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An Affirmative Answer to a Problem Posed by Zălinescu

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Recently, in [5] Zălinescu posed a question about the characterization of the intrinsic core of the Minkowski sum of two graphs associated with two maximal monotone operators. In this note we give an affirmative answer.

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1. Introduction

We suppose throughout this note that X is a real reflexive Banach space with norm $\|\cdot\|$ and dual product $\langle\cdot,\cdot\rangle$. We now introduce some notation. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* or *multifunction* whose graph is defined by

$$\operatorname{gra} A := \{(x,x^*) \in X \times X^* \mid x^* \in Ax\}.$$

The *domain* of A is $\operatorname{dom} A := \{x \in X \mid Ax \neq \varnothing\}$. Recall that A is *monotone* if for all $(x,x^*),(y,y^*) \in \operatorname{gra} A$ we have

$$\langle x-y, x^*-y^*\rangle \geq 0,$$

and A is *maximal monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusions).

The *Fitzpatrick function* of A (see [1]) is given by

$$F_A\colon (x,x^*) \mapsto \sup_{(a,a^*)\in\operatorname{gra} A} \left(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\right). \tag{1}$$

For a function $f\colon X \to \left]-\infty,+\infty\right]$, the *domain* is $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x,x^*\rangle - f(x))$ is the *Fenchel conjugate* of f.

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Given $F : X \times X^* \to \,]-\infty, +\infty]$, we say F is a *representative* of a maximal monotone operator A if F is lower semicontinuous and convex with $F \geq \langle \cdot, \cdot \rangle$, $F^* \geq \langle \cdot, \cdot \rangle$ and

$$\operatorname{gra} A = \{(x, x^*) \mid F(x, x^*) = \langle x, x^* \rangle\}.$$

Following [2], it will be convenient to set $F^\intercal \colon X^* \times X :\to \,]-\infty, +\infty] : (x^*, x) \mapsto F(x, x^*)$, where $F \colon X \times X^* \to \,]-\infty, +\infty]$, and similarly for a function defined on $X^* \times X$.

We define $\widehat{F}$ (see [5]) by

$$\widehat{F}(x, x^*) := F(x, -x^*).$$

Let $a = (x, x^*), b = (y, y^*) \in X \times X^*$, we also set (see [4]) by

$$\lfloor a, b \rfloor = \langle x, y^* \rangle + \langle y, x^* \rangle.$$

Given a subset D of X, $\overline{D}$ is the *closure*, $\operatorname{conv} D$ is the *convex hull*, and $\operatorname{aff} D$ is the *affine hull*. The *conic hull* of D is denoted by $\operatorname{cone} D := \{\lambda x \mid \lambda \geq 0, x \in D\}$. The *indicator function* $\iota_D : X \to \,]-\infty, +\infty]$ of D is defined by

$$x \mapsto \begin{cases} 0, & \text{if } x \in D; \\ +\infty, & \text{otherwise.} \end{cases}$$

The *intrinsic core* or *relative algebraic interior* of D, written as ${}^iD$ in [6], is

$${}^iD := \{a \in D \mid \forall x \in \operatorname{aff}(D - D),\, \exists \delta > 0,\, \forall \lambda \in [0, \delta] : a + \lambda x \in D\}.$$

We define ${}^{ic}D$ by

$${}^{ic}D := \begin{cases} {}^iD, & \text{if } \operatorname{aff} D \text{ is closed;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Zălinescu posed the following problem in [5]: Let $A, B : X \rightrightarrows X^*$ be maximal monotone. Is the implication

$${}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] \neq \varnothing \quad \Rightarrow \quad {}^{ic}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right] \neq \varnothing$$

true? Theorem 2.7 provides an affirmative answer to this question. It further shows that these two sets actually are equal.

2. Main result

Definition 2.1 (Fitzpatrick family). Let $A \colon X \rightrightarrows X^*$ be a maximal monotone operator. The associated *Fitzpatrick family* $\mathcal{F}_A$ consists of all functions $F \colon X \times X^* \to \,]-\infty, +\infty]$ that are lower semicontinuous and convex, and that satisfy $F \geq \langle \cdot, \cdot \rangle$, and $F = \langle \cdot, \cdot \rangle$ on $\operatorname{gra} A$.

Fact 2.2 (Fitzpatrick, see [1, Theorem 3.10]). *Let $A\colon X \rightrightarrows X^*$ be a maximal monotone operator. Then for every $(x, x^*) \in X \times X^*$,*

$$\begin{aligned} F_A(x,x^*) &= \min\{F(x,x^*) \mid F \in \mathcal{F}_A\} \\ \text{and} \quad F_A^{*\intercal}(x,x^*) &= \max\{F(x,x^*) \mid F \in \mathcal{F}_A\}. \end{aligned} \tag{2}$$

Fact 2.3 (Simons, see [4, Lemma 20.4(b)]). *Let $A : X \rightrightarrows X^*$ be maximal monotone and $a := (x, x^*) \in X \times X^*$ with $\langle x, x^*\rangle = 0$. Suppose that there exists $u \in \mathbb{R}$ such that*

$$\lfloor \operatorname{gra} A, a\rfloor = \{u\}.$$

Then

$$\lfloor \operatorname{dom} F_A, a\rfloor = \{u\}.$$

Theorem 2.4. *Let $A, B : X \rightrightarrows X^*$ be maximal monotone. Then*

$$\overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]} = \overline{\operatorname{aff}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right]}. \tag{3}$$

Proof. We do and can suppose $(0,0) \in \operatorname{gra} A$ and $(0,0) \in \operatorname{gra} B$. We first show

$$\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right] \subseteq \overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}. \tag{4}$$

Suppose to the contrary that there exists $c \in X \times X^*$ such that $c \in [\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}]$ but $c \notin \overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}$. By the Separation Theorem, there exist $a := (x, x^*) \in X \times X^*$ and $\delta \in \mathbb{R}$ such that

$$\lfloor a, c\rfloor > \delta > \sup\left\{\lfloor a, e\rfloor \mid e \in \overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}\right\}. \tag{5}$$

Since $(0,0) \in \operatorname{gra} A$, $(0,0) \in \operatorname{gra} B$ and $\overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}$ is a closed subspace, we have $\delta > 0$ and $\lfloor a, b - d\rfloor = 0, \forall b \in \operatorname{gra} A$, $\forall d \in \operatorname{gra}(-B)$. Thus,

$$\lfloor a, \operatorname{gra} A\rfloor = \{0\} = \lfloor a, \operatorname{gra}(-B)\rfloor = \lfloor (-x, x^*), \operatorname{gra} B\rfloor. \tag{6}$$

By $(0,0) \in \operatorname{gra} A$ and $(0,0) \in \operatorname{gra} B$ again,

$$F_A(a) = F_A(x,x^*) = 0, \qquad F_B(-x,x^*) = 0. \tag{7}$$

Since $F_A(x,x^*) \geq \langle x, x^*\rangle$ and $F_B(-x,x^*) \geq \langle -x, x^*\rangle$, by (7), $\langle x, x^*\rangle = 0$. Thus by (6) and Fact 2.3,

$$\lfloor a, \operatorname{dom} F_A\rfloor = \{0\} = \lfloor (-x, x^*), \operatorname{dom} F_B\rfloor = \lfloor a, \operatorname{dom} \widehat{F_B}\rfloor.$$

Thus, $\lfloor a, \operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\rfloor = \{0\}$, which contradicts (5). Hence

$$\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right] \subseteq \overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}. \tag{8}$$

And thus $\overline{\operatorname{aff}[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}]} \subseteq \overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]}$. Hence

$$\overline{\operatorname{aff}[\operatorname{gra} A - \operatorname{gra}(-B)]} = \overline{\operatorname{aff}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right]}.$$

$\square$

Fact 2.5 (Zălinescu, see [5, Lemma 2 and Theorem 3]). *Let $A, B : X \rightrightarrows X^*$ be maximal monotone, and let F_1, F_2 be representatives of A, B, respectively. Then*

$$^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right] = {}^{ic}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))\right]$$

and

$$\begin{aligned} {}^{ic}\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right] &\subseteq \left[\operatorname{gra} A - \operatorname{gra}(-B)\right] \\ \subseteq \ \operatorname{conv}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right] &\subseteq \left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right]. \end{aligned}$$

If $^{ic}[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}] \neq \varnothing$, *then*

$$^{ic}\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right] = {}^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right] = {}^{ic}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))\right]. \tag{9}$$

Remark 2.6. If X is finite-dimensional, the intrinsic core of a convex set $D \subseteq X$ is the same as the relative interior of D in the sense of Rockafellar [3]. Then $^{ic}[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}] = {}^{i}[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}] \neq \varnothing$ by [3, Theorem 6.2]. Thus, (9) always holds.

Our main result comes the following which provides an affirmative answer to the question posed by Zălinescu.

Theorem 2.7. *Let $A, B : X \rightrightarrows X^*$ be maximal monotone such that $^{ic}[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))] \neq \varnothing$. Then*

$$^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right] = {}^{ic}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))\right] = {}^{ic}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right]. \tag{10}$$

Moreover, if F_1, F_2 are representatives of A, B, respectively, then

$$^{ic}\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right] = {}^{ic}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))\right] = {}^{ic}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right]. \tag{11}$$

Proof. Let $a \in {}^{ic}[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))]$. Then we have $a \in [\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}]$ and $\operatorname{cone}[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B)) - a]$ is a closed subspace. By Theorem 2.4,

$$\begin{aligned} &\operatorname{cone}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B)) - a\right] \subseteq \operatorname{cone}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B} - a\right] \\ &\subseteq \operatorname{aff}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B} - a\right] = \operatorname{aff}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right] - \{a\} \\ &\subseteq \overline{\operatorname{aff}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]} - \{a\} \subseteq \overline{\operatorname{aff}\left[\operatorname{gra} A - \operatorname{gra}(-B) - a\right]} \\ &= \overline{\operatorname{aff}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B)) - a\right]} \subseteq \operatorname{cone}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B)) - a\right]. \end{aligned}$$

Hence $\operatorname{cone}[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B} - a] = \operatorname{cone}[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B)) - a]$ is a closed subspace. Thus $a \in {}^{ic}[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}]$. By Fact 2.5,

$$^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right] = {}^{ic}\left[\operatorname{conv}(\operatorname{gra} A - \operatorname{gra}(-B))\right] = {}^{ic}\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right].$$

And by Fact 2.2,

$$\operatorname{conv}[\operatorname{gra} A - \operatorname{gra}(-B))] \subseteq \left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right] \subseteq \left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right].$$

Similar to the proof above, see that (11) holds. □

Remark 2.8. The referee pointed out the following alternative proof of equation (10) in Theorem 2.7. Let $C := \operatorname{gra} A - \operatorname{gra}(-B)$ and $D := \operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}$. Since ${}^{ic}(\operatorname{conv} C) \neq \varnothing$, we have that $\operatorname{aff}(\operatorname{conv} C) = \operatorname{aff} C$ is closed. By Fact 2.2, $C \subseteq D$ and D is convex. Then we have that $\operatorname{conv} C \subseteq D$. Using Theorem 2.4, we get

$$\operatorname{aff}(\operatorname{conv} C) \subseteq \operatorname{aff} D \subseteq \overline{\operatorname{aff} C} = \operatorname{aff}(\operatorname{conv} C).$$

Therefore, $\operatorname{aff}(\operatorname{conv} C) = \operatorname{aff} D$ and so $\operatorname{aff} D$ is closed. Thus by $\operatorname{conv} C \subseteq D$ again, ${}^{i}(\operatorname{conv} C) \subseteq {}^{i}D$ and hence ${}^{ic}(\operatorname{conv} C) \subseteq {}^{ic}D$. By Fact 2.5, (10) holds.

Theorem 2.9. *Let $A, B : X \rightrightarrows X^*$ be maximal monotone, and F_1, F_2 be representatives of A, B, respectively. Then*

$$^{ic}\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right] = {}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] = {}^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]. \quad (12)$$

Proof. We consider two cases.

Case 1. ${}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] = \varnothing$. Assume that ${}^{ic}[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}] \neq \varnothing$. Then by Fact 2.5, ${}^{ic}[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}] = {}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] = \varnothing$. This a contradiction.

Case 2. ${}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] \neq \varnothing$. Apply Theorem 2.7.

Combining the above results, we see that (12) holds. □

Corollary 2.10. *Let $A, B : X \rightrightarrows X^*$ be maximal monotone, and F_1, F_2 be representatives of A, B, respectively. Assume ${}^{ic}\left[\operatorname{conv}\left(\operatorname{gra} A - \operatorname{gra}(-B)\right)\right] \neq \varnothing$. Then*

$$\overline{\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right]} = \overline{\operatorname{conv}\left[(\operatorname{gra} A - \operatorname{gra}(-B)\right]} = \overline{\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]}. \quad (13)$$

In particular,

$$\overline{\left[\operatorname{dom} F_A - \operatorname{dom} \widehat{F_B}\right]} = \overline{\operatorname{conv}\left[(\operatorname{gra} A - \operatorname{gra}(-B)\right]} = \overline{\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]}.$$

Proof. Given a convex set $D \subseteq X$, assume that ${}^{ic}D \neq \varnothing$, then $\overline{{}^{ic}D} = \overline{D}$. By Theorem 2.9,

$$\begin{aligned}\overline{\left[\operatorname{dom} F_1 - \operatorname{dom} \widehat{F_2}\right]} &= \overline{\operatorname{conv}\left[(\operatorname{gra} A - \operatorname{gra}(-B)\right]}\\ &= \overline{{}^{ic}\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]} \subseteq \overline{\left[\operatorname{gra} A - \operatorname{gra}(-B)\right]}\end{aligned}$$

Hence (13) holds. □

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