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About the Regularity of Average Distance Minimizers in $\mathbb{R}^2$

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We focus on the following irrigation problem introduced in [4]

$$\min \mathcal{F}(\Sigma) := \int_\Omega \mathrm{dist}(x,\Sigma)\, \mathrm{d}\mu(x),$$

where Ω is an open subset of $\mathbb{R}^2$, μ is a probability measure and where the minimum is taken over all the sets $\Sigma \subset \Omega$ such that Σ is compact, connected, and $\mathcal{H}^1(\Sigma) \leq \alpha_0$ for a given positive constant α_0. In this paper we seek for some conditions to find in Σ some pieces of C^1 (or more) regular curves. We prove that it is the case in the ball B when $\Sigma \cap B$ contains no corner points. More generally we prove that the Left and Right tangents half lines of Σ (that exist everywhere out of endpoints and triple points) are semicontinuous. We also discuss how the regularity is linked with the pull back measure $\psi := k\sharp\mu$ where k is the projection on Σ. In particular $\Sigma \cap B$ is $C^{1,\alpha}$ when ψ is regular with respect to $\mathcal{H}^1$ with density in a certain L^p. We also prove that Σ is locally a Lipschitz graph away from triple points and endpoints, and that the mean curvature of Σ is a measure that is explicited in terms of measure ψ.

Contents

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $\alpha_0 > 0$ a fixed constant, and μ a given probability measure on Ω. In this paper we study the regularity of the following minimization problem

$$\min_{\Sigma \in \mathcal{A}} \mathcal{F}(\Sigma) := \int_{\Omega} \mathrm{dist}(x, \Sigma)\, \mathrm{d}\mu(x) \tag{1}$$

where the minimum is taken over the family $\mathcal{A}$ of all the compact and connected sets $\Sigma \subset \Omega$ satisfying the length constraint $\mathcal{H}^1(\Sigma) \leq \alpha_0$. This problem also known as the "irrigation problem", was introduced by G. Buttazzo, E. Oudet and E. Stepanov in [4] and then in [5] in a more general formulation in terms of optimal mass transport problem with "free Dirichlet regions". In the sequel we will call Σ an optimal set for the problem (1).

An easy interpretation of the Problem (1) is the following. One could consider Σ as being a ressource of limited length (for instance some water in pipes) that one wants to place in the domain Ω in such a way that the average cost for people living in Ω to reach the resource Σ is minimal, according to the density of population given by the measure μ. We refer to [5, 11, 9, 4, 3] for some more detailed interpretations of Problem (1).

In [5], the topological description of minimizers is studied and it has been proved in particular that Σ has no loops and is a finite union of Lipschitz arcs, that meet by number of three at some finite number of triple junctions. Concerning the regularity, it is only proved in [5] that Σ is Ahlfors-Regular.

Then in [10], F. Santambrogio and P. Tilli restrict themselves to the simpler formulation (1), which in the end is not so restrictive according to some later results [11], and they characterize the blow up limits of the minimal set Σ in order to prove some regularity. They prove that any blow up sequence of the minimal set $\Sigma_r := \frac{1}{r}(\Sigma \cap B(x,r) - x)$ converges in $B(0,1)$ when $r \to 0$, and the limit could be either a radius (x is an endpoint), a diameter (x is ordinary point), three radius making angles of 120 degrees (x is a triple junction), or two radius making an angle different from 180 degrees (x is a corner point).

F. Santambrogio and P. Tilli [10] also found a sufficient condition for having $C^{1,1}$ regularity in a neighborhood of a point $x \in \Sigma$, involving the diameter of the set of points that are projected on $\Sigma \cap B(x,r)$. Since this condition is satisfied in a small enough neighborhood of any triple point, they obtain that any triple point admits a small neighborhood in which the three pieces of curve of Σ are $C^{1,1}$.

Very recently, P. Tilli [12] proved that for any $C^{1,1}$ simple curve Σ of length less than π times the inverse of the infimum of its curvature, one can find an open set Ω containing Σ in such a way that Σ is a minimizer for the problem (1) in Ω with μ equals to the Lebesgue measure. This fact implies that no further regularity is possible for Σ and that $C^{1,1}$ is optimal.

Recall that by "corner point" we mean a point in Σ for which the blow up limit is a union of two radius with a strict angle (different from 180 degrees). Although it is not difficult to find some examples of domains Ω where any minimizer Σ necessarily contains a triple point, it remains an open question as to whether a minimizer could actually contain corner points. On the other hand in [2], the first order Euler-Lagrange equation is computed (see Section 5 below) and the existence of stationary sets Σ that contain corner points is shown.

Now let us describe the contributions of this paper. One of our main result is that away from triple points, Σ is locally at least as regular as the graph of a convex fonction, namely that the Right and Left tangent maps admit some Right and Left limits at every point and are semicontinuous. More precisely, for a given parametrization γ of an injective Lipschitz arc $\Gamma \subset \Sigma$, by existence of blow up limits one can define the Left and Right tangent half-lines at every point $x \in \Gamma$ by

$$T_R(x) := x + \mathbb{R}^+ . \lim_{h \to 0^+} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h}$$

$$T_L(x) := x + \mathbb{R}^+ . \lim_{h \to 0^+} \frac{\gamma(t_0 - h) - \gamma(t_0)}{h}$$

Then we have the following.

Theorem 1.1. *Let $\Gamma \subset \Sigma$ be an open injective Lipschitz arc. Then the Right and Left tangent maps $x \mapsto T_R(x)$ and $x \mapsto T_L(x)$ are semicontinuous, i.e. for every $y_0 \in \Gamma$,*

$$\lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_L(y) = T_L(y_0) \quad \text{and} \quad \lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_R(y) = T_R(y_0).$$

In addition the limit from the other side exists and we have

$$\lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_L(y) = T_R(y_0) \quad \text{and} \quad \lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_R(y) = T_L(y_0).$$

An interesting and immediate consequence is the following result.

Corollary 1.2. *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains neither corner points nor triple points. Then Γ is locally a C^1 regular curve.*

The strategy to prove Theorem 1.1 is to use on one hand that when the diameter of transported set is small we have $C^{1,1}$ regularity (thank to [10]), and on the other hand when the diameter is big Σ stays under some very large "tangent circles" that makes Σ similar to a convex set locally. The difficulty is to glue together all the regions where we control the tangents from one argument or another. This is what we do in Section 4.

In Section 5 and 6 we try to exploit the Euler Equation to get some regularity. In [2], G. Buttazzo, E. Mainini and E. Stepanov give the first order equation for the penalized functional

$$\mathcal{F}(\Sigma) + \lambda \mathcal{H}^1(\Sigma).$$

In Section 5 we prove the existence of a λ_0 such that the Euler equation for the original problem with length constraint is the same as the penalized one. The method, that was already used by F. Santambrogio and P. Tilli to characterize the blow up limits in [10], is to estimate what we loose or win in the average distance functional by adding or erasing a piece of curve at an endpoint. In particular we obtain an explicit value for λ_0 depending on the mass of transport rays arriving at any endpoint and which corresponds to the "shape derivative" of $\mathcal{F}$.

As an application of the Euler equation, in Section 6 we give a "tilt estimate". In other words, we obtain a local control on the oscillations of the tangent lines of Σ with respect to a fixed line.

In Section 7 we apply Theorem 1.1 to find in Σ some Lipschitz graphs (see Theorem 7.2) and applying the Euler Equation and the tilt estimate on those graphs we obtain some results that are summarized in the following statement.

Theorem 1.3. *For every point $x \in \Sigma$ which is neither an endpoint nor a triple point, one can find a radius r, a line $\pi \subset \mathbb{R}^2$ containing x and a 5-Lipschitz function $f : \pi \to \pi^\perp$ such that*

$$\Sigma \cap B(x, r/4) = \{(x, f(x)), x \in \pi\} \cap B(x, r/4), \quad \textit{and}$$

$$\int_{\pi \cap B(x, \frac{r}{16})} |f'(t)|^2 dt \leq C r \psi(B(x, r))^2.$$

Moreover, f' satisfies the equation

$$-\frac{d}{dt}\left(\frac{f'}{\sqrt{1 + |f'|^2}}\right) = \psi_0$$

on $B(x, \frac{r}{16}) \cap \pi$. Here $\frac{d}{dt}$ is the derivative in the distributional sense and ψ_0 is a measure that verifies

$$|\psi_0| \leq (p \circ k)\sharp\mu$$

where $p : \mathbb{R}^2 \to \pi$ is the projection on π and k is a mesurable selection of the projection multimap onto Σ.

As a complement of Theorem 1.3, we also discuss how the regularity is linked with the behavior of the measure ψ. In particular we have the following.

Theorem 1.4. *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains no triple points and such that $\psi|_\Gamma$ is absolutely continuous with respect to $\mathcal{H}^1$ with density in $L^p(\Gamma, d\mathcal{H}^1)$. Then Γ is locally a $C^{1,\alpha}$ curve with $\alpha = \frac{p-1}{p}$.*

This last result is proved independently from all the other sections (in particular does not use the Euler equation), and this is why Theorem 1.4 is actually proved at the very beginning in Section 3. It can be seen as an introduction to understand why the regularity of Σ is difficult to obtain. We also get a reverse statement, namely that if Σ is $C^{1,1}$ regular then ψ is absolutely continuous with respect to $\mathcal{H}^1$ with density in L^∞.

2. Preliminaries

Throughout Σ will refer (as above) to an optimal set for the problem (1). The existence of a minimizer is an easy consequence of Blaschke and Gołąb Theorems and is proved in [5]. It is also proved in [5] that it is not restrictive to assume that Ω is convex. We will denote by d the euclidian distance in $\mathbb{R}^2$ and by d_H the Hausdorff distance. With any minimizer Σ we associate a fixed measurable selection of the projection multimap $k : \Omega \to \Sigma$, that is, for every $x \in \Omega$

$$d(x, \Sigma) = d(x, k(x)).$$

Then we introduce the image measure $\psi := k\sharp\mu$ which is defined for any Borel set $A \subset \mathbb{R}^2$ by

$$\psi(A) := \mu(k^{-1}(A)).$$

By abuse of notation we will sometimes simply denote $k^{-1}(x)$ instead of $k^{-1}(\{x\})$.

For $x \in \Sigma$ we will say that R_x is a transport ray ending at x if R_x is a segment in Ω bounded by x, and having maximal length for the property that every point $y \in R_x$ satisfies $\mathrm{dist}(y, \Sigma) = \mathrm{dist}(y, x)$.

Recall that we already know by [5] that Σ is a finite union of injective Lipschitz arcs meeting at some finite number of triple points. We also know that for any endpoint x of Σ it holds $\psi(x) > c$ for a positive constant c. If we exclude the endpoints and triple junctions, thank to [10] we have a characterization of the blow up limits at point x in terms of $\psi(x)$. Indeed, if x is neither an endpoint nor a triple junction, then x is a corner point if and only if x is an atom for ψ, that is $\psi(x) > 0$. Otherwise it is an "ordinary point" (i.e. the blow up limit is a diameter).

In the sequel we denote by $\mathbb{T}_\Sigma$ the set of triple points of Σ and by $\mathbb{E}_\Sigma$ its endpoints.

2.1. Standard facts on compact connected 1-dimensional sets

Here we recall some standard properties on compact connected 1-dimensional sets that can be found in [6].

Proposition 2.1. *Let $\Sigma \subset \mathbb{R}^N$ be a compact and connected set such that $\mathcal{H}^1(\Sigma) < +\infty$. Then there is a C_N-Lipschitz surjective mapping $f : [0, L] \to \Sigma$. As a consequence, Σ is arcwise connected and rectifiable. Moreover, for each choice of $x_0, y_0 \in \Sigma$ with $x_0 \neq y_0$, we can find an injective Lipschitz mapping $f : [0, 1] \to \Sigma$ such that $f(0) = x_0$ and $f(1) = y_0$.*

Thank to Proposition 2.1, our minimizer Σ is already rectifiable. Further, we will see that Σ is actually "uniformly rectifiable" in the sense of David and Semmes. This will follow from the fact that any minimizer Σ is Ahlfors-Regular as it is proved in [5]. Let us give some more definitions.

Definition 2.2. A set Σ is said to be an Ahlfors-regular set (of dimension 1), if there exists a constant C and a positive radius r_0 such that for every point $x \in \Sigma$ and every $r < r_0$,

$$rC^{-1} \leq \mathcal{H}^1(\Sigma \cap B(x, r)) \leq Cr.$$

In [5] it is proved that any minimizer Σ is Ahlfors regular. More precisely, there is an $r_0 > 0$ such that for every $x \in \Sigma$ and any $r < r_0$,

$$r \leq \mathcal{H}^1(\Sigma \cap B(x,r)) \leq 3\pi r. \tag{2}$$

There is a lot of equivalent definitions of Uniform rectifiability but we will choose the one with Ahlfors-regular curves.

Definition 2.3. An Ahlfors-regular curve with constant $\leq C$ is a set of the form $\Sigma = z(I)$ where $I \subset \mathbb{R}$ is a closed interval (not reduced to one point) and $z : I \to \mathbb{R}^N$ is a Lipschitz function such that

$$|z(x) - z(y)| \leq |x - y| \quad \text{for } x, y \in I$$

and

$$\mathcal{H}^1(\{x \in I; z(x) \in B(y,r)\}) \leq Cr$$

for all $y \in \mathbb{R}^N$ and $r > 0$.

Definition 2.4. Let $\Sigma \subset \mathbb{R}^N$ be an Ahlfors-regular set of dimension 1. We say that Σ is uniformly rectifiable when Σ is contained in some Ahlfors-regular curve.

Theorem 2.5 ([8]). *Every 1 dimensional connected Ahlfors-regular set is uniformly rectifiable.*

We deduce the following fact that will be used in Section 3.2.

Corollary 2.6. *Any minimizer Σ is uniformly rectifiable.*

2.2. Useful estimates and standard assumptions

We will use some estimates that are proved in [10], and that come from comparing Σ with a competitor made by replacing a piece of Σ by a segment.

Lemma 2.7 ([10]). *There exist a constant C satisfying the following properties. Let $\Gamma \subset \Sigma$ be a closed injective arc, with endpoints x, y, such that $\Gamma \backslash \{x,y\}$ contains no triple junctions of Σ and $C\psi(\Gamma\backslash\{x,y\}) < \frac{1}{2}$. Then*

$$\begin{aligned} \mathcal{H}^1(\Gamma) &\leq |x-y| + C\psi(\Gamma\backslash\{x,y\}) d_H(\Gamma, [x,y]), \\ d_H(\Gamma,[x,y]) &\leq C\psi(\Gamma\backslash\{x,y\})|x-y|, \\ \mathcal{H}^1(\Gamma) &\leq |x-y|(1 + C\psi(\Gamma\backslash\{x,y\})^2), \qquad (3) \\ \mathcal{H}^1(\Gamma) &\leq 2|x-y|. \qquad (4) \end{aligned}$$

It will be convenient in the sequel to work in some balls where $\Sigma \cap \partial B(x,r)$ consists in exactly 2 points. For this purpose, let us recall some results that are still contained in [10].

For any $x_0 \in \Sigma$ consider a branch of Σ starting at x_0 consisting of a Lipschitz curve $\gamma : [0,T] \to \Sigma$, parameterized by arclength, such that $\gamma(0) = x_0$ and $\gamma(T)$ is either an endpoint or a triple point of Σ. We may also assume that γ contains neither endpoints nor triple junctions in its relative interior.

Theorem 2.3 of [10] says the following.

Lemma 2.8 ([10]). *Consider $x \in \Sigma$ and $r > 0$ such that $B(x,r)$ contains no endpoint and triple junction other than, possibly, x itself. For any $s < r$, set*

$$t_1 := \min\{t \geq 0; \gamma(t) \in \partial B(x,s)\}, \qquad t_2 := \max\{t \leq T; \gamma(t) \in \partial B(x,s)\}.$$

If $C_1\psi(\gamma(0,t_2]) < 1$, then $t_1 = t_2$.

Lemma 2.8 is generally used together with the following fact which is Lemma 2.4 of [10].

Lemma 2.9 ([10]). *For any $x \in \Sigma$ there exists $r(x) > 0$ such that for all $r < r(x)$ the ball $B(x,r)$ contains no triple junction nor endpoint other than, possibly, x itself, and $C_1\psi(\gamma((0,t_2])) < 1$.*

In the sequel, for any $x \in \Sigma$ we will denote $r(x)$ the maximum radius satisfying the assumptions of Lemma 2.9. In particular, for every $x \in \Sigma \backslash \mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma$ and for all $r < r(x)$ we have

$$\sharp \Sigma \cap \partial B(x,r) = 2.$$

In [10], a uniform version of the above result is stated, saying that in fact one can take a common radius $r(x) = r_0$ for every $x \in \Sigma_1$, where $\Sigma_1 \subset \Sigma$ is compactly contained in the complement of atoms of mass at least $(2C_1)^{-1}$ and of triple junctions and endpoints, r_0 depending now on Σ_1. In this paper we will need this slightly different version of the preceding results.

Proposition 2.10. *For every compact set Σ_1 compactly contained in $\Sigma \backslash \mathbb{T}_\Sigma$, there exists a constant $C_2 := C_2(\Sigma_1)$ and a radius $r_0 := r_0(\Sigma_1)$ such that for all $x \in \Sigma_1$ and $r < r_0$,*

$$\psi(B(x,r)) \leq C_2 \Rightarrow r \leq r(x).$$

Proof. We argue by contradiction as in the proof of Lemma 2.5 of [10]. If the proposition is not true, then there exists a sequence of points $x_n \in \Sigma_1$ and a sequence of radii r_n such that $\psi(B(x_n,r_n))$ tends to 0, r_n tends to 0 and does not satisfy the assumptions of Lemma 2.9. Observe that for n big enough, $B(x_n,r_n)$ contains no endpoints nor triple points. Indeed, it is easy to exclude endpoints as soon as $\psi(B(x_n,r_n))$ gets smaller than $\min\{\psi(\{x\}); x \in \mathbb{E}_\Sigma\}$. For triple points, it suffice to wait until r_n gets small enough with respect to $\mathrm{dist}(\Sigma_1, \mathbb{T}_\Sigma) > 0$. Possibly by extracting a subsequence we may assume that x_n converges to a point x in Σ_1 and since $\psi(B(x_n,r_n)) \to 0$ we deduce that

$$\psi(\{x\}) = 0. \tag{5}$$

We also know that x_n is never a triple junction. That means that for every x_n, exactly two branches of Lipschitz arcs are starting from x_n and meet $\partial B(x_n,r_n)$ at least once and at different points (because Σ has no loops). Assume by contradiction that

$$\sharp\{\Sigma \cap \partial B(x_n,r_n)\} > 2. \tag{6}$$

We denote by γ_n^1 and γ_n^2 the two corresponding parameterizations and $\gamma_n^1([0,t_2^1(n)])$ and $\gamma_n^1([0,t_2^2(n)])$ the two branches of "first return" in $B(x_n,r_n)$. From Lemma 2.8 we know that one of the $\psi(\gamma_n^i([0,t_2^i(n)]))$ is greater than C otherwise (6) would not be true. By extracting a further subsequence we may assume that $\psi(\gamma_n^1([0,t_2^1(n)])) > C$ for all n and arguing as in the proof of [10] Theorem 2.5. we obtain that $\gamma_n^1([0,t_2^1(n)])$ converges for the Hausdorff distance to x which must be an atom of mass at least C and contradicts (5). □

Let us introduce a quantity which will measure the flatness of Σ in the ball $B(x,r)$, defined for $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ and $r < r(x)$ by

$$\beta(x,r) := \frac{d_H(\Sigma \cap B(x,r), [z,z'])}{|z-z'|}$$

where z and z' are the two points of $\partial B(x,r) \cap \Sigma$. The notation is given compared to the well known P. Jones β-numbers.

For simplicity, when there is no possible confusion we will use the notation $\psi(x,r)$ instead of $\psi(B(x,r))$. By Lemma 2.7 we directly have

$$\beta(x,r) \leq C\psi(x,r). \tag{7}$$

Finally, we end this preliminary section by recalling the basic steps that lead to the regularity result of [10] since we will also need the intermediate estimates. The next proposition is a direct consequence of the proof of Lemma 2.10 of [10]. We let the details to the reader.

Proposition 2.11 ([10]). *For all $x \in \Sigma$ and r such that there exists a line $\pi \subset \mathbb{R}^2$ satisfying*

$$d_H(\Sigma \cap B(x,2r), \pi \cap B(x,2r)) \leq \frac{r}{100}$$

we have

$$\psi(x,r) \leq Cr\,\mathrm{diam}(k^{-1}(B(x,r_0))) + Cr^{-1}d_H(\Sigma \cap B(x,2r), \pi \cap B(x,2r)). \tag{8}$$

In particular if $r < r(x)$ and $\psi(x,r)$ is small enough then

$$\psi(x,r) \leq C(r + \beta(x,2r)).$$

As it is shown in [10] (Theorem 2.11.), the last estimate can be iterated in the case when $r < r(x)$ in order to obtain the following result which will be also needed later.

Proposition 2.12 ([10]). *Let $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ and $r < r(x)$. If $\mathrm{diam}(k^{-1}(B(x,r_0))) < 1/(2C)$ then there exists r_0 depending on Σ such that*

$$\psi(x,r) \leq Cr \quad \forall r \leq \min(r_0, r(x)) \tag{9}$$

where C is a constant depending only on Σ, Ω and μ.

Observe that (9) together with (7) leads to some $C^{1,1}$ regularity.

3. The measure ψ

In the next sections we will see how ψ is linked with the mean curvature of Σ. Therefore it is natural to think that some good control on ψ will give some regularity on Σ. This is what we do in this section.

3.1. The regularity is equivalent to the behavior of ψ

Let us first prove that the regularity of Σ implies some decay on $\psi(x, r)$.

Proposition 3.1. *If $\Sigma \cap B(x_0, r_0)$ is a $C^{1,\alpha}$ regular curve then there exists $r_1 \leq r_0$ such that $\psi(x, r) \leq Cr^\alpha$ for all $x \in \Sigma \cap B(x_0, r_0/2)$ and $r \leq r_1$.*

Proof. Assume that $\Sigma' := \Sigma \cap B(x_0, r_0)$ is a $C^{1,\alpha}$ regular curve γ parameterized by arclenght. Let $x \in \Sigma \cap B(x_0, r_0/2)$ and $r \leq r_0/2$. Then for all $y \in B(x, r)$ one has

$$\gamma(t) - \gamma(0) = \int_0^t \gamma'(s)ds$$

with $\gamma(0) = x$ and $\gamma(t) = y$. Further,

$$\begin{aligned} \gamma(t) - \gamma(0) &= \int_0^t \gamma'(s) - \gamma'(0)ds + \int_0^t \gamma'(0)ds, \\ |\gamma(t) - \gamma(0) - t\gamma'(0)| &\leq \int_0^t |\gamma'(s) - \gamma'(0)|ds \\ &\leq Ct^{1+\alpha} \end{aligned}$$

which implies

$$\operatorname{dist}(y, T(x)) \leq C|x - y|^{1+\alpha} \leq Cr^{1+\alpha}$$

where $T(x_0)$ is the tangent line at x_0. Since y is an arbitrary point lying in $\Sigma \cap B(x, r)$, by (8) we conclude that $\psi(x, r) \leq Cr^\alpha$ for r small enough depending on r_0 an other constants. $\square$

Now we prove the reverse statement.

Proposition 3.2. *Assume that $x_0 \in \Sigma$ and $r_0 > 0$ are such that $B(x_0, r_0)$ contains no triple points nor endpoints and such that $\psi(x, r) \leq Cr^\alpha$ for all $x \in \Sigma \cap B(x_0, r_0/2)$ and $r < r_0/2$. Then $\Sigma \cap B(x_0, r_0/2)$ is a $C^{1,\alpha}$ regular curve.*

Proof. We denote $r_0(\Sigma_1)$ the radius given by Proposition 2.10 with $\Sigma_1 := \Sigma \cap B(x_0, r_0)$. We also denote $r_2 \leq \min(r_0(\Sigma_1), r_0/2)$ a radius such that $Cr_2^\alpha \leq C_2(\Sigma_1)$ in such a way that for all $x \in \Sigma \cap B(x_0, r_0/2)$, $r_2 \leq r(x)$. Now from $\psi(x, r) \leq Cr^\alpha$ for all $r \leq r_2$ we obtain by (7) that $\beta(x, r) \leq Cr^\alpha$ for all $x \in \Sigma_1$ and $r < r_2$.

For every $x \in B(x_0, r_0/2)$ and $r < r_2$, we denote $\pi_{x,r}$ the line through the two points of $\Sigma \cap \partial B(x, r)$. We claim that $\pi_{x,r}$ converges to some tangent line π_x at x when r goes to 0. To see this, let us introduce for two lines π_{x,s_1} and π_{x,s_2} the distance

$$\operatorname{dist}(\pi_{x,s_1}, \pi_{x,s_2}) := d_H(\bar{\pi}_{x,s_1} \cap B(0, 1), \bar{\pi}_{x,s_2} \cap B(0, 1))$$

where d_H the Hausdorff distance, and $\bar{\pi}_{x,r}$ is the line parallel to $\pi_{x,r}$ through the origin. In other words $\mathrm{dist}(\pi_{x,s_1}, \pi_{x,s_2}) \simeq \alpha(\pi_{x,s_1}, \pi_{x,s_2})$ where α is the smallest angle between the two lines π_{x,s_1} and π_{x,s_2} thus endowed with this distance the set of lines in $\mathbb{R}^2$ centered at the origin is a complete metric space. Now since $\beta(x,r) \leq Cr^\alpha$ we claim that for any $s_1 < s_2$,

$$\mathrm{dist}(\pi_{x,s_1}, \pi_{x,s_2}) \leq C s_2^\alpha. \tag{10}$$

Indeed, for all $s \leq r_2$ it is clear that

$$\mathrm{dist}(\pi_{x,s/2}, \pi_{x,s}) \leq C s^\alpha. \tag{11}$$

Now if k is such that $2^{-(k+1)} s_2 < s_1 \leq 2^{-k} s_2$ we have

$$\mathrm{dist}(\pi_{x,s_1}, \pi_{x,s_2}) \leq C \sum_{j=0}^{k} \mathrm{dist}(\pi_{x,2^{-(j+1)}r_2}, \pi_{x,2^{-j}r_2}) \leq C \sum_{j=0}^{k} 2^{-j\alpha} s_2^\alpha \leq C s_2^\alpha$$

which proves (10).

Now (10) says that $\bar{\pi}_{x,r}$ is a Cauchy sequence and converges to some line $\bar{\pi}_x$ centered at the origin. Moreover, if π_x denotes the line parallel to $\bar{\pi}_x$ passing through x, for all $r < r_0$ we have that

$$\mathrm{dist}(\pi_x, \pi_{x,r}) \leq C r^\alpha$$

and

$$\begin{aligned} d_H(\Sigma \cap B(x,r), \pi_x \cap B(x,r)) &\leq 2 d_H(\pi_{x,r} \cap B(x,r), \pi_x \cap B(x,r))) \\ &\leq 4r\, \mathrm{dist}(\pi_{x,r}, \pi_x) \leq C r^{\alpha+1} \end{aligned}$$

thus π_x is a tangent line at x.

So $\Sigma \cap B(x_0, r_0/2)$ admits a tangent line π_x at every point x. To prove that $\Sigma \cap B(x, r_0/2)$ is a $C^{1,\alpha}$ regular curve, it suffice to show that the map $x \mapsto \pi_x$ is Hölder regular. Let y and z be two different points of $\Sigma \cap B(x_0, r_0/2)$ and let $\rho := |y - z|$. Assume first that $\rho \leq r_2/10$. We have that

$$\begin{aligned} \mathrm{dist}(\pi_y, \pi_z) \leq&\ \mathrm{dist}(\pi_y, \pi_{y,2\rho}) + \mathrm{dist}(\pi_{y,2\rho}, \pi_{z,2\rho}) + \mathrm{dist}(\pi_{z,2\rho}, \pi_z) \\ &\leq C\rho^\alpha + \mathrm{dist}(\pi_{y,2\rho}, \pi_{z,2\rho}). \end{aligned} \tag{12}$$

Now observe that taking a point z' between y and z and applying (10) at this point with $r = 4\rho$ we have that

$$\mathrm{dist}(\pi_{y,2\rho}, \pi_{z,2\rho}) \leq C[\beta(z', 4\rho) + \mathrm{dist}(\pi_{y,2\rho}, \pi_{z',4\rho}) + \mathrm{dist}(\pi_{z,2\rho}, \pi_{z',4\rho})] \leq C\rho^\alpha. \tag{13}$$

Therefore, (12) and (13) imply

$$\mathrm{dist}(\pi_y, \pi_z) \leq C|y - z|^\alpha. \tag{14}$$

Now if $\rho \geq r_2/10$ (14) is also true up to change C (depending on r_2), which means that $\Sigma \cap B(x_0, r_0/2)$ is $C^{1,\alpha}$. $\square$

As an application we can state the following.

Theorem 3.3. *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains no triple points and such that $\psi|_\Gamma$ is absolutely continuous with respect to $\mathcal{H}^1$ with density in $L^p(\Gamma, d\mathcal{H}^1)$. Then Γ is locally a $C^{1,\alpha}$ curve with $\alpha = \frac{p-1}{p}$.*

Proof. Let $x \in \Gamma$. Since Γ is open, we may assume that there is a ball $B(x, r)$ such that $r < r(x)$ and $\psi|_{\Sigma\cap B(x,r)}$ is absolutely continuous with respect to $\mathcal{H}^1$ in $B(x, r)$ and its density belongs to L^p. Then for all $x \in \Sigma \cap B(x, r/2)$ Hölder inequality gives, for all $y \in B(x, r)$ and $s < r/2$,

$$\psi(y, s) := \psi(B(y, s)) \leq \|\psi\|_p \mathcal{H}^1(\Sigma \cap B(y, s))^{\frac{1}{p'}} \leq Cs^{\frac{1}{p'}}$$

thus Proposition 3.2 applies which proves that Γ is $C^{1,\alpha}_{loc}$ with $\alpha = 1 - \frac{1}{p}$. □

As far as the reverse implication is concerned, we can prove the following.

Proposition 3.4. *If $\Sigma \cap B(x, r)$ is a $C^{1,1}$ regular curve then $\psi|_{\Sigma_r}$ is absolutely continuous with respect to $\mathcal{H}^1$ in $B(x, r/2)$ and its density belongs to L^∞.*

Proof. According to Theorem 2.56 of [1], it is enough to find a constant M such that for every $y \in \Sigma \cap B(x, r/2)$,

$$\limsup_{s\to 0} \frac{\psi(y, s)}{s} \leq M \tag{15}$$

and this is the case when $\Sigma \cap B(x, r)$ is $C^{1,1}$, because then arguing as for Proposition 3.1 we easily have that $\beta(y, s) \leq Cs$ for every $y \in \Sigma \cap B(x, r/2)$ and $s < r_2(\Sigma \cap B(x, r))$ thus $\psi(y, s) \leq Cr$ by (8). □

3.2. $\psi(x, t)d\mathcal{H}^1(x)\frac{dt}{t}$ satisfies a Carleson measure condition

As Proposition 3.1 and 3.2 say, the regularity of Σ depends on the behavior of $\psi(x, r)$ with respect to r. The next proposition shows that in average, $\psi(x, t)^2$ is very small with respect to t at every scale, at least sufficiently small to make a certain integral converging. In other words $\psi(x, t)^2\chi_{[x,r(x)]}(t)d\mathcal{H}^1(x)\frac{dt}{t}$ is a Carleson Measure.

Proposition 3.5. *For all $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$. Then there exists $r_0(x) \leq r(x)$ such that*

$$\int_{y\in\Sigma\cap B(x,r)} \int_{0<t<r} \psi(y, t)^2 d\mathcal{H}^1(y)\frac{dt}{t} \leq Cr \quad \forall r \in (0, r_0(x)).$$

Proof. Since Σ is a uniformly rectifiable set of dimension 1 in $\mathbb{R}^2$, there is a constant C (see [7]) such that

$$\int_{y\in\Sigma\cap B(x,r)} \int_{0<t<r} \beta(y, t)^2 d\mathcal{H}^1(y)\frac{dt}{t} \leq Cr \tag{16}$$

for $x \in \Sigma$ and $r \in (0, r_0)$. Actually the β in (16) is normally the one of P. Jones which is slightly different than our β but smaller than 2 times ours thus (16) holds.

We denote by $\Sigma_r := \Sigma \cap B(x,r)$. Now possibly by taking a smaller r_0 (depending on $r(x)$) and using inequality (8) we compute

$$\begin{aligned}&\int_{y\in\Sigma_r}\int_{0<t<r}\psi(y,t)^2 d\mathcal{H}^1(y)\frac{dt}{t}\\ \leq C&\int_{y\in\Sigma_r}\int_{0<t<r}(t^2+t\beta(y,t)+\beta(y,t)^2)d\mathcal{H}^1(y)\frac{dt}{t}\leq Cr.\end{aligned}$$

□

4. About the diameter of the transported set and applications

In [10], it is proved that Σ is $C^{1,1}$ provided that the diameter of the transported set is small. In the next section we are interested in the opposite situation, when the diameter is very large. In this case Σ stays under some very large "tangent circles" that makes it close to be a convex graph.

4.1. Considerations for large diameters

We first want to give a notion of *Right* and *Left* tangents at a point $x \in \Sigma$ when its blow up is a line or a corner. To do this, we need to give an orientation on Σ to say in which direction Σ is followed.

Definition 4.1. For any injective parametrization $\gamma : [0,T] \to \Sigma$ of a piece of Σ that contains no triple point we define the Right and Left Tangent at point $x = \gamma(t_0) \in \gamma(]0,T[)$ associated to γ and denote by $T_R(x)$ and $T_L(x)$ the half lines

$$T_R(x) := x + \mathbb{R}^+.\lim_{h\to 0^+}\frac{\gamma(t_0+h)-\gamma(t_0)}{h},$$

$$T_L(x) := x + \mathbb{R}^+.\lim_{h\to 0^+}\frac{\gamma(t_0-h)-\gamma(t_0)}{h}.$$

Remark 4.2. Notice that the existence of Right and Left tangents comes from the existence of blow up limits (line or corner) at each points and that the dependance on γ is only relying on orientation.

In the sequel we will prove that $x \mapsto T_R(x)$ and $x \mapsto T_L(x)$ admits some Left and Right limits at every point (see Theorem 4.11) and are semi-continuous but let us check first that it is the case in basic situations when the diameter of transported set is under control.

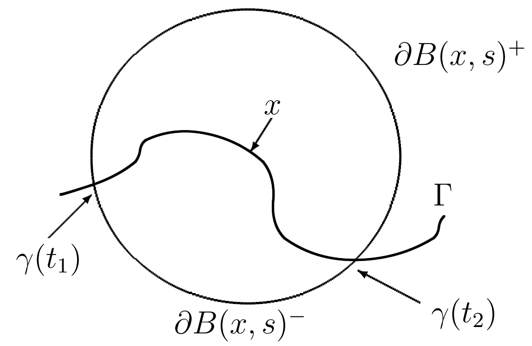
For any parametrization $\gamma : [0,T] \to \Sigma$ we will use the notation

$$x <_\gamma y$$

to say that $x = \gamma(t)$ and $y = \gamma(t')$ with $t < t'$.

Since our result is local, it is not restrictive to consider the situation in a small ball $B(x,r)$ around $x := \gamma(t) \in \Sigma$. We need to define a sort of local orientation on Σ.

Recall that for every $x \in \Sigma$ which is neither a triple point, nor an endpoint, Lemma 2.9 gives a radius $r(x)$ such that $\sharp \Sigma \cap \partial B(x,r) = 2$ for all $r < r(x)$. It follows that for all $r < r(x)$, $B(x,r) \backslash \Sigma$ is cut by Σ in exactly two connected components. Suppose now that $x \in \Sigma$ and $r \in (0, r(x))$ are such that $\Sigma \cap B(x,r) = \Gamma \cap B(x,r)$ where $\Gamma := \gamma([0,T])$. Let $s < r$ be given and let $\gamma(t_1)$ and $\gamma(t_2)$ be the two points of $\partial B(x,s) \cap \Gamma$. Assume in addition that $t_1 < t_2$. Then we denote $\partial B(x,s)^{\pm}$ the two connected components of $\partial B(x,s) \backslash \Gamma$, in such a way that $\partial B(x,s)^+$ is corresponding to the piece of circle obtained when we start at $\gamma(t_1)$ and follow the circle in the clockwise sense as in the following picture.



Then we define $B(x,s)^{\pm}$ as being the connected components of $B(x,s) \backslash \Gamma$ labeled in such a way that the boundary of $B(x,s)^+$ meets $\partial B(x,s)^+$. Observe that by this way, if $s' < s$ then $\partial B(x,s')^+ \subset B(x,s)^+$. This follows from the fact that the points z_s and z'_s lying on $\Sigma \cap \partial B(x,s)$ are continuous with respect to s. It is worth mentioning that the orientation does not depend on point x, in other words if $B(x,s)$ and $B(x',s')$ are both contained in $B(x_0,r_0)$ with $r_0 < r(x_0)$, then $B(x,s)^+ \cap B(x',s')^- = \emptyset$ and viceversa.

General Assumptions 4.3. We will say that we are under General Assumptions 4.3 in $B(x_0,r_0)$ when $\gamma : [0,T] \to \Sigma$ is a given parametrization as in Definition 4.1, $\Sigma \cap B(x_0,r_0) = \gamma[t_1,t_2]$ for some $t_1, t_2 \in [0,T]$ and $\gamma([t_1,t_2])$ contains neither triple points nor endpoints. We also assume that $r_0 \leq r(x_0)$. In this situation we have an orientation, namely $B(x_0,s)^{\pm}$ are well defined for all $s \leq r_0$. We also denote $\Gamma := \gamma([0,T])$.

Notice that for every $x \in \Sigma \backslash (\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ one can always find a parametrization γ and a radius r in such a way that $B(x,r)$ satisfies General Assumptions 4.3.

Definition 4.4. Assume that we are under General Assumptions 4.3 in $B(x_0,r_0)$. Then for every $y \in \Gamma \cap B(x_0,r_0)$ and for every transport ray R_y ending at y we say that R_y is coming from above if $R_y \cap B(x_0,r_0)^+ \neq \emptyset$ and we say that R_y is coming from below if $R_y \cap B(x_0,r_0)^- \neq \emptyset$. If R_y and R_z are two different transport rays that are both coming from below or both coming from above we will say that R_y and R_z are coming from the same direction. We denote $k^{-1}(y)^+$ the family of transported Rays ending at y and coming from above and $k^{-1}(y)^-$ the ones coming from below.

Remark 4.5. Of course a non empty ray cannot comes from above and below at the same time. The definition of Above and Below depends only on the orientation given by γ.

We will need this elementary fact which was already used in a slightly different version in [10].

Proposition 4.6. *Assume that Ω is convex and that we are under General Assumptions* 4.3 *in $B(x_0, r_0)$. Then $x \mapsto \mathrm{diam}(k^{-1}(x)^{\pm})$ are upper-semicontinuous for $x \in \Sigma \cap B(x_0, r_0)$.*

Proof. It is enough to prove the Proposition for $\mathrm{diam}(k^{-1}(x)^{+})$. Assume the contrary. Namely, there exists $\delta > 0$ and a sequence of points x_n that converges to x_∞ in $\Sigma \cap B(x_0, r_0)$ and such that

$$\mathrm{diam}(k^{-1}(x_n)^{+}) \geq \mathrm{diam}(k^{-1}(x_\infty)^{+}) + \delta. \tag{17}$$

Let y_n be a sequence of points in $k^{-1}(\{x_n\})^{+}$ such that $d(x_n, y_n) = \mathrm{diam}(k^{-1}(x_n)^{+})$. Up to a subsequence we can assume that y_n converges to a certain y_∞, and by continuity of $x \mapsto \mathrm{dist}(x, \Sigma)$ we deduce that $y_\infty \in k^{-1}(x_\infty)$. Moreover y_∞ is still coming from above. Then from (17) and

$$d(y_n, x_n) \leq d(y_n, x_\infty) + d(x_\infty, x_n)$$

we obtain

$$\mathrm{diam}(k^{-1}(x_\infty)^{+}) + \delta \leq d(y_n, x_\infty) + d(x_\infty, x_n),$$

thus passing to the limit it comes

$$\mathrm{diam}(k^{-1}(x_\infty)) + \delta \leq d(y_\infty, x_\infty)$$

which is a contradiction. □

For all $x \in \Sigma$ and R_x a transported ray arriving at x we will denote $\nu(R_x)$ the unit "normal" vector oriented by R_x and defined by the identity

$$R_x = x + [0, \mathcal{H}^1(R_x)].\nu(R_x).$$

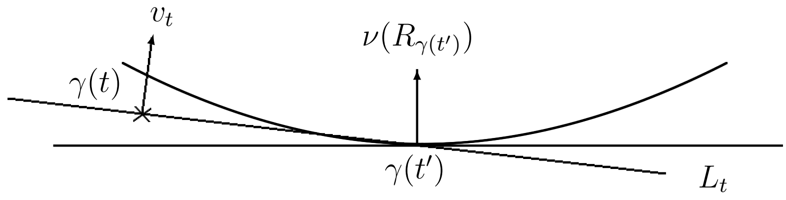
In most of our next arguments we will need the following "key lemma".

Lemma 4.7. *Assume that we have General Assumptions* 4.3 *in $B(x_0, r_0)$ with $x_0 \in \Sigma$ and a parametrization γ. Then for every $C_3 > r_0$ the following holds. Let $t < t'$ be such that $\gamma(t)$ and $\gamma(t')$ lie in $B(x_0, r_0)$ and admit some transport rays R_t and $R_{t'}$ that are both coming from above and satisfying* $\min(\mathcal{H}^1(R_t), \mathcal{H}^1(R_{t'})) \geq C_3$. *Then:*

$$\mathrm{Angle}(\nu(R_{\gamma(t')}), \nu(R_{\gamma(t)})) \leq 2 \arcsin\left(\frac{1}{C_3}|\gamma(t) - \gamma(t')|\right) \tag{18}$$

where $\mathrm{Angle}(v, w)$ *denotes the oriented angle between the two vectors v and w.*

Proof. Let t and t' be as in the statement of the Lemma. We know that $\gamma(t)$ is under a circle of radius bigger than C_3 "tangent" to $\gamma(t')$ and viceversa. Let us assume without loss of generality that $\gamma(t')$ is the origin and $R^{\perp}_{\gamma(t')}$ is the first axis. Let L_t be the line containing the two points $\gamma(t)$ and $\gamma(t')$ and let v_t be the unit vector orthogonal to L_t pointing in the "above" direction, which means pointing in the clockwise sense on the circle $B(\gamma(t'), |\gamma(t) - \gamma(t')|)$. The only way for the angle $\mathrm{Angle}(\nu(R_{\gamma(t')}), v_t)$ to be positive is when $\gamma(t)$ has negative first coordinate and positive second coordinate as in the following picture



and since $\gamma(t)$ must be at the same time lying under the "tangent" circle associated to $\gamma(t')$ we deduce that

$$\text{Angle}(\nu(R_{\gamma(t')}), v_t) \leq \arcsin\left[\frac{1}{C_3}|\gamma(t)-\gamma(t')|\right].$$

By the same kind of argument considering this time the circle associated to $\gamma(t)$ we also have that

$$\text{Angle}(v_t, \nu(R_{\gamma(t)})) \leq \arcsin\left[\frac{1}{C_3}|\gamma(t)-\gamma(t')|\right]$$

which all together gives (18), and the Lemma is proved. $\square$

Now we can state a first regularity result.

Proposition 4.8. *Assume that we have General Assumptions* 4.3 *in* $B(x_0, r_0)$ *and that*

$$\inf\{\text{diam}(t^{-1}(y)^+); y \in \Sigma \cap B(x_0, r_0)\} > 0. \tag{19}$$

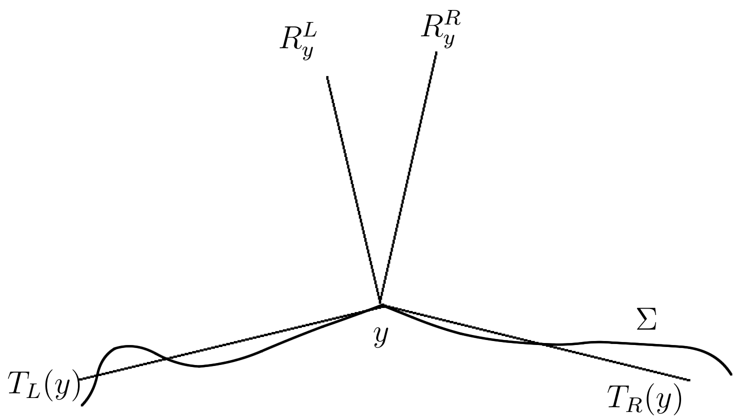
then $x \mapsto T_R(x)$ *and* $x \mapsto T_L(x)$ *are semicontinuous, i.e. for every* $y_0 \in B(x_0, r_0)$,

$$\lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_L(y) = T_L(y_0) \quad \textit{and} \quad \lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_R(y) = T_R(y_0). \tag{20}$$

In addition the limit from the other side exists and we have

$$\lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_L(y) = T_R(y_0) \quad \textit{and} \quad \lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_R(y) = T_L(y_0). \tag{21}$$

Proof. Up to change the orientation it is enough to prove the result for T_L. For any corner point $y \in \Sigma \cap B(x, r)$ let us denote R_y^R and R_y^L the two transported Rays orthogonal to $T_R(y)$ and $T_L(y)$ as in the following picture



Under assumption (19), if R_y^R and R_y^L are not empty they can only arrive from above. We denote $\mathcal{R}_1$ the union of all the R_y^R and R_y^L for y a corner point in $\Sigma \cap B(x_0, r_0)$. Then, for every ordinary point $y \in \Sigma \cap B(x_0, r_0)$ we denote R_y^+ the single ray coming from above and arriving at y and we denote $\mathcal{R}_2$ the union of all the R_y^+ for all ordinary point y. Finally, we denote $\mathcal{R} := \mathcal{R}_1 \cup \mathcal{R}_2$.

We claim that (19) implies the following stronger condition

$$\inf\{\mathcal{H}^1(R_y); R_y \in \mathcal{R}\} \geq \delta/2 \tag{22}$$

where

$$\delta := \inf\{\mathrm{diam}(t^{-1}(y)^+); y \in \Sigma \cap B(x, r)\} > 0.$$

Indeed, if y is an ordinary point then $\mathrm{diam}(t^{-1}(y)^+) = \mathcal{H}^1(R_y^+)$ so the problem could only occur at corner points. Now let y be a corner point and assume by contradiction that $\mathcal{H}^1(R_y^R) < \delta/2$ (the argument will work by the same way for R_y^L). Then by semicontinuity of the length of transported rays (Proposition 4.8), all the transported rays coming from above and arriving in a sufficiently small neigborhood at the right hand side of y still has a length strictly less than δ. Now to get a contradiction with (19) it suffice to choose an ordinary point z in this neighborhood for which we know that the lenght of R_z^+ is exactly $\mathrm{diam}(t^{-1}(z)^+) \geq \delta$. It is always possible to find such a point z because ordinary points of Σ have full $\mathcal{H}^1$ measure.

Now to prove the existence of limit we will use the "key lemma". Let y_0 be a fixed point in $B(x_0, r_0)$. Since the result is local we can restrict ourself to $B(y_0, s)$ for a radius s small as we want. For instance we can take $s \leq \delta/100$. Now by convention, when y is an ordinary point we set $R_y^R = R_y^L = R_y^+$ and we define for $y \in B(y_0, s)$ and $y \leq_\gamma y_0$ the function

$$\theta(y) := \mathrm{Angle}(\nu(R_{y_0}^L), \nu(R_y^L)).$$

We want to prove that $\theta(y)$ has a limit when $y \to y_0$, and $y <_\gamma y_0$. Let

$$M := \sup\{\theta(y); y \in B(y_0, s) \text{ and } y <_\gamma y_0\}.$$

It is clear that

$$\limsup_{\substack{y \to y_0 \\ y <_\gamma y_0}} \theta(y) \leq M. \tag{23}$$

Now by definition of M, for every $\varepsilon > 0$ one can find $y_\varepsilon \in \Sigma \cap B(y_0, s)$ such that $\theta(y_\varepsilon) \geq M - \varepsilon$. On the other hand for all $y >_\gamma y_\varepsilon$ Lemma 4.7 implies

$$\theta(y_\varepsilon) \leq \theta(y) + 2 \arcsin\left(\frac{1}{C_3}|y - y_\varepsilon|\right)$$

which leads to

$$M - \varepsilon \leq \liminf_{\substack{y \to y_0 \\ y <_\gamma y_0}} \theta(y)$$

and since ε is arbitrary, combining the last inequality with (23) and letting ε goes to 0 we obtain that the Left limit of $\theta(y)$ exists and is equal to M, which means that the Left limit

$$\lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_L(y)$$

exists. For the existence of Right limit of T_L one can argue by the same way using this time the infimum instead of supremum.

Let us prove now that

$$\lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} T_L(y) = T_L(y_0). \tag{24}$$

The proof of

$$\lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_L(y) = T_R(y_0) \tag{25}$$

will follow by the same argument.

We already know that

$$\limsup_{\substack{y \to y_0 \\ y <_\gamma y_0}} \theta(y) \leq \theta(y_0) = 0$$

so it is enough to prove the reverse inequality, for which we argue as follows. Let y_k be a sequence of points converging to y_0 and let z_k be a corresponding sequence of points belonging to a transport Ray $R^L_{y_k}$ ending at y_k. By continuity of $x \mapsto \mathrm{dist}(x, \Sigma)$ we obtain that the z_k converges to a point z which belongs to $k^{-1}(y_0)^+$. This implies that

$$\limsup_{\substack{y \to y_0 \\ y <_\gamma y_0}} \theta(y) \geq 0$$

which ends the proof. □

Remark 4.9. A consequence of the proposition just proved is that if we assume Σ to contain no corner points in $B(x,r)$, then under assumption (19) Σ is C^1 in $B(x,r)$ because in this case $T_L(x) = T_R(x)$ at every point.

Now if (19) holds from above and below at the same time we have more regularity as it is stated in the following proposition.

Proposition 4.10. *Assume that we have General Assumption* 4.3 *in* $B(x_0, r_0)$. *If*

$$\inf\{\min(\mathrm{diam}(t^{-1}(x)^+), \mathrm{diam}(t^{-1}(x)^-)); x \in \Sigma \cap B(x,r)\} > 0 \tag{26}$$

then Σ *is* $C^{1,1}$ *in* $B(x_0, r_0/2)$.

Proof. Observe that under assumption (26), for every point $y \in \Sigma \cap B(x_0, r_0/2)$ we have that Σ is lying in the complement of two circles with radius uniformly bounded from below and tangent to each other at y. From this fact one can find a radius r_1 such that $\beta(y,r) \leq Cr$ for all $r < r_1$ and the proposition follows from the same argument as for Proposition 3.2. □

4.2. A regularity result

This paragraph is devoted to the proof of the following result.

Theorem 4.11. *For any minimizer* Σ *and for every injective and open arc* $\Gamma \subset \Sigma$, *the Right and Left Tangents* $T_R(x)$ *and* $T_L(x)$ *admit some Right and Left limits at every point* $x \in \Gamma$ *and are semicontinuous. More precisely* (20) *and* (21) *holds for every point* $y_0 \in \Gamma$.

To prove Theorem 4.11 we will first need a precision about the $C^{1,1}$ regularity result of [10].

Lemma 4.12. *Let Σ_1 be compactly contained in $\Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$. Let $x \in \Sigma_1$ be such that* $\mathrm{diam}(k^{-1}(x)) < \min(C, C_2(\Sigma_1))$ *and let $I \subset \Sigma$ be an "interval" in Σ_1 (i.e. an injective Lipschitz image of* $[0,1]$*) containing x maximal for the property that*

$$\sup_{y\in I} \mathrm{diam}(k^{-1}(y)) \leq \min(C, C_2(\Sigma_1))$$

and let $z \in \bar{I}\backslash I$. Then Σ is $C^{1,1}$ regular up to z, with Lipschitz constant for the derivative depending only on Σ, Ω and μ, in particular does not depend on I and x.

Proof. The Lemma is an easy consequence of the regularity result of [10] so let us give only a sketch of the proof. Assume that $\bar{I}$ is parameterized by an injective map $\gamma : [0,1] \to \bar{I}$ and assume that $z = \gamma(0)$. We already know by the result of [10] that γ is $C^{1,1}$ in the interior of I. Moreover, by existence of blow up limits we know that γ' exists at 0. Denoting $T_R(z)$ the half tangent line at z, and using Lemma 2.7 we have that

$$\frac{1}{\delta} \sup_{t\in[0,\delta]} \mathrm{dist}(\gamma(t), T_R(z)) \leq C\psi(\gamma((0,\delta]).$$

On the other hand, one can easily prove the estimate

$$\psi(\gamma((0,\delta]) \leq Cr \quad \forall \delta < \delta_0 \tag{27}$$

by a small modification of the proof of Proposition 2.12. Indeed, the only difference is to find an analogous "one-sided" version of inequality (8). This is done by delimiting one side of the domain $k^{-1}(\gamma(0,\delta))$ with exactly the same argument as for the original proof of (8), and for the other side the rays are delimited by the line orthogonal to the left tangent $T_L(z)$. Then the proof of (27) follows by the same way as the proof of Proposition 2.12, the iteration still works since the diameter of transported set is small enough for the points of $\gamma((0,\delta]$ by our assumptions. We left the details to the reader.

Once (27) is proved, the desired $C^{1,1}$ regularity follows by the same argument as in the proof of Proposition 3.1. □

We are now ready to prove our regularity result.

Proof of Theorem 4.11. We can assume that we are working on Σ_1 compactly contained in the complement of $\mathbb{T}_\Sigma$ and $\mathbb{E}_\Sigma$ since we already know by [10] that the curves that compose Σ are $C^{1,1}$ in a neighborhood of any triple point. Let C_0 be the constant depending on Ω, μ and Σ that comes from the regularity result of [10] (i.e. that implies $C^{1,1}$ regularity whenever $\mathrm{diam}(k^{-1}(\{x\})) < C_0$) and let $C < \min(C_0, C_2(\Sigma_1))$. Since the result is local, when x_0 is not an endpoint we can work under General Assumption 4.3 in a ball $B(x_0, r_0)$, and we can assume that $r_0 < C/100$.

Then let us decompose $\Gamma \cap B(x_0, r_0)$ in a disjoint union

$$\Gamma \cap B(x_0, r_0) := O_1 \cup A^+ \cup A^- \cup F$$

where

$$O_1 := \{x \in \Gamma \cap B(x_0, r_0); \operatorname{diam}(t^{-1}(x)) < C\},$$
$$A^+ := \{x \in \Gamma \cap B(x_0, r_0); \operatorname{diam}(t^{-1}(x)^-) < C/4 \text{ and } \operatorname{diam}(t^{-1}(x)^+) \geq C/2\},$$
$$A^- := \{x \in \Gamma \cap B(x_0, r_0); \operatorname{diam}(t^{-1}(x)^+) < C/4 \text{ and } \operatorname{diam}(t^{-1}(x)^-) \geq C/2\},$$
$$F := \Sigma \cap B(x_0, r_0) \backslash (O_1 \cup A^+ \cup A^-).$$

In particular in F, all the points have very big transported sets from above and below. By semicontinuity of the diameter of transported set (Proposition 4.6), we get that O_1, $O_1 \cup A_+$ and $O_1 \cup A^-$ are relatively open sets in $\Gamma \cap B(x_0, r_0)$ thus F is relatively closed by definition. We will first prove that taken separately in the interior of all the above sets, Σ is C^1 regular. Indeed by definition of constant C, from [10] we directly know that O_1 is $C^{1,1}_{loc}$, which means that the maps T_L and T_R are continuous (even Lipschitz) in O_1. In the interior of F, A^+ and A^-, we know that the maps T_L and T_R are semicontinuous by Proposition 4.8. Now we have to glue together those sets to prove that T_L and T_R are semicontinuous everywhere. Up to a change of orientation it is enough to prove the result for only T_L.

Let us consider O_1 as a countably union of disjoints "intervals" like

$$O_1 := \sum_{i \in \mathbb{N}} \gamma(]t_i, t_{i+1}[)$$

with $t_i < t_{i+1}$. We already know that γ is $C^{1,1}$ in each of the $I_i := \gamma([t_i, t_{i+1}])$ (also up the the boundaries of each interval thanks to Lemma 4.12). Now we will enlarge the set of points in which T_L is semicontinuous to progressively achieve the semicontinuity everywhere. Let us start with the open set $A^+ \cup O_1$. Let $y_0 \in A^+ \backslash O_1$ (otherwise γ is $C^{1,1}$ in the neighborhood of x and we have nothing to prove). For r small enough we know that $B(x, r) \cap \Sigma \subset A^+ \cup O_1$. We want to prove that $T_L(y)$ tends to $T_L(y_0)$ when $y \to y_0$ and $y <_\gamma y_0$. We use the same notations as for the proof of Proposition 4.8, i.e. we denote the oriented angle

$$\theta(y) := \operatorname{Angle}(\nu(R^L_{y_0}), \nu(R^L_y))$$

and we want to prove that $\theta(y)$ tends to $\theta(y_0) = 0$.

First of all, since $y_0 \in A^+$, for all subsequence $y_n \to y_0$ with $y_n <_\gamma y_0$ and such that for all $n > 0$, $y_n \in A^+ \cap B(y_0, r)$ we can prove that $\theta(y_n)$ converges to $\theta(y_0) = 0$ arguing exactly as for Proposition 4.8. This means that for every $\varepsilon > 0$ there exists $r_\varepsilon < r$ such that

$$\sup\{|\theta(y)|; y \in A^+ \cap B(y_0, r_\varepsilon) \text{ and } y <_\gamma y_0\} < \varepsilon. \tag{28}$$

Now we have to control the angle for points in O_1. Since $y_0 \in A^+$, we deduce that

$$I_i \cap B(y_0, r_\varepsilon) \neq \emptyset \Rightarrow I_i \subset \{y; y <_\gamma y_0\} \text{ or } I_i \subset \{y; y >_\gamma y_0\}.$$

Then by Lemma 4.12, for all $y \in I_i$ with $I_i \subset O_1 \cap \{y; y <_\gamma y_0\} \cap B(y_0, r_\varepsilon)$, one can estimate (possibly taking a smaller r_ε)

$$|\theta(y) - \theta(y_i)| \leq C\mathcal{H}^1(I) \leq \varepsilon$$

where y_i is the right hand side bound of the interval I_i, in other words the point maximal in I_i for the order $<_\gamma$. Then it comes, for all $y \in O_1 \cap B(x, r_\varepsilon) \cap \{y <_\gamma y_0\}$,

$$|\theta(y)| \leq |\theta(y) - \theta(y_i)| + |\theta(y_i)| \leq 2\varepsilon$$

because in particular $y_i \in A^+$ so that we can apply (28) to estimate $|\theta(y_i)|$, and finally we have proved that

$$\lim_{\substack{y \to y_0 \\ y <_\gamma y_0}} \theta(y) = 0$$

which implies the semicontinuity of T_L in $O_1 \cup A^+$.

By a similar argument we can also prove that for every $y_0 \in O_1 \cup A^+$,

$$\lim_{\substack{y \to y_0 \\ y >_\gamma y_0}} T_R(y) = T_R(y_0). \tag{29}$$

Thus reversing the orientation and applying (29) we obtain the semicontinuity for T_L, when $y <_\gamma y_0$ at every point y_0 of $A^- \cup O_1$ as well. Since $A^\pm \cup O_1$ are open sets we have proved that T_L is semicontinuous in $O_1 \cup A^+ \cup A^-$. To have the semicontinuity of T_L everywhere it remains to prove the semicontinuity at points of F.

We already know that T_L is semicontinuous in the interior of F (Σ is even $C^{1,1}$ in this case by Lemma 4.10). So it is enough to prove semicontinuity at point $y_0 \in F \cap \overline{A^+ \cup O_1 \cup A^-}$. But this will be done by the same arguments as before. All we have to prove is that for r_ε small enough and for all points $y \in B(y_0, r_\varepsilon) \cap \{y <_\gamma y_0\}$, we have that $|\theta(y)| \leq \varepsilon$. Since the point $y_0 \in F$ is achieved by two large transport rays from above and below at the same time, we can control the angle of tangents for every point $y \in B(y_0, r_\varepsilon) \cap \{y <_\gamma y_0\}$ by considering the four different cases whenever y lie in O_1, A^+, A^- or F. Indeed, in each situation between O_1, A^+ and A^- we can use one of the arguments that we already used before to prove semicontinuity in $O_1 \cup A^+ \cup A^-$, and for points of F we can use either the argument associated to A^+ or the one of A^-.

In conclusion we have proved that (20) holds at every point of Γ, and the proof of (21) works by the same way. $\square$

As an immediate consequence of Theorem 4.11 we can state the following interesting result.

Corollary 4.13. *Let Σ be an optimal set for the problem* (1) *and let B be a ball such that $\Sigma \cap B$ contains only ordinary points. Then $\Sigma \cap B$ is locally a C^1 regular curve.*

5. Euler-Lagrange equation

We will need the equation of first derivative that one can find in [2]. We refer for instance to [1] page 355 for the definition and classical properties of the tangential divergence $\mathrm{div}^\Sigma \Phi$.

Proposition 5.1 ([2]). *For every compact and connected set $\Sigma \subset \Omega$ and for every $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ one has*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{F}((\mathrm{Id} + \varepsilon\Phi)(\Sigma))\Big|_{\varepsilon=0} = \int_{\mathbb{R}^2} \left\langle \Phi(k(x)), \frac{k(x) - x}{|k(x) - x|} \right\rangle \mathrm{d}\mu(x). \tag{30}$$

As a consequence, for a given $\lambda > 0$, if Σ is a minimizer for the functional

$$\mathcal{G}(\Sigma') := \int_\Omega d(x, \Sigma') \, \mathrm{d}\mu(x) + \lambda \mathcal{H}^1(\Sigma') \tag{31}$$

over all compact and connected sets $\Sigma' \subset \Omega$, then for all $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ one has

$$\int_{\mathbb{R}^2} \left\langle \Phi(k(x)), \frac{k(x) - x}{|k(x) - x|} \right\rangle \mathrm{d}\mu(x) + \lambda \int_\Omega \mathrm{div}^\Sigma \Phi \, \mathrm{d}\mathcal{H}^1 = 0. \tag{32}$$

We would like to apply equation (32) to the minimizers of our functional $\mathcal{F}$ defined in (1) with length constraint instead of the penalized functional $\mathcal{G}$. The following Proposition was suggested to the Author by F. Santambrogio and says that one can find a λ_0 such that the two first order equations for the two minimizing problems are the same. To get a similar result one could also try to apply the classical Lagrange multipliers theorem on a suitable Banach space of diffeomorphisms to the functional $J(\varphi) := \mathcal{F}(\varphi(\Sigma))$ but the Fréchet differentiability of such functional at $\varphi_0 := \mathrm{Id}$ is not clear. Moreover, despite of the technical difficulties of the proof of the next proposition, the idea is very intuitive and perhaps more instructive as well since it gives the explicit value of λ_0 in terms of measure ψ at any endpoint x_0.

To be more precise, let x_0 be an endpoint of Σ that we will assume, up to a translation, being the origin. Following [10], let us denote by ν the image measure of $\mu \llcorner k^{-1}(\{x_0\})$ by the application $x \mapsto \frac{x}{\|x\|}$ and define the vector

$$\bar{v} := \int_{S^1} v \, \mathrm{d}\nu(v).$$

By [10] Theorem 3.2. we know that Σ admits a tangent line at x_0 which direction is given by the vector $-\bar{v}$. Now we define the constant

$$\lambda_0 := \int_{S^1} v . \frac{\bar{v}}{\|\bar{v}\|} \, \mathrm{d}\nu(v) = \|\bar{v}\|. \tag{33}$$

Proposition 5.2. *Let Σ be a minimizer for the problem* (1) *and x_0 be one of its endpoint. Then Equation* (32) *holds with $\lambda = \lambda_0$ defined in* (33) *and for every $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ compactly supported in the complement of $\{x_0\}$.*

The idea is to quantify how much one can win or loose in the functional adding a piece of segment of size r starting at the endpoint x_0 or erasing a piece of curve of size r from the same endpoint. We will prove that the two operations have a cost in $\lambda_0 r + o(r)$ and this is the purpose of the two next lemmas. Actually one can find similar computations in the proof of Theorem 3.5 of [10] (with a more elliptic redaction) but we would like to re-write here the arguments in full details for the convenience of the reader.

Let x_0 be an endpoint of Σ that we still assume being the origin, and let $\Phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be a given diffeomorphism supported in a compact K contained in the complement of $\{0\}$. We denote C_K^∞ the family of diffeomorphisms $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ supported in K satisfying $\varphi(K) \subset K$. Let us define

$$\varphi_\varepsilon := \mathrm{Id} + \varepsilon\Phi.$$

Notice that if ε is small enough, then $\varphi_\varepsilon \in C_K^\infty$. For every ε we denote k_ε a measurable selection of the projection multimap onto $\Sigma_\varepsilon := \varphi_\varepsilon(\Sigma)$.

Let us define now ν_ε the image measure of $\mu \llcorner k_\varepsilon^{-1}(\{0\})$ by the application $x \mapsto \frac{x}{\|x\|}$ and introduce the constant

$$\lambda_\varepsilon := \int_{S^1} v.\frac{\bar{v}}{\|\bar{v}\|}\, \mathrm{d}\nu_\varepsilon(v).$$

It is not difficult to see that $\lambda_\varepsilon \to \lambda_0$ when $\varepsilon \to 0$.

Lemma 5.3. *One can find an r_0 such that for every $r < r_0$ there exists a set Σ_ε^r such that*

$$\Sigma_\varepsilon^r \cap K = \Sigma_\varepsilon, \qquad \mathcal{H}^1(\Sigma_\varepsilon^r) = \mathcal{H}^1(\Sigma_\varepsilon) + r \quad \textit{and}$$
$$\mathcal{F}(\Sigma_\varepsilon) - \mathcal{F}(\Sigma_\varepsilon^r) = r\lambda_\varepsilon + o(r) \tag{34}$$

where $o(r)$ depends on Φ but not on ε.

Proof. Up to a rotation we can assume that $\bar{v} = e_1$. Defining the line $L := \mathbb{R}^-.\bar{v}$, by [10] we know that

$$\frac{1}{r} d_H(\Sigma \cap B(x_0, r), L \cap B(x_0, r)) \to 0$$

when $r \to 0$, and this is the same for $\varphi(\Sigma)$ for all $\varphi \in C_K^\infty$ since they do not move any points near the origin. Let P^+ be the half space

$$P^+ := \{(x, y) \in \mathbb{R}^2; x \geq 0\}. \tag{35}$$

We claim that for all ε,

$$k_\varepsilon^{-1}(\{0\}) \subseteq P^+.$$

Indeed, suppose the contrary, namely that there exists a point x such that $k_\varepsilon(x) = 0$ and $x \notin P^+$. Then, since $\{0\}$ admits $\mathbb{R}^- e_1$ as left-tangent line it would be better for x to be projected onto a point $y \in \Sigma \cap \partial B(0, s)$ for s small enough which is a contradiction.

We define

$$\Sigma_\varepsilon^r := \Sigma_\varepsilon \cup L_r^+$$

where $L_r^+ := [0, r] \times \{0\}$. We have to compute the winning in the functional $\mathcal{F}$ in terms of r but independently from ε small enough, say less than ε_0.

Let $D_r := (L_r^+ \times \mathbb{R}) \cap \Omega$. For every point $x \in P^+ \backslash D_r$ one has $d(x, L_r^+) = d(x, x_r)$ where $x_r := (r, 0)$. Then, a simple computation yields, for $r \to 0$

$$\|x - x_r\|^2 = \|x\|^2 - 2\langle x, x_r\rangle + o(r) = \|x\|^2 \left(1 - 2\left\langle \frac{x}{\|x\|^2}, x_r \right\rangle + o(r)\right). \tag{36}$$

Therefore, we obtain that for all $x \in k_\varepsilon^{-1}(\{0\})\backslash D_r$,

$$d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r) = \|x\| - \|x - x_r\| = r\left\langle \frac{x}{\|x\|}, e_1 \right\rangle + o(r)$$

where $o(r)$ does not depend on ε. On the other hand, let us define

$$A_r^\varepsilon := \left\{x \in \Omega\backslash k_\varepsilon^{-1}(\{0\}); d(x,\Sigma_\varepsilon^r) = d(x, L_r^+)\right\}$$

and

$$A_r := \bigcup_{\varepsilon \le \varepsilon_0} A_r^\varepsilon.$$

We claim that

$$\sup_{\varepsilon \le \varepsilon_0} \int_{A_r^\varepsilon} d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r)\, \mathrm{d}\mu(x) = o(r). \tag{37}$$

To see this, observe that $A_r^\varepsilon \subset P^+$ thus, using (36), for $x \in A_r^\varepsilon$ we have

$$0 \le d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r) \le \|x\| - d(x,\Sigma_\varepsilon^r) = \|x\| - d(x,x_r) = r\left\langle \frac{x}{\|x\|}, e_1 \right\rangle + o(r)$$

where $o(r)$ does not depend on ε. Then

$$\left|\int_{A_r^\varepsilon} d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r)\, \mathrm{d}\mu(x)\right| \le r\mu(A_r^\varepsilon) + o(r) \le r\mu(A_r) + o(r)$$

and we conclude by observing that $\mu(A_r) \to 0$ thus (37) is true.

Finally, since $\int_{D_r} d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r) = o(r)$ we have

$$\begin{aligned}\mathcal{F}(\Sigma_\varepsilon) - \mathcal{F}(\Sigma_\varepsilon^r) &= \int_{k_\varepsilon^{-1}(\{0\})\backslash D_r} d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r)\, \mathrm{d}\mu(x) \\ &\qquad + \int_{A_r^\varepsilon} d(x,\Sigma_\varepsilon) - d(x,\Sigma_\varepsilon^r)\, \mathrm{d}\mu(x) + o(r) \\ &= \lambda_\varepsilon r + o(r)\end{aligned}$$

which proves the Lemma. □

Now we want to do the same while removing this time from Σ_ε a piece of size r, and estimate the loss in terms of r independently from ε. Compared to Lemma 5.3 we will this time prove only an inequality which will be enough to prove Proposition 5.2.

Lemma 5.4. *One can find an r_0 such that for every $r < r_0$ there exists a set Σ_ε^r such that*

$$\Sigma_\varepsilon^r \cap K = \Sigma_\varepsilon, \qquad \mathcal{H}^1(\Sigma_\varepsilon^r) = \mathcal{H}^1(\Sigma_\varepsilon) - r \quad \textit{and}$$

$$F(\Sigma_\varepsilon^r) - F(\Sigma_\varepsilon) \le r\lambda_\varepsilon + o(r) \tag{38}$$

where $o(r)$ depends on Φ but not on ε.

Proof. The proof is very similar to the one of Lemma 5.3. The only difference is that here we don't consider a piece of segment but a piece of curve that converges to a segment with speed $o(r)$.

Let $\gamma : [0,T] \to \mathbb{R}^2$ be a parametrization by arclength of the Lipschitz curve starting at x_0, such that $\gamma(0) = x_0$ and $\gamma(T)$ is a triple point or endpoint. By [10], for t small enough, we know that

$$\sharp\{\partial B(x_0,t) \cap \Sigma\} = 1.$$

We deduce the existence of a radius t_r defined by $\Sigma \cap \partial B(x_0,t_r) = \gamma(r)$ and satisfying $\gamma([0,r]) = \Sigma \cap B(x_0,t_r)$. Moreover since the blow up limit at x_0 is a radius we also know that $r = t_r + o(r)$ when $r \to 0$. We assume r_0 small enough in such a way that $B(x,t_r) \cap K = \emptyset$ for all $r < r_0$ and we define

$$\Sigma_\varepsilon^r := \Sigma_\varepsilon \backslash \gamma([0,r]).$$

By construction we automatically get $\mathcal{H}^1(\Sigma_\varepsilon^r) = \mathcal{H}^1(\Sigma_\varepsilon) - r$.

Now we want to compute what we have lost in the functional $\mathcal{F}$. We still suppose $x_0 = \{0\}$ and the tangent line at x_0 being $\mathbb{R}^-.e_1$. We denote P^+ the half space defined in (35). As before, we know that for every ε small enough, $k_\varepsilon^{-1}(\{0\}) \subset P^+$. Let us denote $x_r := \Sigma \cap \partial B(0,t_r)$ and $\bar{x}_r = p_1(x_r)$ where p_1 is the projection on the first axis. We know that $\|x_r - \bar{x}_r\| = o(r)$ and $\|\bar{x}_r\| = t_r = r - o(r)$. By a computation similar to (36) and using that $o(r) = o(t_r)$ we obtain that for all $x \in P^+$,

$$\|x - \bar{x}_r\|^2 = \|x\|^2 \left(1 - 2\left\langle \frac{x}{\|x\|^2}, \bar{x}_r \right\rangle + o(r)\right)$$

which implies

$$\|x - \bar{x}_r\| - \|x\| = r\left\langle \frac{x}{\|x\|}, e_1 \right\rangle + o(r).$$

Then, since $\|x_r - \bar{x}_r\| = o(r)$ we deduce

$$\|x - x_r\| - \|x\| = r\left\langle \frac{x}{\|x\|}, e_1 \right\rangle + o(r).$$

Now we compute

$$\begin{aligned}
\mathcal{F}(\Sigma_\varepsilon^r) - \mathcal{F}(\Sigma_\varepsilon) &= \int_{k_\varepsilon^{-1}(B(0,t_r))} d(x,\Sigma_\varepsilon^r) - d(x,\Sigma_\varepsilon)\, \mathrm{d}\mu(x) \\
&\leq \int_{k_\varepsilon^{-1}(\{0\})} d(x,x_r) - d(x,0)\, \mathrm{d}\mu(x) + r\int_{k_\varepsilon^{-1}(B(0,t_r)\backslash\{0\})} \mathrm{d}\mu(x) \\
&\leq \int_{k_\varepsilon^{-1}(\{0\})} \|x - x_r\| - \|x\|\, \mathrm{d}\mu(x) + r\psi(B(0,t_r)\backslash\{0\}) \\
&\leq r\lambda_\varepsilon + o(r)
\end{aligned}$$

which ends the proof. □

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. Let Φ be given and consider $\Sigma_\varepsilon := \varphi_\varepsilon(\Sigma)$ where as before $\varphi_\varepsilon = \mathrm{Id} + \varepsilon\Phi$. Assume first that $\mathcal{H}^1(\Sigma_\varepsilon) - \mathcal{H}^1(\Sigma) = -r_\varepsilon < 0$. Then, denoting Σ_ε^+ the set given by Lemma 5.3 applied with $r := r_\varepsilon$ and using that Σ is a minimizer for $\mathcal{F}$ we obtain that

$$\begin{aligned}\mathcal{F}(\Sigma) &\leq \mathcal{F}(\Sigma_\varepsilon^+)\\ &= \mathcal{F}(\Sigma_\varepsilon) - \lambda_\varepsilon r_\varepsilon - o(r_\varepsilon)\\ &= \mathcal{F}(\Sigma_\varepsilon) + \lambda_\varepsilon[\mathcal{H}^1(\Sigma_\varepsilon) - \mathcal{H}^1(\Sigma)] + o(r_\varepsilon).\end{aligned}$$

Now if $\mathcal{H}^1(\Sigma_\varepsilon) - \mathcal{H}^1(\Sigma) = r_\varepsilon > 0$ we can use Lemma 5.4 to find a set Σ_ε^- satisfying the length constraint so by minimality of Σ,

$$\begin{aligned}\mathcal{F}(\Sigma) &\leq \mathcal{F}(\Sigma_\varepsilon^-)\\ &\leq \mathcal{F}(\Sigma_\varepsilon) + \lambda_\varepsilon r_\varepsilon + o(r_\varepsilon)\\ &\leq \mathcal{F}(\Sigma_\varepsilon) + \lambda_\varepsilon[\mathcal{H}^1(\Sigma_\varepsilon) - \mathcal{H}^1(\Sigma)] + o(r_\varepsilon).\end{aligned}$$

In conclusion, using that $r_\varepsilon = O(\varepsilon)$ and that $\lambda_\varepsilon \to \lambda_0$ we have proved for every ε,

$$\mathcal{F}(\Sigma) - \mathcal{F}(\Sigma_\varepsilon) + \lambda_0[\mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma_\varepsilon)] \leq o(\varepsilon).$$

Now dividing by $\pm\varepsilon$ and passing to the limit, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\mathcal{F}(\Sigma_\varepsilon) + \lambda_0\mathcal{H}^1(\Sigma_\varepsilon)\right]\Big|_{\varepsilon=0} = 0$$

and we conclude using (30) and the classical fact that the derivative of $\mathcal{H}^1(\Sigma_\varepsilon)$ is the mean curvature. □

Remark 5.5. The constant λ_0 does not depend on the choice of endpoint x_0. Indeed, if the constant λ_1 associated to a different endpoint $x_1 \neq x_0$ was greater or lower, one could get a contradiction with the minimality of Σ adding or erasing a little piece of curve at one endpoint and do the opposite operation at the other endpoint in order to diminish the functional $\mathcal{F}$.

6. Tilt estimate

In this section we control the oscillation of the tangent lines π_x to Σ with respect to a fixed line π, also called "the tilt". When π_1 and π_2 are two lines in $\mathbb{R}^2$, we denote by $\alpha(\pi_1, \pi_2) \in [0, \frac{\pi}{2}]$ the smallest angle between them.

For any $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ and $r < r(x)$ we denote by $\pi_{x,r}$ the line that contains the segment $[z, z']$, where z and z' are, as usual, the two points of $\partial B(x,r) \cap \Sigma$. For $\mathcal{H}^1$-a.e. $y \in \Sigma$ we also denote by π_y the approximate tangent line centered at y. Finally, we denote $\alpha(y) := \alpha(\pi_y, \pi_{x,r})$. The definition of $\alpha(y)$ depends in particular on x and r but we do not make it explicit to lighten the notations. A first easy estimate is the following

$$\int_{\Sigma\cap B(x,r)} 1 - \cos(\alpha(y))\, \mathrm{d}\mathcal{H}^1(y) \leq Cr\psi(x,r)\beta(x,r). \tag{39}$$

Indeed, let $\gamma : [-T,T]$ be a parametrization of $\Sigma_r := \Sigma \cap B(x,r)$. Assume without loss of generality that the segment $S := [z,z']$ is contained in the first axis of $\mathbb{R}^2$ and that $\gamma(-T) = z$, $\gamma(T) = z'$ with $z < z'$. Then by setting $\gamma(t) := (x(t), y(t))$, using Lemma 2.7 we have

$$\begin{aligned}\int_{-T}^{T} \sqrt{x'(t)^2 + y'(t)^2} - x'(t)dt &= \mathcal{H}^1(\Sigma_r) - (z'-z)\\ &= \mathcal{H}^1(\Sigma_r) - \mathcal{H}^1(S)\\ &\leq C\psi(x,r)d_H(\Sigma_r, S)\\ &\leq Cr\psi(x,r)\beta(x,r).\end{aligned}$$

On the other hand the area formula shows that

$$\begin{aligned}\int_{-T}^{T} \sqrt{x'(t)^2 + y'(t)^2} - x'(t)\ \mathrm{d}t &= \int_{-T}^{T} \left(1 - \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right)\sqrt{x'(t)^2 + y'(t)^2}\ \mathrm{d}t\\ &= \int_{\Sigma_r} (1 - \langle \tau(y), e_1\rangle)\ \mathrm{d}\mathcal{H}^1(y)\\ &\geq \int_{\Sigma\cap B(x,r)} 1 - \cos(\alpha(y))\ \mathrm{d}\mathcal{H}^1(y)\end{aligned}$$

where $\tau(y)$ is the unit tangent vector at point x (oriented by the parametrization γ) and e_1 is the first vector of basis, so (39) follows.

The next proposition gives a slightly better estimate than (39) proved by a variational argument.

Proposition 6.1. *For all $\tau \in (0,1)$, $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ and $r < r(x)$ we have*

$$\int_{\Sigma\cap B(x,\tau r)} \sin^2(\alpha(y))\ \mathrm{d}\mathcal{H}^1(y) \leq C(\tau)r\psi(x,r)\beta(x,r)$$

where $\alpha(y)$ is the angle between π_y and $\pi_{x,r}$.

Proof. Without loss of generality we may assume that $\pi := \pi_{x,r}$ is the first axis. Let us choose $\Phi(z) := \eta(z)^2(\pi^\perp(z))$, where $\pi^\perp$ is the projection on the second axis and where $\eta \in C_c^1(B(x,r))$, $0 \leq \eta \leq 1$, $\eta = 1$ on $B(\tau r)$ and $|\nabla\eta| \leq 2/r(1-\tau)$.

For every line π' let $e_{\pi'}$ be a unit vector in the direction of π' and denote by $M_{\pi'}$ the orthogonal projection on $\mathbb{R}e_{\pi'}$. We maintain that

$$\|M_\pi - M_{\pi'}\| = \sin(\alpha(\pi',\pi)) \tag{40}$$

where the norm in the left side is the euclidian norm of linear operators and $\alpha(\pi',\pi) \in [0, \frac{\pi}{2}]$ is, as usual, the smallest angle between the lines $\bar{\pi}'$ and π. To show (40), let $z \in \mathbb{R}^2$ be of unit norm and let (a,b) be its coefficients in the orthonormal basis $\{e_\pi, e_{\pi^\perp}\}$. Then $\|M_{\pi'}(z) - M_\pi(z)\|^2 = \|a(M_{\pi'}(e_\pi) - e_\pi) + bM_{\pi'}(e_{\pi^\perp})\|^2 = \|bM_{\pi'}(e_{\pi^\perp}) - aM_{\pi'^\perp}(e_\pi)\|^2 = (a^2+b^2)\|M_{\pi'}(e_{\pi^\perp})\|^2 = \langle e_{\pi'}, e_{\pi^\perp}\rangle^2 = \cos^2(\alpha(\pi', \pi^\perp)) = \sin^2(\alpha(\pi',\pi))$, so (40) holds.

Now let us compute the tangential divergence of Φ. Since the first component of Φ is 0 and the second is equal to $\eta(z)^2 z_2$ we have

$$\mathrm{div}^{\pi'}\Phi(z) = \langle \nabla^{\pi'}(\eta(z)^2 z_2), e_2\rangle$$

and

$$\nabla^{\pi'}(\eta(z)^2 z_2) = [2\eta(z)z_2\langle \nabla\eta(z), e_{\pi'}\rangle + \eta(z)^2\langle e_2, e_{\pi'}\rangle].e_{\pi'}.$$

Thus

$$\begin{aligned}
\mathrm{div}^{\pi'}\Phi(z) &= 2\eta(z)z_2\langle \nabla\eta(z), e_{\pi'}\rangle\langle e_2, e_{\pi'}\rangle + \eta(z)^2\langle e_2, e_{\pi'}\rangle^2 \\
&= 2\eta(z)\langle M_{\pi'}(\nabla\eta(z)), \pi^\perp(z)\rangle + \eta(z)^2\sin^2(\alpha(\pi,\pi')) \\
&= 2\eta(z)\langle (M_{\pi'} - M_\pi)(\nabla\eta(z)), \pi^\perp(z)\rangle + \eta(z)^2\sin^2(\alpha(\pi,\pi')) \\
&\geq \eta(z)^2\sin^2(\alpha(\pi,\pi')) - \left[\frac{1}{t}\eta(z)^2\|M_{\pi'} - M_\pi\|^2\|\nabla\eta(z)\|^2 + t|\pi^\perp(z)|^2\right]
\end{aligned}$$

hence setting $t := 2\|\nabla\eta(z)\|^2$ and using (40) we get

$$\mathrm{div}^{\pi'}\Phi(z) \geq \frac{1}{2}\eta(z)^2\sin^2(\alpha(\pi,\pi')) - \frac{8}{r^2(1-\tau)^2}\|\pi^\perp(z)\|^2. \tag{41}$$

Therefore, applying the above inequality with π' the approximate tangent line at point x and recalling (by 2) that $\mathcal{H}^1(B(x,r)\cap\Sigma) \leq 3\pi r$ we obtain

$$\begin{aligned}
\int_{B(x,\tau r)} \sin^2(\alpha(z))\,\mathrm{d}\mathcal{H}^1 &\leq 2\int_{B(x,r)\cap\Sigma} \mathrm{div}^\Sigma\Phi\,\mathrm{d}\mathcal{H}^1 + \frac{C}{(1-\tau)^2}r\beta(x,r)^2 \\
&\leq 2\int_{B(x,r)\cap\Sigma} \mathrm{div}^\Sigma\Phi\,\mathrm{d}\mathcal{H}^1 + C(\tau)r\beta(x,r)\psi(x,r)
\end{aligned}$$

by (7). On the other hand, since $B(x,r)$ does not contain any endpoint, by Proposition 5.2 we have that

$$\begin{aligned}
\int_\Omega \mathrm{div}^\Sigma\Phi\,\mathrm{d}\mathcal{H}^1 &\leq \frac{1}{\lambda_0}\int_{\mathbb{R}^2}\left|\left\langle \Phi(k(z)), \frac{k(z)-z}{|k(z)-z|}\right\rangle\right|\,\mathrm{d}\mu(x) \\
&\leq \int_{k^{-1}(B(x,r))} \eta(k(z))^2\|\pi^\perp(k(z))\|\,\mathrm{d}\mu(z) \\
&\leq Cr\beta(x,r)\psi(x,r)
\end{aligned}$$

so the proof is complete. □

7. Σ is locally a Lipschitz graph

In this last section we prove that away from triple points Σ is locally a graph. We begin with some precisions about corner points.

7.1. About the aperture of corners

Using the first order equation, one can rely the aperture of any corner point x_0 in terms of measure ψ. Following the notations of Section 6.1, for any atom $x \in \Sigma$ for the measure ψ (i.e. x is either a corner point of endpoint) we define ν_x the image measure of $\mu \llcorner k^{-1}(x)$ by the application $y \mapsto \frac{y-x}{\|y-x\|}$ and the vector

$$\bar{v}(x) := \int_{S^1+x} (v - x)\, \mathrm{d}\nu_x(v).$$

Then we denote

$$\lambda(x) := \|\bar{v}(x)\|.$$

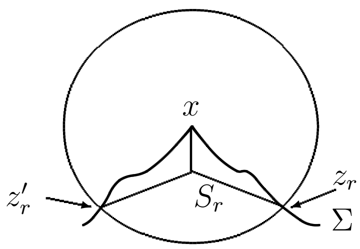
It is clear that $\lambda(x) \leq \psi(\{x\})$ and recall that $\lambda_0 := \lambda(x_0)$ where x_0 is any endpoint of Σ. For any corner point x let us denote $\theta(x)$ the smallest angle between the two rays of the bow up limit at point x. Then, by the proof of Theorem 3.7 of [10] we have the following very nice identity:

$$\lambda_0 \cos\left(\frac{\theta(x)}{2}\right) = \lambda(x).$$

The next proposition gives a lower bound on the aperture of any corner point and will be needed to find some pieces of graphs. This is probably a well known fact but as far as the author knows, it was never explicitly written before in the literature.

Proposition 7.1. *For any corner point x of Σ it holds $\theta(x) \geq \frac{2\pi}{3}$.*

Proof. The proof is fairly simple, relying on the fact that if the aperture is too small, one can replace Σ by a suitable Steiner connection to win some length. Indeed, let x be a corner point with aperture $\theta := \theta(x) < \frac{2\pi}{3}$. For any $r \in (0, r(x))$ let z_r and z_r' be the two points of $\Sigma \cap \partial B(x,r)$ and let S_r be the Steiner minimal set connecting the points z_r, z_r' and x.



Since the blow up limit converges to a union of two rays of aperture $\theta < \frac{2\pi}{3}$, we deduce that

$$\mathcal{H}^1(\Sigma \cap B(x,r)) = 2r + o(r)$$

and

$$\mathcal{H}^1(S_r) = l(\theta)r + o(r)$$

where $l(\theta)$ is the length of the Steiner connection corresponding to an exact angle of aperture θ in the unit ball. In particular $l(\theta) < 2$, and for r small enough we have that $\mathcal{H}^1(\Sigma \cap B(x,r)) > \mathcal{H}^1(S_r)$. This allows us to take as a competitor for Σ the set

$$\Sigma_r := \Sigma \backslash B(x,r) \cup S_r \cup L_r^+$$

where L_r^+ is a piece of segment added at any endpoint of Σ as in the proof of Lemma 5.3, and satisfying

$$\mathcal{H}^1(L_r^+) = \mathcal{H}^1(\Sigma \cap B(x,r)) - \mathcal{H}^1(S_r) = (2 - l(\theta))r + o(r).$$

Then it comes

$$\begin{aligned}\mathcal{F}(\Sigma) &\leq \mathcal{F}(\Sigma_r)\\ &\leq \mathcal{F}(\Sigma) + r\psi(B(x,r)\backslash\{x\}) - \lambda_0[2 - l(\theta)]r + o(r),\\ &\lambda_0[2 - l(\theta)]r \leq r\psi(B(x,r)\backslash\{x\}) + o(r)\end{aligned}$$

which implies a contradiction for r small enough because $\psi(B(x,r)\backslash\{x\})$ tends to 0. □

7.2. Construction of the graph

A consequence of Theorem 4.11, is that Σ is locally a graph. We still denote $\mathbb{T}_\Sigma$ the set of triple points. For any ordinary point we denote π_y the tangent line at y (which is defined $\mathcal{H}^1$-a.e. on Σ), and when y is a corner point we also denote π_y the line through y orthogonal to the mediatrice of the corner resulting from taking the blow up limit at y.

Proposition 7.2. *For all $x \in \Sigma\backslash(\mathbb{T}_\Sigma \cup \mathbb{E}_\Sigma)$ there exists r depending on x, and there exists a 5-Lipschitz function $f : \pi_x \to \mathbb{R}$ with graph denoted by $\Gamma_f := \{(t, f(t)); t \in \pi_x\}$ which has the following properties*

$$\Sigma \cap B(x, r/4) = \Gamma_f \cap B(x, r/4), \tag{42}$$

$$\int_{\pi_x \cap B(x,\frac{r}{16})} |f'(t)|^2 dt \leq Cr\psi(x,r)^2. \tag{43}$$

Proof. Let $\gamma := [-T,T] \to \mathbb{R}^2$ be a parametrization of $\Sigma_r := \Sigma \cap B(x,r)$, where $r < r(x)$ the usual radius given by Lemma 2.9. Assume without loss of generality that π_x is the first axis of $\mathbb{R}^2$ x is the origin. We denote p_1 the orthogonal projection on the first axis, and (γ_1, γ_2) the coordinates of γ.

For $\mathcal{H}^1$-a.e. $y \in \Sigma \cap B(x,r)$ we denote $\alpha(y)$ the smallest non oriented angle between the lines π_y and π_x. In particular by the area formula one has

$$\int_{\gamma(a,b)} \sin(\alpha(y))^2 d\mathcal{H}^1(y) = \int_a^b \frac{\gamma_2'(t)^2}{\sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}} dt.$$

From Theorem 4.11 we know that for every $\varepsilon > 0$ there exists a radius r such that

$$\left|\sin^2(\alpha(y)) - \sin^2\left(\frac{\pi - \theta(x)}{2}\right)\right| \leq \varepsilon \text{ for } \mathcal{H}^1 \text{ a.e. } y \in \Sigma \cap B(x,r)$$

which implies using Proposition 7.1 that

$$\sin^2(\alpha(y)) \leq \frac{3}{4} + \varepsilon \text{ for } \mathcal{H}^1 \text{ a.e. } y \in \Sigma \cap B(x,r).$$

Let us choose $\varepsilon = 1/400$. Since Σ admits two half tangent lines at x, up to a smaller choice of r we may assume that $\gamma([-T,0]) \cap B(x,r) \subset \mathbb{R}^-_* \times \mathbb{R}$, $\gamma([0,T]) \cap B(x,r) \subset \mathbb{R}^+_* \times \mathbb{R}$ and $\gamma([-T,T]) \cap \{0\} \times \mathbb{R} = \{x\}$. Taking if necessary a smaller radius r we may also assume that $C\psi(\gamma([-T,T]\backslash\{0\}))^2 \leq 1/400$ and $d_H(\gamma([-T,T], L \cap B(x,r)) \leq r/400$ where $L := T_R(x) \cup T_L(x)$ is the blow up limit at point x. Then using (3), for every $a, b \in [0,T]$ such that $\gamma([a,b]) \subset B(x,r)$ we have

$$\begin{aligned}
\int_a^b \frac{\gamma_2'(t)^2}{\sqrt{\gamma_1'(t)^2+\gamma_2'(t)^2}}dt &= \int_{\gamma([a,b])} \sin(\alpha(y))^2 \mathcal{H}^1(y) \\
&\leq (3/4 + 1/400)\mathcal{H}^1(\gamma([a,b])) \\
&\leq (8/10)|\gamma(a) - \gamma(b)|. \qquad (44)
\end{aligned}$$

because under our assumptions, $\mathcal{H}^1(\gamma([a,b])) \leq (1+\frac{1}{400})|\gamma(a)-\gamma(b)|$. Now we denote $F := \Sigma \cap B(x,r/4)$. We claim that

$$\sharp\{z' \in F; p_1(z) = p_1(z')\} = 1 \quad \forall z \in F. \qquad (45)$$

Let us denote $F^- := \gamma([-T,0]) \cap B(x,r/4)$ and $F^+ := \gamma([0,T]) \cap B(x,r/4)$. To prove (45), it is enough to prove that

$$\sharp\{z' \in F^\pm; p_1(z) = p_1(z')\} = 1 \quad \forall z \in F^\pm.$$

It suffice to consider the case of F^+ (the proof for F^- will follow by the same way). Assume the contrary, namely that there is $z, z' \in F^+$ such that $p_1(z) = p_1(z')$. Let $I \subset [0,T]$ be such that $\gamma(I)$ is the arc that goes from z to z' and fix $r_0 := |z-z'| \leq r/2$. We know that $\gamma(I) \cap \partial B(z, r_0) = z'$, in particular $\gamma(I)$ is contained in $B(z,r_0)$, and we have

$$\int_I |\gamma_2'(t)|dt \geq r_0. \qquad (46)$$

On the other hand by (44), (3)

$$\begin{aligned}
\int_I |\gamma_2'(t)|dt &\leq \left(\int_I \sqrt{\gamma_1'^2(t)+\gamma_2'(t)^2}dt\right)^{\frac{1}{2}} \left(\int_I \frac{|\gamma_2'(t)|^2}{\sqrt{\gamma_1'^2(t)+\gamma_2'(t)^2}}dt\right)^{\frac{1}{2}} \\
&\leq \mathcal{H}^1(\gamma(I))^{\frac{1}{2}} \left(\int_{\gamma(I)} \sin(\alpha(y))^2 \, \mathrm{d}\mathcal{H}^1(y)\right)^{\frac{1}{2}} \\
&\leq r_0 \sqrt{\frac{401}{400} . \frac{8}{10}} \\
&\leq \frac{9}{10} r_0 \qquad (47)
\end{aligned}$$

which gives a contradiction with (46). Therefore, one can define the application $f : p_1(F) \to \mathbb{R}$ such that $(t, f(t)) \in F$ for all $t \in p_1(F)$.

Further, using a similar argument as before we claim that for all t and t' in $p_1(F)$ we have that

$$|f(t) - f(t')| \leq 5|t - t'|. \qquad (48)$$

Indeed assume by contradiction that (48) is not true, thus there is t and t' such that $|f(t) - f(t')| > |t - t'|$. It is enough to consider the case when $t, t' \leq 0$ or $t, t' \geq 0$ (the general case follows from taking 0 as an intermediate point between t and t'). We denote $z := (t, f(t))$, $z' := (t', f(t'))$, and $r_1 := |z - z'| \geq \sqrt{6}|t - t'|$. As before, let $J \subset [0, T]$ be such that $\gamma(J)$ is the arc that goes from z to z'. We know that $\gamma(J) \cap \partial B(z, r_1) = z'$, in particular $\gamma(J)$ is contained in $B(z, r_1) \subset B(x, 3r/4)$, and we have

$$\int_J |\gamma_2'(t)|dt \geq \sqrt{r_1^2 - |t - t'|^2} \geq \sqrt{\frac{5}{6}} r_1 > \frac{9}{10} r_1. \tag{49}$$

On the other hand arguing as for (47),

$$\int_J |\gamma_2'(t)|dt \leq \frac{9}{10} r_1 \tag{50}$$

which gives a contradiction with (49) so (48) is proved.

Therefore, by a standard extension argument one can find a 5-Lipschitz function $\tilde{f}$ on $\mathbb{R}$ that is equal to f on $p_1(F)$, that satisfies (42) and that we will still denote by f instead of $\tilde{f}$.

It remains to prove (43). Observe that by our assumptions since $x = 0$, $d_H(\gamma([-T, T] \cap B(x, r), T \cap B(x, r)) \leq r/400$, using Proposition 7.1 and the fact that $\Sigma \cap B(x, r/4)$ is connected we also have that

$$p_1(\Sigma \cap B(x, r/4)) \supseteq \left[-\frac{r}{16}, \frac{r}{16}\right].$$

On the other hand since

$$d_H(\Sigma \cap B(x, r), L \cap B(x, r)) \leq Cr\psi(B(x, r) \backslash \{x\}),$$

we deduce that

$$\alpha(\pi_x, \pi_{x,r}) \leq C\psi(B(x, r) \backslash \{x\}). \tag{51}$$

Now since f is 5-Lipschitz, applying Proposition 6.1 with $\tau = \frac{1}{4}$, using (7) and (51) we get

$$\begin{aligned}
\int_{[-\frac{r}{16}, \frac{r}{16}]} f'(t)^2 dt &\leq \sqrt{6} \int_{[-\frac{r}{16}, \frac{r}{16}]} \frac{f'(t)^2}{\sqrt{1 + f'(t)^2}} dt \\
&\leq \sqrt{6} \int_{F \cap B(x, \frac{1}{4} r)} \sin^2(\alpha(y)) \, \mathrm{d}\mathcal{H}^1(y) \\
&\leq Cr\psi(x, r)^2
\end{aligned}$$

thus (43) holds and the proposition is proved. □

7.3. The equation of curvature

To complete the proof of Theorem 1.3 we will give some further remarks about the first order equations applied to f.

Given η, let us take $\Phi(x,y) := (0, \eta(x)\chi(y))$ with $\chi \in C^1_0([-\delta,\delta])$, $\chi = 1$ on $(-\frac{\delta}{2}, \frac{\delta}{2})$ and $\delta > 0$ is chosen in such a way that $\mathrm{supp}(\Phi) \subset \mathrm{supp}(\eta) \times (-\delta,\delta) \subset B(x_0, r_0)$. By applying Proposition 5.2 with this choice of diffeomorphism Φ we obtain that

$$\int_\Omega \mathrm{div}^\Sigma \Phi \, \mathrm{d}\mathcal{H}^1 = \frac{1}{\lambda_0} \int_{\mathbb{R}^2} \left\langle \Phi(k(y)), \frac{y - k(y)}{|y - k(y)|} \right\rangle \mathrm{d}\mu(y) \tag{52}$$

Now for $\mathcal{H}^1$ a.e. $x \in \Sigma \cap \Gamma$, a direct computation gives

$$\mathrm{div}^\Sigma \Phi(z) = \frac{\eta'(z) f'(z)}{1 + |f'(z)|^2}$$

thus by the area formula we obtain

$$\int_I \frac{\eta'(z) f_n'(z)}{\sqrt{1 + |f_n'(z)|^2}} \, \mathrm{d}\mathcal{H}^1 = \frac{1}{\lambda_0} \int_{\mathbb{R}^2} \eta(\pi(k(y))) \left\langle e_2, \frac{y - k(y)}{|y - k(y)|} \right\rangle \mathrm{d}\mu(y). \tag{53}$$

An immediate consequence of the above equation is that the derivative of $t \mapsto \frac{f'(t)}{\sqrt{1+|f'(t)|^2}}$ in the distributional sense is a measure. Indeed, we can also write this equation in a more natural form using disintegration. Consider the linear form T that associate for every $\eta \in C_c(I)$ the quantity

$$T(\eta) := \int_{\mathbb{R}^2} \eta(\pi(k(y))) \left\langle e_2, \frac{y - k(y)}{|y - k(y)|} \right\rangle \mathrm{d}\mu(y)$$

Since $|T(\eta)| \leq C\|\eta\|_\infty$ by the Riesz theorem one can find a measure ψ_0 on I such that $T(\eta) = \int_I \eta(t) \, \mathrm{d}\psi_0(t)$. Then (53) becomes

$$-\frac{d}{dt}\left(\frac{f'(t)}{\sqrt{1 + |f'(t)|^2}} \right) = \psi_0$$

Furthermore it is interesting to link ψ_0 with ψ. Actually, we easily have that

$$|\psi_0| \leq \pi \sharp \psi.$$

In particular, $\frac{f'(t)}{\sqrt{1+|f'(t)|^2}} \in BV(I)$ and the jump set is concentrated on corner points so that we have

$$-\frac{d}{dt}\left(\frac{f'(t)}{\sqrt{1 + |f'(t)|^2}} \right) = H(t)dt + \sum_{(t,f(t)) \in \mathrm{Corner}} c_t \delta_t + H^{\mathrm{Cant}}$$

where H^{Cant} is the cantor part, $\|H(t)\|_{L^1(I)} \leq \psi(B(x,r))$ and $|c_t| \leq \psi(\{(t, f(t))\})$ for any atom $(t, f(t))$ of ψ.

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