. Moreover, we shall say that a function f is *evenly convex at* $\bar{x}\in\mathbb{R}^n$ if $f(\bar{x})=f^{ec}(\bar{x})$.

Lemma 3.9. *Let* $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ *be a function with* $\operatorname{epi} f\neq\emptyset$. *Then,* $\binom{0_n}{1}\in O^+(\operatorname{eco}(\operatorname{epi} f))$.

Proof. Given $\binom{\bar{x}}{\bar{\lambda}}\in\operatorname{epi} f$, we have that $\binom{\bar{x}}{\bar{\lambda}+\varepsilon}\in\operatorname{epi} f\subset\operatorname{eco}(\operatorname{epi} f)$, for all $\varepsilon\geq 0$. Then, according to [4, Proposition 3.4], we obtain $\binom{0_n}{1}\in O^+(\operatorname{eco}(\operatorname{epi} f))$. □

Proposition 3.10. *Let* $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ *be a function and let* $x\in\mathbb{R}^n$. *Then*

$$f^{ec}(x)=\inf\left\{a\in\mathbb{R}\mid\begin{pmatrix}x\\a\end{pmatrix}\in\operatorname{eco}(\operatorname{epi} f)\right\}\tag{15}$$

and $\operatorname{eco}(\operatorname{epi} f)\subset\operatorname{epi} f^{ec}$.

Proof. When $\operatorname{epi} f=\emptyset$, the result is obvious. So, we can suppose that $\operatorname{epi} f\neq\emptyset$. According to Lemma 3.9, we have that $\binom{0_n}{1}\in O^+(\operatorname{eco}(\operatorname{epi} f))$ and, since $\operatorname{eco}(\operatorname{epi} f)$ is a non-empty evenly convex set, by Proposition 2.13, we obtain that

$$g(x):=\inf\left\{a\in\mathbb{R}\mid\begin{pmatrix}x\\a\end{pmatrix}\in\operatorname{eco}(\operatorname{epi} f)\right\}$$

is an evenly convex function. Moreover, as $\operatorname{epi} f\subset\operatorname{eco}(\operatorname{epi} f)$, $\binom{x}{f(x)}\in\operatorname{eco}(\operatorname{epi} f)$, for all $x\in\mathbb{R}^n$, and, consequently, $g(x)\leq f(x)$, for all $x\in\mathbb{R}^n$. Hence, g is an evenly convex function minorizing f and, therefore, $g\leq f^{ec}$.

On the other hand, since $f^{ec}\leq f$ and f^{ec} is an evenly convex function, we have

$$\operatorname{eco}(\operatorname{epi} f)\subset\operatorname{eco}(\operatorname{epi} f^{ec})=\operatorname{epi} f^{ec}$$

and, considering

$$f^{ec}(x)=\inf\left\{a\in\mathbb{R}\mid\begin{pmatrix}x\\a\end{pmatrix}\in\operatorname{epi} f^{ec}\right\},$$

it is obvious that $f^{ec}\leq g$. □

Proposition 3.11. *Let* $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ *be a convex function. Then, the following statements hold:*

(*i*) $\operatorname{epi}_S f^{ec}\subset\operatorname{eco}(\operatorname{epi} f)\subset\operatorname{epi} f^{ec}$.
(*ii*) $\bar{f}\leq f^{ec}\leq f$.
(*iii*) $\operatorname{dom} f\subset\operatorname{dom} f^{ec}\subset\operatorname{dom}\bar{f}\subset\operatorname{cl}(\operatorname{dom} f)$.
(*iv*) $\operatorname{dom} f^{ec}\subset\operatorname{eco}(\operatorname{dom} f)\subset\operatorname{cl}(\operatorname{dom} f)$.
(*v*) $\operatorname{rint}(\operatorname{dom} f^{ec})=\operatorname{rint}(\operatorname{dom} f)$.

Proof. The statements are trivial when $f \equiv +\infty$. Otherwise, we can suppose that $\operatorname{epi} f \neq \emptyset$ and $\operatorname{dom} f \neq \emptyset$.

(i) Let $\binom{\bar{x}}{\bar{\lambda}} \in \operatorname{epi}_S f^{ec}$. Then, $f^{ec}(\bar{x}) < \bar{\lambda}$ and, by (15), there exists $a \in \mathbb{R}$ such that $\binom{\bar{x}}{a} \in \operatorname{eco}(\operatorname{epi} f)$ and $f^{ec}(\bar{x}) \leq a < \bar{\lambda}$. According to Lemma 3.9, we can conclude that

$$\begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \bar{x} \\ a \end{pmatrix} + (\bar{\lambda} - a) \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \operatorname{eco}(\operatorname{epi} f).$$

The second inclusion has been obtained in Proposition 3.10.

(ii) As f is convex, $\bar{f}$ is a lsc convex function. Then, $\bar{f}$ is an evenly convex function minorizing f and, therefore, $\bar{f} \leq f^{ec} \leq f$.

(iii) As a consequence of (ii), we have that $\operatorname{dom} f \subset \operatorname{dom} f^{ec} \subset \operatorname{dom} \bar{f}$. Moreover, denoting by $\operatorname{proy}_{\mathbb{R}^n}$ the projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ given by $\operatorname{proy}_{\mathbb{R}^n} \binom{x}{r} = x$, by [15, Theorem 6.6], we have that $\operatorname{proy}_{\mathbb{R}^n}(\operatorname{cl}(\operatorname{epi} f)) \subset \operatorname{cl}(\operatorname{proy}_{\mathbb{R}^n}(\operatorname{epi} f))$, so that

$$\operatorname{dom} \bar{f} = \operatorname{proy}_{\mathbb{R}^n}(\operatorname{epi} \bar{f}) = \operatorname{proy}_{\mathbb{R}^n}(\operatorname{cl}(\operatorname{epi} f)) \subset \operatorname{cl}(\operatorname{proy}_{\mathbb{R}^n}(\operatorname{epi} f)) = \operatorname{cl}(\operatorname{dom} f).$$

(iv) In the same way, applying [6, Proposition 2.4], we obtain

$$\begin{aligned} \operatorname{dom} f^{ec} &= \operatorname{proy}_{\mathbb{R}^n} \operatorname{epi} f^{ec} = \operatorname{proy}_{\mathbb{R}^n} \operatorname{eco}(\operatorname{epi} f) \subset \operatorname{eco}(\operatorname{proy}_{\mathbb{R}^n} \operatorname{epi} f) \\ &= \operatorname{eco}(\operatorname{dom} f) \subset \operatorname{cl}(\operatorname{dom} f). \end{aligned}$$

(v) Since $\operatorname{rint}(\operatorname{dom} f) \subset \operatorname{dom} f \subset \operatorname{dom} f^{ec} \subset \operatorname{cl}(\operatorname{dom} f)$, by [15, Corollary 6.3.1], we obtain $\operatorname{rint}(\operatorname{dom} f^{ec}) = \operatorname{rint}(\operatorname{dom} f)$. □

The following two results characterize even convexity at a point. Similar characterizations are given in [1] for pointwise even quasiconvexity using sublevel sets instead of epigraphs.

Proposition 3.12. *A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is evenly convex at $\bar{x} \in \mathbb{R}^n$ if and only if, for any $a < f(\bar{x})$, $\binom{\bar{x}}{a} \notin \operatorname{eco}(\operatorname{epi} f)$.*

Proof. If f is evenly convex at $\bar{x} \in \mathbb{R}^n$, then $f^{ec}(\bar{x}) = f(\bar{x})$ and, given $a < f(\bar{x})$, we have that $\binom{\bar{x}}{a} \notin \operatorname{epi} f^{ec}$ and, therefore, $\binom{\bar{x}}{a} \notin \operatorname{eco}(\operatorname{epi} f)$.

Conversely, if $\binom{\bar{x}}{a} \notin \operatorname{eco}(\operatorname{epi} f)$ for every $a < f(\bar{x})$, then, by Proposition 3.10, we have that $f^{ec}(\bar{x}) \geq f(\bar{x})$ and f is evenly convex at $\bar{x}$. □

Corollary 3.13. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function. Then, the following statements hold:*

(i) *f is evenly convex at $\bar{x} \in \mathbb{R}^n$ if and only if, for any $a < f(\bar{x})$, there exists $q \in \mathbb{R}^{n+1}$ such that*

$$\left\langle q, \begin{pmatrix} x \\ \lambda \end{pmatrix} - \begin{pmatrix} \bar{x} \\ a \end{pmatrix} \right\rangle > 0,$$

for all $\binom{x}{\lambda} \in \operatorname{epi} f$.

(ii) *f is evenly convex if and only if it is evenly convex at every $\bar{x} \in \mathbb{R}^n$.*

Proof. (i) It is a consequence of Proposition 3.12 and Proposition 1.1(ii).

(ii) If f is an evenly convex function, then $\operatorname{eco}(\operatorname{epi} f) = \operatorname{epi} f$ and, by 15, we obtain $f^{ec}(\bar{x}) = f(\bar{x})$, for all $\bar{x} \in \mathbb{R}^n$. Conversely, if f is evenly convex at every $\bar{x} \in \mathbb{R}^n$ and we take a point $\binom{x}{a} \notin \operatorname{epi} f$, as $a < f(x)$, applying statement (i), we obtain a hyperplane $H \subset \mathbb{R}^{n+1}$ such that $\binom{xa}{\in} H$ and $H \cap \operatorname{epi} f = \emptyset$. Finally, as a consequence of Proposition 1.1(ii), f is evenly convex. □

References

[1] A. Daniilidis, J.-E. Martínez-Legaz: Characterizations of evenly convex sets and evenly quasiconvex functions, J. Math. Anal. Appl. 273 (2002) 58–66.

[2] W. Fenchel: A remark on convex sets and polarity, Meddel. Lunds Univ. Mat. Sem. Suppl. (1952) 82–89.

[3] M. A. Goberna, V. Jeyakumar, N. Dinh: Dual characterizations of set containments with strict convex inequalities, J. Glob. Optim. 34 (2006) 33–54.

[4] M. A. Goberna, V. Jornet, M. M. L. Rodríguez: On linear systems containing strict inequalities, Linear Algebra Appl. 360 (2003) 151–171.

[5] M. A. Goberna, M. A. López: Linear Semi-infinite Optimization, John Wiley and Sons, Chichester (1998).

[6] M. A. Goberna, M. M. L. Rodríguez: Analyzing linear systems containing strict inequalities via evenly convex hulls, Eur. J. Oper. Res. 169 (2006) 1079–1095.

[7] V. Jeyakumar: Characterizing set containments involving infinite convex constraints and reverse-convex constraints, SIAM J. Optim. 13 (2003) 947–959.

[8] V. Klee: Maximal separation theorems for convex sets, Trans. Amer. Math. Soc. 134 (1968) 133–147.

[9] V. Klee, E. Maluta, E. Zanco: Basic properties of evenly convex sets, J. Convex Analysis 14 (2007) 137–148.

[10] D. G. Luenberger: Quasi-convex programming, SIAM J. Appl. Math. 16 (1968) 1090–1095.

[11] J.-E. Martínez-Legaz: Un Concepto Generalizado de Conjugación, Aplicación a las Funciones Quasiconvexas, Thesis, Universidad de Barcelona (1981).

[12] J.-E. Martínez-Legaz: Quasiconvex duality theory by generalized conjugation methods, Optimization 19 (1988) 603–652.

[13] U. Passy, E. Z. Prisman: Conjugacy in quasi-convex programming, Math. Program. 30 (1984) 121–146.

[14] J.-P. Penot, M. Volle: On quasiconvex duality, Math. Oper. Res. 15 (1990) 597–625.

[15] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton, New Jersey (1970).