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Convexity, Local Simplicity, and Reduced Boundaries of Sets

David G. Caraballo
Department of Mathematics, Georgetown University, St. Mary's Hall, 3rd floor, Washington, D.C. 20057–1233, USA
dgc3@georgetown.edu

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We establish fundamental connections between the convexity of a set K in $\mathbb{R}^n$, its local simplicity, and its reduced boundary in the sense of geometric measure theory. One of the most important results in convex analysis asserts that a closed set with non-empty interior is convex if and only if it has a supporting hyperplane through each topological boundary point. More generally, requiring only non-empty measure-theoretic interior, we prove that a proper closed subset of $\mathbb{R}^n$ is convex if and only if it is locally simple and has a supporting hyperplane at each point of its reduced boundary, so that the convexity information about a closed set K is essentially encoded in its reduced boundary.

We also use techniques of geometric measure theory to refine and generalize other principal theorems about convex sets, standard results on separation and representation which have found significant applications in functional analysis, economics, optimization, control theory, and other areas. Because convexity is closely related to many other topics, our main theorem helps establish connections between reduced boundaries and these other concepts and results as well.

Keywords: Locally simple, convex, reduced boundary, measure-theoretic boundary, topological boundary, interior, exterior, support, separation, density

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1. Introduction

The following is one of the most important and fundamental results in convex analysis:

Theorem 1.1 (cf. [11] Theorem 5.4, [13] Theorem 1.3.3 and Corollary 1.3.5, [14] Theorem 2.4.12). *Suppose $K \subset \mathbb{R}^n$ is closed, with non-empty interior K°. Then K is convex if and only if K has a supporting hyperplane through each point of its topological boundary, $\partial_{top} K$.*

This theorem is quite surprising since it establishes an unexpected connection between the topological boundary and the convexity of a set. Indeed, R. Webster ([14], p. 71) calls it "arguably the single most important property of convex sets."

In this paper, we use techniques from geometric measure theory to refine and generalize this theorem, as well as standard results concerning separation and representation which have found significant applications in functional analysis, economics, optimization, control theory, and other areas ([13], [15]).

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Our main theorem establishes a surprising connection between the convexity of K and the *reduced boundary* $\partial^* K$ of K, as defined by E. De Giorgi (see [4], [5], [6]). It also relates the convexity of K and the local simplicity of K, a density condition introduced by the author in the context of sets of finite perimeter in [3], and formulated more generally below (see Definition 3.1). It is surprising that we may deduce the convexity of K without even considering supporting hyperplanes through points in the set $\partial_{top} K \setminus \partial^* K$.

Theorem 1.1 (Main Theorem). *Suppose K is a Lebesgue measurable proper subset of $\mathbb{R}^n$. Suppose either K is closed or $\mathrm{Cl}_{meas}(K) = K$. Suppose K has non-empty measure-theoretic interior, $\mathrm{Int}_{meas}(K)$ (or suppose K has non-empty interior, K°). Then K is convex if and only if K is locally simple and has a supporting hyperplane at each point p of its reduced boundary $\partial^* K$.*

After giving some preliminary definitions and results, we show (Theorem 3.4) that convexity implies local simplicity, for proper subsets of $\mathbb{R}^n$ having non-empty interior.

Several important separation theorems in convex analysis follow from a key separation result, which asserts that if K is a closed subset of $\mathbb{R}^n$ having non-empty interior K°, and having a supporting hyperplane at each point of its topological boundary $\partial_{top} K$, then any $x \notin K$ can be separated from K by a supporting hyperplane of K through a point $p \in \partial_{top} K$ (see for example [13], Theorems 1.3.3 and 1.3.4).

We use local simplicity to give measure-theoretic versions of this vital separation result (Theorems 4.1 and 4.2), in which we suppose the existence of supporting hyperplanes to K at points p in the reduced boundary of K. Unlike in the standard result, we do not require this supporting hyperplane condition to hold at all points p in the topological boundary. Our conclusion is stronger than that of the standard result, since we show it is possible to separate x from K using a supporting hyperplane through a point p of the reduced boundary.

Additionally, we use local simplicity to establish representation theorems for K in terms of intersecting half-spaces. If $K \subset \mathbb{R}^n$ is closed and has non-empty interior, and if K has a supporting hyperplane at each point p of its topological boundary $\partial_{top} K$, then K is the intersection of all its supporting half-spaces through topological boundary points (cf. [11] Theorem 5.3, [13] Theorem 1.3.3 and Corollary 1.3.5); in particular, it is convex. In Theorems 4.3 and 4.4, we again suppose the existence of supporting hyperplanes to K at points p of the reduced boundary of K. Our representation for K involves the intersection of only the supporting half-spaces through points of the reduced boundary of K.

Finally, in Section 5 we prove our main theorem, Theorem 1.1, characterizing convexity in terms of local simplicity and a supporting hyperplane condition at reduced boundary points. Because convexity is closely related to many other topics – such as other separation theorems, nearest point estimates, etc. – our main theorem helps establish connections between reduced boundaries and these other concepts and results as well.

2. Notation and Boundaries

Throughout this paper, we work in $\mathbb{R}^n$ with $n \geq 2$. We measure volume with n dimensional Lebesgue measure $\mathcal{L}^n$ and surface area with $(n-1)$ dimensional Hausdorff measure $\mathcal{H}^{n-1}$. We let $B(p,r)$ and $U(p,r)$ denote, respectively, the closed and open balls in $\mathbb{R}^n$ with center p and radius $r \in (0,\infty)$, and we set $\alpha(n) = \mathcal{L}^n(B(\mathbf{0},1))$, where $\mathbf{0} = (0,0,\ldots,0)$ is the origin in $\mathbb{R}^n$. If $A, B \subset \mathbb{R}^n$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of A and B.

Given a point $p \in \mathbb{R}^n$ and a unit vector $u \in \mathbb{R}^n$, we define the hyperplane $H(p,u) = \{x : (x-p)\cdot u = 0\}$ and its associated open half-spaces $H_+(p,u) = \{x : (x-p)\cdot u > 0\}$ and $H_-(p,u) = \{x : (x-p)\cdot u < 0\}$. We define the associated closed half-spaces $\overline{H_+(p,u)}$ and $\overline{H_-(p,u)}$ by replacing $>$ and $<$ with $\geq$ and $\leq$ respectively. A hyperplane $H = H(p,u)$ is a *supporting hyperplane* for the set $X \subset \mathbb{R}^n$ at the point $p \in \mathbb{R}^n$ provided $p \in X$, $p \in H$, and either $X \subset \overline{H_+(p,u)}$ or $X \subset \overline{H_-(p,u)}$. A closed half-space in $\mathbb{R}^n$ is called a *supporting half-space* for a set $X \subset \mathbb{R}^n$ if it contains X and if it is bounded by a supporting hyperplane for X.

If $A \subset \mathbb{R}^n$, $\overline{A}$ and A° denote, respectively, the topological closure and topological interior of A in $\mathbb{R}^n$. The interior of A relative to the smallest affine subspace, or flat, containing A is denoted relint A. When $A \subset \mathbb{R}^n$, we let $\partial_{top} A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$ denote the topological boundary of A.

When $X \subset \mathbb{R}^n$ is $\mathcal{L}^n$ measurable, $p \in \mathbb{R}^n$, and $R > 0$, we define the *n dimensional density ratio of X at p in $B(p,R)$* as

$$\Theta^n(X,p,R) = \mathcal{L}^n(X \cap B(p,R)) \;/\; \mathcal{L}^n(B(p,R)),$$

and we set $\Theta^n(X,p) = \lim_{R\to 0^+} \Theta^n(X,p,R)$, provided the limit exists. If $X \subset \mathbb{R}^n$ is $\mathcal{L}^n$ measurable, we let $\partial^* X$ denote the *reduced boundary* of X (see [4], [5], [6]). If $X \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$, the vector $u \in \mathbb{R}^n$ is called a *measure-theoretic exterior unit normal to X at p in the sense of Federer* (cf. [7], [8] 4.5.5) provided $|u| = 1$, $\Theta^n(H_+(p,u) \cap X,\ p) = 0$, and $\Theta^n(H_-(p,u) \setminus X,\ p) = 0$. If no such u exists, we define $n_X(p) = \mathbf{0}$, while if such a u exists it is necessarily unique ([7] Theorem 3.4) and we define $n_X(p) = u$. Whenever X is $\mathcal{L}^n$ measurable and $p \in \partial^* X$, $n_X(p)$ is a unit vector.

Some excellent references for reduced boundaries include [1], [6], [9], [10], and [12]. Also, see [2].

If $X \subset \mathbb{R}^n$ is $\mathcal{L}^n$ measurable, we define the *measure-theoretic interior, measure-theoretic exterior, measure-theoretic boundary,* and *measure-theoretic closure* of X as follows:

$$\begin{gathered}
\mathrm{Int}_{meas}(X) = \{x \in \mathbb{R}^n : \Theta^n(X,x) = 1\}, \\
\mathrm{Ext}_{meas}(X) = \{x \in \mathbb{R}^n : \Theta^n(X,x) = 0\}, \\
\partial^M X = \mathbb{R}^n \setminus (\mathrm{Int}_{meas}(X) \cup \mathrm{Ext}_{meas}(X)), \\
\mathrm{Cl}_{meas}(X) = \mathrm{Int}_{meas}(X) \cup \partial^M X = \{x \in \mathbb{R}^n : \limsup_{R\to 0^+} \Theta^n(X,x,R) > 0\}.
\end{gathered}$$

3. Local Simplicity and Convexity

In [3], we introduced local simplicity in the context of sets of finite perimeter, and we showed that it plays a major role in connecting topological and measure-theoretic properties of sets. Here, we extend the notion to sets having locally finite perimeter, and we establish strong connections with convexity of subsets of $\mathbb{R}^n$.

Definition 3.1. A set $X \subset \mathbb{R}^n$ is called **locally simple** provided X is $\mathcal{L}^n$ measurable, X has locally finite perimeter, $\partial^* X \neq \emptyset$, and there exist continuous functions $\delta, a, A : \mathbb{R}^n \to [0, \infty)$ such that $\delta(x) > 0$ for all $x \in \overline{\partial^* X}$ and

$$0 < a(x) \leq \Theta^n(X, x, R) \leq A(x) < 1 \tag{1}$$

whenever $x \in \overline{\partial^* X}$ and $0 < R < \delta(x)$.

We observe that, if X is locally simple, then so is $\mathbb{R}^n \setminus X$; also, $\mathcal{L}^n(X) > 0$ and $\mathcal{L}^n(\mathbb{R}^n \setminus X) > 0$. The following theorem shows that local simplicity allows us to work with measure-theoretic interiors and exteriors in much the same way we would work with topological interiors and exteriors.

Theorem 3.2. *Suppose $X \subset \mathbb{R}^n$ is locally simple. Then $\mathrm{Int}_{meas}(X)$ and $\mathrm{Ext}_{meas}(X)$ are open.*

Proof. Taking complements in (1), we obtain

$$0 < 1 - A(x) \leq \Theta^n(\mathbb{R}^n \setminus X, x, R) \leq 1 - a(x) < 1 \tag{2}$$

whenever $x \in \overline{\partial^* X}$ and $0 < R < \delta(x)$.

Suppose $x \in \mathrm{Int}_{meas}(X)$. If we had $x \in \overline{\partial^* X} = \overline{\partial^*(\mathbb{R}^n \setminus X)}$, then (2) would imply

$$\limsup_{R \to 0^+} \Theta^n(\mathbb{R}^n \setminus X, x, R) \geq 1 - A(x) > 0,$$

so that $\Theta^n(\mathbb{R}^n \setminus X, x) \neq 0$, contradicting $x \in \mathrm{Int}_{meas}(X)$. Therefore, $x \notin \overline{\partial^* X}$. Let $D = \mathrm{dist}\left(x, \overline{\partial^* X}\right) > 0$. Since $x \in \mathrm{Int}_{meas}(X)$, there exists an $r \in (0, D)$ such that $\mathcal{L}^n(X \cap U(x, r)) > 0$. If we were to have $\mathcal{L}^n((\mathbb{R}^n \setminus X) \cap U(x, r)) > 0$ as well, then the relative isoperimetric inequality ([6] 5.6.2, or [10] 3.7.14) would give $\mathcal{H}^{n-1}(\partial^* X \cap U(x, r)) > 0$, implying $\mathrm{dist}\left(x, \overline{\partial^* X}\right) \leq \mathrm{dist}(x, \partial^* X) < r < D$, a contradiction. Thus, $\mathcal{L}^n((\mathbb{R}^n \setminus X) \cap U(x, r)) = 0$, so that $\mathcal{L}^n(X \cap U(x, r)) = \alpha(n) r^n$. It follows that $\Theta^n(X, z) = 1$, and hence $z \in \mathrm{Int}_{meas}(X)$, for any $z \in U(x, r)$, so that $\mathrm{Int}_{meas}(X)$ is open.

It follows from (2) that $\mathbb{R}^n \setminus X$ is locally simple, and so $\mathrm{Ext}_{meas}(X) = \mathrm{Int}_{meas}(\mathbb{R}^n \setminus X)$ is also open. □

Next, we will prove that the class of locally simple subsets of $\mathbb{R}^n$ includes all convex sets, other than $\mathbb{R}^n$, with non-empty interior. We will first need to establish the following lemma, which ensures that we may replace a convex set with its closure or with its interior without altering it on a set having positive volume.

Lemma 3.3. *Suppose $K \subset \mathbb{R}^n$ is convex. Then*

1) $\mathcal{L}^n(\partial_{top}K) = 0$.
2) $\mathcal{L}^n(K^\circ \triangle K) = 0$.
3) $\mathcal{L}^n\left(K \triangle \overline{K}\right) = 0$.

Proof. Since K is convex, $\partial_{top}K = \partial_{top}\overline{K}$ ([13] 1.1.14), and also $\overline{K}$ is convex ([13] 1.1.9). Since $\overline{K} = K^\circ \cup \partial_{top}K$, it follows that $\partial_{top}K \subset \overline{K}$. Since $\mathcal{L}^n$ is a Radon measure and $\overline{K}$ is $\mathcal{L}^n$ measurable, $\Theta^n\left(\overline{K}, x\right) = 1$ for $\mathcal{L}^n$ almost every $x \in \overline{K}$ (cf. [12] 2.14) and in particular for $\mathcal{L}^n$ almost every $x \in \partial_{top}K$. However, because $\overline{K}$ is convex, there is a supporting hyperplane to $\overline{K}$ at each point $p \in \partial_{top}\overline{K}$ ([13] 1.3.2), so that $\Theta^n\left(\overline{K}, p, r\right) \leq 1/2$ (and in particular $\Theta^n\left(\overline{K}, p\right) = 1$ does not hold) for each $p \in \partial_{top}\overline{K}$, $r > 0$. It follows that $\mathcal{L}^n(\partial_{top}K) = \mathcal{L}^n\left(\partial_{top}\overline{K}\right) = 0$. Finally, $\mathcal{L}^n\left(K^\circ \triangle \overline{K}\right) = \mathcal{L}^n\left(\overline{K} \setminus K^\circ\right) = 0$, by *1)*, and so *2)* and *3)* follow immediately. □

Theorem 3.4. *If $K \subset \mathbb{R}^n$ is convex, with non-empty interior K°, then either K is locally simple, or $K = \mathbb{R}^n$.*

Proof. Let K be a convex subset of $\mathbb{R}^n$ having non-empty interior K°. If $K = \mathbb{R}^n$ we're done, so suppose $K \neq \mathbb{R}^n$ and choose $x \in \mathbb{R}^n \setminus K$. Since $K \setminus K^\circ \subset \partial_{top}K$ has Lebesgue measure zero (Lemma 3.3), it is Lebesgue measurable, and therefore so is $K = K^\circ \cup (K \setminus K^\circ)$. K has locally finite perimeter since it is convex. K and $\{x\}$ can be separated by a hyperplane ([13] 1.3.4), so $\mathcal{L}^n(\mathbb{R}^n \setminus K) > 0$. Since K° is non-empty, K contains a ball and so $\mathcal{L}^n(K) > 0$. The relative isoperimetric inequality ([6] 5.6.2, or [10] 3.7.14) implies that $\mathcal{H}^{n-1}(\partial^* K) > 0$, so that in particular $\partial^* K$ is non-empty.

We must now establish continuous density ratio bounds at points in $\partial^* K$. Because K° is non-empty, translating if necessary we may assume $\mathbf{0} = (0, 0, \ldots, 0) \in K^\circ$, so that $B(\mathbf{0}, 2R) \subset K^\circ$ for some $R > 0$. We fix R for the remainder of the proof and let $U' = U(\mathbf{0}, R)$.

Suppose $p \in \partial^* K \subset \partial_{top}K$. Since K has a supporting hyperplane through p, it follows that $\Theta^n(K, p, r) \leq 1/2$ for all $r > 0$. Next, we note that $\partial^* K \cap K^\circ = \emptyset$, since $\Theta^n(K, x) = 1/2$ for $x \in \partial^* K$ and $\Theta^n(K, x) = 1$ for $x \in \mathrm{Int}_{meas}(K)$, which contains K°. Thus,

$$p \in \partial^* K \subset \mathbb{R}^n \setminus K^\circ \subset \mathbb{R}^n \setminus B(\mathbf{0}, 2R),$$

which implies $|p| > 2R$.

Suppose $w \in \mathbb{R}^n$ satisfies $|w| > R$, so that $\mathrm{dist}(w, U') > 0$. For $x \in \mathbb{R}^n$ with $x \neq w$, we let $\overrightarrow{wx} = \{(1 - \lambda) w + \lambda x : \lambda \geq 0\}$ denote the ray through x with initial point w, and we let

$$\mathrm{Cone}(w, U') = \{x \in \mathbb{R}^n : \overrightarrow{wx} \cap U' \neq \emptyset\}$$

denote the cone with vertex w through the ball U'. For w satisfying $|w| > R$, we define

$$f(w) = \mathcal{L}^n(\mathrm{Cone}(w, U') \cap B(w, 1)).$$

We note that f is continuous at each w in its domain. By symmetry, $f(v) = f(w)$

provided $|v| = |w|$. Letting $p_0 = (2R, 0, 0, \ldots, 0) \in \mathbb{R}^n$, it then follows that

$$a(w) = \begin{cases} (1/\alpha(n))\ f(w), & \text{if } |w| > 2R \\ (1/\alpha(n))\ f(p_0), & \text{if } |w| \le 2R \end{cases} \tag{3}$$

is continuous and positive for all $w \in \mathbb{R}^n$.

Since K is convex, $\overline{K}$ is convex and satisfies $\mathcal{L}^n\left(K \triangle \overline{K}\right) = 0$ (Lemma 3.3). Since $\overline{K}$ contains both p and U', by convexity it must contain the convex hull of their union, $\operatorname{conv}(\{p\} \cup U')$. This set, in turn, contains $\operatorname{Cone}(p, U') \cap U(p, R)$, since $|p| > 2R$. Thus, $\operatorname{Cone}(p, U') \cap B(p, r) \subset \overline{K} \cap B(p, r)$ whenever $0 < r < R$. For each such r,

$$\mathcal{L}^n(K \cap B(p, r)) = \mathcal{L}^n\left(\overline{K} \cap B(p, r)\right) \ge \mathcal{L}^n(\operatorname{Cone}(p, U') \cap B(p, r)) = f(p) \cdot r^n.$$

The last equality follows from the fact that $\operatorname{Cone}(p, U') \cap B(p, r)$ and $\operatorname{Cone}(p, U') \cap B(p, 1)$ are homothetic, with scale factor r (cf. [14] 6.2.15). Thus, for each $p \in \partial^* K$, and for each $0 < r < R$, we have

$$\Theta^n(K, p, r) = \frac{\mathcal{L}^n(K \cap B(p, r))}{\alpha(n)\, r^n} \ge \frac{f(p) \cdot r^n}{\alpha(n)\, r^n} = a(p).$$

It follows that K satisfies Definition 3.1 with $\delta(p) = R$, $A(p) = 1/2$, and $a(p)$ as given in (3). □

Remark 3.5. When K is bounded, we can take $a(x)$ and $A(x)$ to be constants in (1).

4. Separation and Representation

It is well-known that, if K is a closed subset of $\mathbb{R}^n$ having non-empty interior K°, and having a supporting hyperplane at each point of its topological boundary $\partial_{top} K$, then any $x \notin K$ can be separated from K by a supporting hyperplane of K through a point $p \in \partial_{top} K$ (see for example [13] Theorems 1.3.3 and 1.3.4).

We will now use local simplicity to give a measure-theoretic version of this important standard result, in which we suppose only that K has a supporting hyperplane at each point $p \in \partial^* K \subset \partial_{top} K$. We also suppose only that $\operatorname{Int}_{meas}(K) \neq \emptyset$, which is more general than $K^\circ \neq \emptyset$. Our conclusion is stronger, since we will show it is possible to separate x from K using a supporting hyperplane through a point p of the reduced boundary.

Theorem 4.1. *Suppose $K \subset \mathbb{R}^n$ is locally simple and closed, with non-empty measure-theoretic interior $\operatorname{Int}_{meas}(K)$. Suppose K has a supporting hyperplane at each point p of its reduced boundary $\partial^* K$. If $x \in \mathbb{R}^n \setminus K$, then K has a supporting hyperplane $H = H(p, u)$, with $p \in \partial^* K$, such that $K \subset \overline{H_-(p, u)}$ and $x \in H_+(p, u)$.*

Proof. Since K is locally simple, K is non-empty and $K \neq \mathbb{R}^n$. Suppose $x \in \mathbb{R}^n \setminus K$. Since K is closed, we may select r so that $0 < r < \operatorname{dist}(x, K)$. By assumption, there exists $y \in \operatorname{Int}_{meas}(K)$. Theorem 3.2 then implies $U(y, R) \subset \operatorname{Int}_{meas}(K)$ for some

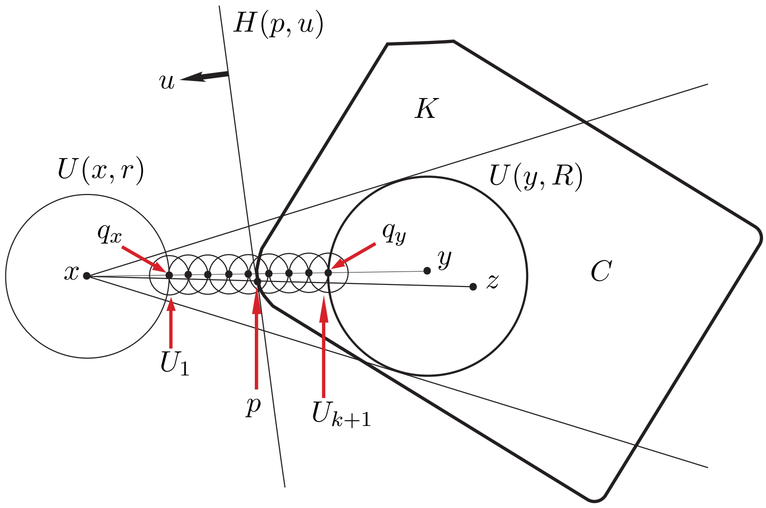

Figure 4.1.

$R > 0$. Let $\overline{xy} = \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$, and define $q_x = \overline{xy} \cap \partial_{top} U(x, r)$ and $q_y = \overline{xy} \cap \partial_{top} U(y, R)$. $q_x \in \mathbb{R}^n \setminus K$ by our choice of r, so that $q_x \neq q_y$ in particular.

For any $p \in \mathbb{R}^n$ with $p \neq x$, let $\overrightarrow{xp} = \{(1-\lambda)x + \lambda p : \lambda \ge 0\}$ denote the ray through p with initial point x. We then define the cone with vertex x through the open ball $U(y, R)$ as follows:

$$C = \text{Cone}(x,\ U(y, R)) = \{p \in \mathbb{R}^n : \overrightarrow{xp} \cap U(y, R) \neq \emptyset\}. \tag{4}$$

Fix $s \in \mathbb{R}$ so that $0 < s < \text{dist}(q_x, \mathbb{R}^n \setminus C)$ and so that $|q_y - q_x|/s = k$ is a positive integer. Define the vector $w = s(q_y - q_x)/|q_y - q_x| = (q_y - q_x)/k$. For each $i = 1, 2, \ldots, k+1$, let $U_i = U(q_x + (i-1)w, s)$, so that $U_1 = U(q_x, s)$ and $U_{k+1} = U(q_y, s)$ in particular. (See Figure 4.1.)

Since $r < \text{dist}(x, K)$, we must have $\mathcal{L}^n(K \cap U(x, r)) = 0$, which implies

$$\mathcal{L}^n((\mathbb{R}^n \setminus K) \cap U(x, r)) = \alpha(n) r^n,$$

and therefore $\mathcal{L}^n((\mathbb{R}^n \setminus K) \cap U_1) > 0$. Similarly, $\mathcal{L}^n(K \cap U(y, R)) = \alpha(n) R^n$, and therefore $\mathcal{L}^n(K \cap U_{k+1}) > 0$. It follows that $\mathcal{L}^n(K \cap U_i) > 0$ and $\mathcal{L}^n((\mathbb{R}^n \setminus K) \cap U_i) > 0$ for some $1 \le i \le k+1$. By the relative isoperimetric inequality ([6] 5.6.2, or [10] 3.7.14), $\mathcal{H}^{n-1}(\partial^* K \cap U_i) > 0$, so that in particular we may choose $p \in \partial^* K \cap U_i$.

Let $H(p, u)$ be a supporting hyperplane to K at p, such that $K \subset \overline{H_-(p, u)}$. By (4), there exists $z \in \overrightarrow{xp} \cap U(y, R)$, with $p \in \text{relint}\,\overline{xz}$. We cannot have $x \in H_-(p, u)$, since then $p \in \overline{xz} \subset H_-(p, u)$, which contradicts $p \in H(p, u)$. We also cannot have $x \in H(p, u)$, for then we would have $\overrightarrow{xp} \in H(p, u)$ and therefore $z \in H(p, u)$, a contradiction. □

We can now state and prove a more general version of Theorem 4.1:

Theorem 4.2. *Suppose $K \subset \mathbb{R}^n$ is locally simple and K is closed or* $\text{Cl}_{meas}(K) = K$. *Suppose* $\text{Int}_{meas}(K) \neq \emptyset$ *(or suppose $K^\circ \neq \emptyset$). Suppose K has a supporting hyperplane at each point p of its reduced boundary $\partial^* K$. If $x \in \mathbb{R}^n \setminus K$, then K has* *a supporting hyperplane* $H = H(p,u)$, *with* $p \in \partial^* K$, *such that* $K \subset \overline{H_-(p,u)}$ *and* $x \in H_+(p,u)$.

Proof. If $x \in K^\circ$, then $U(x,r) \subset K^\circ$ for some $r > 0$, so that $\Theta^n(K,x) = 1$ and thus $x \in \mathrm{Int}_{meas}(K)$. Therefore, if $K^\circ \neq \emptyset$ we have $\mathrm{Int}_{meas}(K) \neq \emptyset$, as in the hypotheses of Theorem 4.1. Also, if $K \subset \mathbb{R}^n$ is locally simple and $\mathrm{Cl}_{meas}(K) = K$, Theorem 3.2 implies that

$$K = \mathrm{Cl}_{meas}(K) = \mathrm{Int}_{meas}(K) \cup \partial^M K = \mathbb{R}^n \setminus \mathrm{Ext}_{meas}(K) \tag{5}$$

is closed, as in the hypotheses of Theorem 4.1. The result now follows from Theorem 4.1. □

If $K \subset \mathbb{R}^n$ is closed and has non-empty interior, and if K has a supporting hyperplane at each point p of its topological boundary $\partial_{top} K$, then K is convex, and in fact it equals the intersection of all its supporting half-spaces through topological boundary points (cf. [11] Theorem 5.3, [13] Theorem 1.3.3 and Corollary 1.3.5). We will again use local simplicity to give a measure-theoretic version of this important standard result, in which we suppose only that K has a supporting hyperplane at each point p of the reduced boundary of K. We also suppose only that $\mathrm{Int}_{meas}(K) \neq \emptyset$, which is more general than $K^\circ \neq \emptyset$. Our representation theorem involves the intersection of all supporting half-spaces through points of the reduced boundary of K only.

Theorem 4.3. *Suppose* $K \subset \mathbb{R}^n$ *is locally simple and closed, with non-empty measure-theoretic interior* $\mathrm{Int}_{meas}(K)$. *Suppose* K *has a supporting hyperplane at each point* p *of its reduced boundary* $\partial^* K$. *Then* $K = \cap \overline{H_-(p,u)}$, *where the intersection is taken over all supporting hyperplanes* $H(p,u)$ *to* K *at points* $p \in \partial^* K$, *with unit vectors* u *chosen so that* $K \subset \overline{H_-(p,u)}$. *In particular,* K *is convex.*

Proof. Since K is locally simple, $K \neq \mathbb{R}^n$, so we may choose $x \in \mathbb{R}^n \setminus K$. Since K is a subset of each of the $\overline{H_-(p,u)}$'s, it follows that $K \subset \cap \overline{H_-(p,u)}$, where the intersection is taken over all supporting hyperplanes $H(p,u)$ to K at points $p \in \partial^* K$, with unit vectors u chosen so that $K \subset \overline{H_-(p,u)}$. By Theorem 4.1, K has a supporting hyperplane $H = H(p,u)$, with $p \in \partial^* K$, such that $K \subset \overline{H_-(p,u)}$ and

$$x \in H_+(p,u) = \mathbb{R}^n \setminus \overline{H_-(p,u)} \subset \mathbb{R}^n \setminus \cap \overline{H_-(p,u)}.$$

Thus, $\mathbb{R}^n \setminus K \subset \mathbb{R}^n \setminus \cap \overline{H_-(p,u)}$, since x was arbitrary, so $K = \cap \overline{H_-(p,u)}$ as claimed. Since K is the intersection of closed half-spaces, it must be convex. □

More generally, we have the following:

Theorem 4.4. *Suppose* $K \subset \mathbb{R}^n$ *is locally simple and either* K *is closed or* $\mathrm{Cl}_{meas}(K) = K$. *Suppose* $\mathrm{Int}_{meas}(K) \neq \emptyset$ *(or suppose* $K^\circ \neq \emptyset$*). Suppose* K *has a supporting hyperplane at each point* p *of its reduced boundary* $\partial^* K$. *Then* $K = \cap \overline{H_-(p,u)}$, *where the intersection is taken over all supporting hyperplanes* $H(p,u)$ *to* K *at points* $p \in \partial^* K$, *with unit vectors* u *chosen so that* $K \subset \overline{H_-(p,u)}$. *In particular,* K *is convex.*

Proof. If $K^\circ \neq \emptyset$, then $\mathrm{Int}_{meas}(K) \supset K^\circ$ is also non-empty, as in the hypotheses of Theorem 4.3. Also, if $K \subset \mathbb{R}^n$ is locally simple and $\mathrm{Cl}_{meas}(K) = K$, Theorem 3.2 and (5) imply that K is closed, as in the hypotheses of Theorem 4.3. We now apply Theorem 4.3 to complete the proof. $\square$

5. Proof of the Main Theorem

We can now prove our main theorem, which gives a characterization for convexity in terms of local simplicity and a supporting hyperplane condition at reduced boundary points. This result fundamentally connects convexity, local simplicity, and the reduced boundary.

Proof of Theorem 1.1. Suppose K is a Lebesgue measurable proper subset of $\mathbb{R}^n$. Suppose either $K \subset \mathbb{R}^n$ is closed or $\mathrm{Cl}_{meas}(K) = K$. Suppose $\mathrm{Int}_{meas}(K) \neq \emptyset$ (if $K^\circ \neq \emptyset$, we automatically have $\mathrm{Int}_{meas}(K) \neq \emptyset$).

If K is locally simple and K has a supporting hyperplane at each point p of its reduced boundary $\partial^* K$, then Theorem 4.4 shows that K is convex.

Now suppose K is convex. Since $\mathrm{Int}_{meas}(K) \neq \emptyset$, there exists $y \in \mathrm{Int}_{meas}(K)$, and so $\Theta^n(K, y) = 1$. In particular, $\mathcal{L}^n(K) > 0$. Lemma 3.3, *2)* then implies $\mathcal{L}^n(K^\circ) = \mathcal{L}^n(K) > 0$, so that K° is non-empty. Since we also have $K \neq \mathbb{R}^n$, we can apply Theorem 3.4 to conclude K is locally simple. K must be closed, since if K satisfies merely $\mathrm{Cl}_{meas}(K) = K$, then (5) and Theorem 3.2 imply that K is closed, since K is locally simple. K is therefore closed and convex, with non-empty interior, so K has a supporting hyperplane at each point p of $\partial_{top} K$ (cf. [11] Theorem 5.4, [13] Theorem 1.3.2). The proof is now complete since $\partial^* K \subset \partial_{top} K$. $\square$

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