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Type One Functions and Voronoï's Theorem

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Voronoï's theorem characterizes local maxima of the Hermite invariant $m/\det^{1/n}$ defined on the open cone of positive definite n by n symmetric matrices, where m denotes the arithmetical minimum function. In this paper, we extend Voronoï's theorem to functions of the form m/ϕ when ϕ is a type one function. Moreover, we study the Hermite like constant defined from m/ϕ.

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1. Introduction

Let V_n be the vector space of real $n \times n$ symmetric matrices, P_n the open cone of positive definite matrices in V_n and P_n^{semi} the closure of P_n in V_n. For $a \in P_n^{\mathrm{semi}}$, $m(a)$ denotes the inferior $\inf_{x \in \mathbb{Z}^n \setminus \{0\}} {}^t xax$, where $\mathbb{Z}^n$ is the lattice of integral vectors in an n-dimensional real Euclidean space. The function γ on P_n defined by

$$\gamma(a) = \frac{m(a)}{\det(a)^{1/n}}$$

for $a \in P_n$ is called the Hermite invariant, and its maximum $\gamma_n = \max_{a \in P_n} \gamma(a)$ is known as the Hermite constant. In [6], Poor and Yuen introduced another Hermite like constant c_n, which is defined by

$$c_n = \min_{a \in P_n} \frac{w(a)}{m(a)} = \left(\max_{a \in P_n} \frac{m(a)}{w(a)} \right)^{-1},$$

where $w(a)$ denotes the dyadic trace of a. The constant c_n is connected with γ_n by means of the inequality

$$\frac{1}{c_n} \le \frac{\gamma_n^2}{n}. \tag{1}$$

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A fundamental tool to study γ_n is Voronoï's theorem, which states that γ attains a local maximum on $a \in P_n$ if and only if a is perfect and eutactic. In some proofs of Voronoï's theorem, the convexity of the domain P_n and the concavity of both functions m and $\det^{1/n}$ play key roles. Poor and Yuen [5] investigated a family of such kind of functions as m, $\det^{1/n}$ and w. They named this family type one functions. In general, for a given type one function ϕ, one can consider the function $F_\phi = m/\phi$ and its Hermite like constant $\delta_\phi = \sup_{a\in P_n} F_\phi(a)$. The purpose of this paper is to study δ_ϕ and an analog of Voronoï's theorem for F_ϕ.

Definition and some properties of type one functions are given in Section 2. For a type one function ϕ, the corresponding convex subset

$$K_1(\phi) = \{\, a \in P_n^{\mathrm{semi}} \mid \phi(a) \geq 1 \,\}$$

plays an important role. We will introduce the notion of semikernel for convex subsets of P_n^{semi}, and then prove that the mapping $\phi \mapsto K_1(\phi)$ gives a one to one correspondence between type one functions and semikernels.

An analog of Voronoï's theorem for F_ϕ is considered for differentiable type one functions. For a such ϕ, we will introduce the notion of ϕ-eutaxy for elements in P_n, and prove that F_ϕ attains a local maximum on $a \in P_n$ if and only if a is perfect and ϕ-eutactic.

For any type one class function ϕ, we observe that there exist positive constants C_1 and C_2 such that $C_1 m \leq \phi \leq C_2 w$. Namely, m (resp. w) is the smallest (resp. largest) one among type one class functions up to constant multiples. Thus both Hermite like constants $\delta_\phi = \sup_{a\in P_n} m(a)/\phi(a)$ and $\widehat{\delta}_\phi = \sup_{a\in P_n} \phi(a)/w(a)$ are finite for any type one class function ϕ. If ϕ° denotes the dual type one class function of ϕ, then the equality $\delta_\phi = \widehat{\delta}_{\phi^\circ}$ will be proved in Section 4. In particular, we obtain the following expression of Hermite's constant γ_n:

$$\gamma_n = \delta_{\det^{1/n}} = \widehat{\delta}_{(\det^{1/n})^\circ} = n \sup_{a\in P_n} \frac{\det(a)^{1/n}}{w(a)}.$$

It is convenient to consider the constant $\xi_\phi = \delta_\phi \cdot \widehat{\delta}_\phi$ since it has the invariant property $\xi_{C\phi} = \xi_\phi = \xi_{\phi^\circ}$ for any constant $C > 0$. By definition, we have $\xi_m = \xi_w = 1/c_n$ and $\xi_{\det^{1/n}} = \gamma_n^2/n$. The extreme property of w leads us to the inequality

$$\xi_w \leq \xi_\phi \tag{2}$$

for any type one class function ϕ. Thus the inequality (1) may be viewed as a particular case of (2).

Notation. The vector space V_n is equipped with the inner product $\langle a_1, a_2\rangle = \mathrm{tr}(a_1 a_2)$ for $a_1, a_2 \in V_n$. The identity matrix in V_n is denoted by I_n. For $a \in V_n$, $\sigma(a)$ stands for the operator norm of a, i.e.,

$$\sigma(a) = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \sqrt{\frac{{}^t(ax)ax}{{}^txx}}\,.$$

For a constant $c \in \mathbb{R}$, $\mathbb{R}_{>c}$ and $\mathbb{R}_{\geq c}$ stand for the open interval $(c, +\infty)$ and the closed interval $[c, +\infty)$, respectively.

2. Type one functions and semikernels

We start with definition of type one functions.

Definition. A function $\phi : P_n^{\mathrm{semi}} \to \mathbb{R}_{\geq 0}$ is called a type one function if ϕ satisfies the following conditions:

(TO$_1$) $\phi(\theta a) = \theta\phi(a)$ for all $a \in P_n^{\mathrm{semi}}$ and $\theta \geq 0$,
(TO$_2$) $\phi(a_1 + a_2) \geq \phi(a_1) + \phi(a_2)$ for all $a_1, a_2 \in P_n^{\mathrm{semi}}$,
(TO$_3$) $\phi(a) > 0$ for all $a \in P_n$.

A type one function ϕ is called a type one class function if $\phi({}^t gag) = \phi(a)$ holds for all $a \in P_n^{\mathrm{semi}}$ and $g \in GL_n(\mathbb{Z})$.

It is known that a type one function is continuous on P_n (cf. [4, Proposition 2.2] or [7, Theorem 10.1]).

Example 2.1 The trace tr and the smallest eigenvalue λ_1 are type one functions, but not type one class functions. The reduced determinant $\det^{1/n}$ and the arithmetical minimum m are type one class functions.

For a type one function ϕ, the dual type one function $\phi^\circ : P_n^{\mathrm{semi}} \to \mathbb{R}_{\geq 0}$ is defined to be

$$\phi^\circ(a) = \inf_{b \in P_n} \frac{\langle a, b\rangle}{\phi(b)}.$$

If ϕ is a type one class function, then so is ϕ°. The dual type one class function of m is denoted by w, which is called the dyadic trace. The dual type one class function of $\det^{1/n}$ is $n \det^{1/n}$.

Definition. Let K be a convex subset of P_n^{semi} such that $0 \notin K$, $\mathbb{R}_{\geq 1} \cdot K = K$ and $\mathbb{R}_{>0} \cdot K \supset P_n$.

(1) K is called a kernel if K is closed in P_n^{semi}.
(2) K is called a semikernel if the following three conditions are satisfied:
(SK$_1$) $K \cap (P_n \cup \{0\})$ is closed in $P_n \cup \{0\}$,
(SK$_2$) $\{\,\theta \geq 0 \mid \theta a \in K\,\}$ is closed in $\mathbb{R}_{\geq 0}$ for any $a \in K$,
(SK$_3$) $a + b \in K$ for all $a \in K$ and $b \in P_n^{\mathrm{semi}}$.

Lemma 2.2. *A kernel is a semikernel.*

Proof. Let K be a kernel. It is obvious that K satisfies (SK$_1$) and (SK$_2$), so we show that K satisfies the condition (SK$_3$). Let $a \in K$ and $b \in P_n$. There exists a $\theta > 0$ such that $\theta b \in K$. We have

$$\left(1 - \frac{1}{1+\theta}\right) a + \frac{1}{1+\theta}\theta b \in K,$$

thus

$$a + b \in \frac{1+\theta}{\theta} \cdot K \subset K.$$

This implies (SK$_3$) since K is closed. □

Lemma 2.3. *If K is a semikernel, then K is contained in the closure $\overline{K \cap P_n}$ of $K \cap P_n$ in P_n^{semi}.*

Proof. There exists a $\theta > 0$ such that $\theta I_n \in K$. For any $a \in K$ and $0 \leq \mu \leq 1$, we have $(1-\mu)a + \mu\theta I_n \in K$. Since $(1-\mu)a + \mu\theta I_n \in P_n$ for $0 < \mu \leq 1$, a is an adherent point of $K \cap P_n$. □

The dual $K^{\sqcup}$ of a semikernel K is defined to be

$$K^{\sqcup} = \{\, a \in V_n \mid \langle a, b\rangle \geq 1 \text{ for all } b \in K \,\}.$$

Lemma 2.4. *For any semikernel K, $K^{\sqcup} \subset P_n^{\mathrm{semi}}$ and $K^{\sqcup}$ is a kernel.*

Proof. Let $a \in K^{\sqcup}$ and $x \in \mathbb{R}^n$. Since $x^t x \in P_n^{\mathrm{semi}}$, there exist $b_m \in P_n$ and $\nu_m > 0$ $(m = 1, 2, \ldots)$ such that $\nu_m b_m \in K$ and $b_m \to x^t x$ as $m \to \infty$. Then we have $\langle \nu_m b_m, a\rangle \geq 1$, i.e., $\langle b_m, a\rangle \geq 1/\nu_m > 0$. Therefore,

$${}^t x a x = \langle x^t x, a\rangle = \left\langle \lim_{m\to\infty} b_m, a \right\rangle = \lim_{m\to\infty} \langle b_m, a\rangle \geq 0$$

holds for all $x \in \mathbb{R}^n$, and hence $a \in P_n^{\mathrm{semi}}$. This proves $K^{\sqcup} \subset P_n^{\mathrm{semi}}$. It is obvious that $K^{\sqcup}$ is a closed convex subset, $\mathbb{R}_{\geq 1} \cdot K^{\sqcup} \subset K^{\sqcup}$ and $0 \notin K^{\sqcup}$. We have to show $\mathbb{R}_{>0} \cdot K^{\sqcup} \supset P_n$. Since $0 \notin K$ and $K \cap (P_n \cup \{0\})$ is closed in $P_n \cup \{0\}$, there exists a $\mu_K > 0$ depending only on K such that

$$K \cap (P_n \cup \{0\}) \subset \{\, b \in P_n \cup \{0\} \mid \sigma(b) \geq \mu_K \,\}.$$

By taking the closures of both sides and Lemma 2.3,

$$K \subset \{\, b \in P_n^{\mathrm{semi}} \mid \sigma(b) \geq \mu_K \,\}.$$

Let $a \in P_n$ and $\lambda_1(a)$ the smallest eigenvalue of a. Then we have

$$\langle a, b\rangle \geq \langle \lambda_1(a) I_n, b\rangle = \lambda_1(a)\mathrm{tr}(b) \geq \lambda_1(a)\sigma(b) \geq \lambda_1(a)\mu_K$$

for all $b \in K$. This implies $a \in \lambda_1(a)\mu_K \cdot K^{\sqcup}$, i.e., $a \in \mathbb{R}_{>0} \cdot K^{\sqcup}$. □

If K is a kernel, then $K^{\sqcup\sqcup}$ coincides with K (see [1, p. 128]).

For a type one function ϕ, we set

$$K_1(\phi) = \{\, a \in P_n^{\mathrm{semi}} \mid \phi(a) \geq 1 \,\}.$$

It is easy to check that $K_1(\phi)$ is a semikernel. If ϕ is upper semicontinuous on P_n^{semi}, then $K_1(\phi)$ is a kernel. We recall that ϕ is said to be upper semicontinuous at a if

$$\phi(a) = \limsup_{b\to a} \phi(b) = \lim_{\epsilon\downarrow 0}(\sup\{\phi(b) \mid \sigma(a-b) \leq \epsilon,\ b \in P_n^{\mathrm{semi}}\}).$$

Conversely, for a semikernel K, define the function $\psi(K, \cdot) : P_n^{\mathrm{semi}} \to \mathbb{R}_{\geq 0}$ by

$$\psi(K, a) = \max(\{\, \theta > 0 \mid a \in \theta \cdot K \,\} \cup \{0\}).$$

The existence of this maximum follows from the condition (SK_2).

Lemma 2.5. *For any semikernel K, $\psi(K,\cdot)$ is a type one function. If K is a kernel, then $\psi(K,\cdot)$ is upper semicontinuous on P_n^{semi}.*

Proof. Let $a \in P_n^{\mathrm{semi}}$. If $\psi(K,a)=\alpha$, then

$$\{\,\theta > 0 \mid a \in \theta \cdot K\,\} \cup \{0\} = [0,\alpha]\,,$$

and hence

$$\{\,\theta > 0 \mid \mu a \in \theta \cdot K\,\} \cup \{0\} = [0,\mu\alpha]$$

for any $\mu \geq 0$. Therefore, we get $\psi(K,\mu a) = \mu\alpha = \mu\psi(K,a)$.

For $a_1, a_2 \in P_n^{\mathrm{semi}}$, set $\alpha_1 = \psi(K,a_1)$ and $\alpha_2 = \psi(K,a_2)$. If $\alpha_1 > 0$ and $\alpha_2 > 0$, then $(1/\alpha_1)a_1 \in K$, $(1/\alpha_2)a_2 \in K$ and

$$\begin{aligned}
a_1 + a_2 &= \alpha_1 \cdot \frac{1}{\alpha_1}a_1 + \alpha_2 \cdot \frac{1}{\alpha_2}a_2 \\
&= (\alpha_1+\alpha_2)\frac{\alpha_1\cdot(1/\alpha_1)a_1 + \alpha_2\cdot(1/\alpha_2)a_2}{\alpha_1+\alpha_2} \in (\alpha_1+\alpha_2)\cdot K.
\end{aligned}$$

This gives

$$\psi(K,a_1+a_2) \geq \alpha_1+\alpha_2 = \psi(K,a_1)+\psi(K,a_2)\,.$$

If $\alpha_1 > 0$ and $\alpha_2 = 0$, then $(1/\alpha_1)a_1 \in K$, $(1/\alpha_1)a_2 \in P_n^{\mathrm{semi}}$, and

$$a_1 + a_2 = \alpha_1\left\{(1/\alpha_1)a_1 + (1/\alpha_1)a_2\right\} \in \alpha_1 \cdot K.$$

This gives

$$\psi(K,a_1+a_2) \geq \alpha_1 = \alpha_1+\alpha_2 = \psi(K,a_1)+\psi(K,a_2)\,.$$

If $\alpha_1 = \alpha_2 = 0$, then we have immediately

$$\psi(K,a_1+a_2) \geq 0 = \alpha_1+\alpha_2 = \psi(K,a_1)+\psi(K,a_2)\,.$$

For $a \in P_n$, there exists a $\theta > 0$ such that $a \in \theta\cdot K$. Then we have $\psi(K,a) \geq \theta > 0$. This completes the proof of the first half of the Lemma.

We assume K is a kernel. Let $a \in P_n^{\mathrm{semi}}$ and $\theta = \psi(K,a)$. Choose a sequence $\{a_n\}$ in P_n^{semi} such that $a_n \to a$ as $n \to \infty$. For any $\epsilon > 0$, $P_n^{\mathrm{semi}} \setminus (\theta+\epsilon)\cdot K$ is open in P_n^{semi} and contains a. Therefore, $\psi(K,a_n) < \theta+\epsilon$ for sufficiently large n, and hence

$$\limsup_{n\to\infty} \psi(K,a_n) \leq \theta+\epsilon\,.$$

Since this holds for any $\epsilon > 0$, we have

$$\limsup_{n\to\infty} \psi(K,a_n) \leq \theta = \psi(K,a)$$

for any sequence $\{a_n\}$ such that $a_n \to a$. Thus $\psi(K,\cdot)$ is upper semicontinuous at a. □

Proposition 2.6. *For any type one function ϕ and any semikernel K, one has*

$$\psi(K_1(\phi),\cdot) = \phi \quad \textit{and} \quad K_1(\psi(K,\cdot)) = K.$$

Proof. For any $a \in P_n^{\mathrm{semi}}$, we have

$$\begin{aligned}\psi(K_1(\phi), a) &= \max\left(\{\,\theta > 0 \mid a \in \theta \cdot K_1(\phi)\,\} \cup \{0\}\right)\\ &= \begin{cases}\alpha & \text{if } \phi(a) = \alpha > 0\\ 0 & \text{if } \phi(a) = 0\end{cases}\\ &= \phi(a),\end{aligned}$$

so that $\psi(K_1(\phi), \cdot) = \phi$.

If K is a semikernel, then

$$a \in K_1(\psi(K, \cdot)) \Leftrightarrow \psi(K, a) \geq 1 \Leftrightarrow a \in 1 \cdot K = K,$$

and hence $K_1(\psi(K, \cdot)) = K$. □

By Proposition 2.6, type one functions and upper semicontinuous type one functions are bijectively corresponding to semikernels and kernels, respectively.

Lemma 2.7. *For any semikernel K, $\psi(K, \cdot)^\circ = \psi(K^\sqcup, \cdot)$.*

Proof. For any $a \in P_n^{\mathrm{semi}}$, we have

$$\psi(K, a)^\circ = \inf_{b \in P_n} \frac{\langle a, b\rangle}{\psi(K, b)} = \inf_{b \in P_n} \left\langle a, \frac{1}{\psi(K, b)} b \right\rangle = \inf_{b \in K \cap P_n} \langle a, b\rangle,$$

and

$$\psi(K^\sqcup, a) = \max\left(\{\,\theta > 0 \mid a \in \theta \cdot K^\sqcup\,\} \cup \{0\}\right) = \inf_{b \in K} \langle a, b\rangle.$$

From Lemma 2.3, it follows

$$\inf_{b \in K} \langle a, b\rangle \leq \inf_{b \in K \cap P_n} \langle a, b\rangle = \inf_{b \in \overline{K \cap P_n}} \langle a, b\rangle \leq \inf_{b \in K} \langle a, b\rangle,$$

and hence $\psi(K, \cdot)^\circ = \psi(K^\sqcup, \cdot)$. □

Corollary 2.8. *For any type one function ϕ, ϕ° is upper semicontinuous on P_n^{semi}.*

Proof. This immediately follows from Lemmas 2.4, 2.5, 2.7 and Proposition 2.6. □

If ϕ is a type one function and upper semicontinuous on P_n^{semi}, then we have

$$\phi^{\circ\circ} = \psi(K_1(\phi), \cdot)^{\circ\circ} = \psi(K_1(\phi)^\sqcup, \cdot)^\circ = \psi(K_1(\phi)^{\sqcup\sqcup}, \cdot) = \psi(K_1(\phi), \cdot) = \phi$$

because of $K^{\sqcup\sqcup} = K$ for a kernel K. The next proposition shows a relation between ϕ and $\phi^{\circ\circ}$ for general ϕ.

Proposition 2.9. *For any type one function ϕ, we have*

$$\begin{cases}\phi^{\circ\circ}(a) = \phi(a) & \text{if } a \in P_n\\ \phi^{\circ\circ}(a) \geq \phi(a) & \text{if } a \in P_n^{\mathrm{semi}} \setminus P_n.\end{cases}$$

Proof. By the definition of a dual semikernel, we get $K_1(\phi) \subset K_1(\phi)^{\sqcup\sqcup}$, i.e., $K_1(\phi) \subset K_1(\phi^{\circ\circ})$. This implies $\phi^{\circ\circ}(a) \geq \phi(a)$ for all $a \in P_n^{\mathrm{semi}}$. It remains that we show $\phi^{\circ\circ}(a) \leq \phi(a)$ for all $a \in P_n$. Since the closure $\overline{K_1(\phi)}$ of $K_1(\phi)$ in P_n^{semi} is a kernel, we have

$$K_1(\phi^{\circ\circ}) = K_1(\phi)^{\sqcup\sqcup} \subset \overline{K_1(\phi)}^{\sqcup\sqcup} = \overline{K_1(\phi)}.$$

If we set $\alpha = \phi^{\circ\circ}(a)$, then, by $a \in \alpha \cdot K_1(\phi^{\circ\circ}) \subset \alpha \cdot \overline{K_1(\phi)}$, we have $(1/\alpha)a \in \overline{K_1(\phi)}$. Therefore, there is a sequence $b_m \in K_1(\phi) \cap P_n$ $(m = 1, 2, \ldots)$ such that $b_m \to (1/\alpha)a$ as $m \to \infty$. Since ϕ is continuous on P_n, we have

$$\phi\left(\frac{1}{\alpha}a\right) = \phi\left(\lim_{m\to\infty} b_m\right) = \lim_{m\to\infty} \phi(b_m) \geq 1,$$

namely, $\phi(a) \geq \alpha = \phi^{\circ\circ}(a)$. □

Remark. Proposition 2.9 follows from the general theory of convex functions. A type one function ϕ is extended to the whole V_n by putting formally $\phi(a) = -\infty$ if $a \notin P_n^{\mathrm{semi}}$. This ϕ satisfies

$$\phi(a_1 + a_2) \geq \phi(a_1) + \phi(a_2)$$

for all $a_1, a_2 \in V_n$. Then $\phi^{\circ\circ}$ is the closure of ϕ and $\phi^{\circ\circ}(a) \geq \phi(a)$ holds for all $a \in V_n$. (See [7, §7 and §15], but one has to modify definitions and results in [7] to concave functions).

3. A generalization of Voronoï's theorem

In this section, we generalize Voronoï's theorem to $F_\phi = m/\phi$ for some type one function ϕ. For $a \in P_n$, $S(a)$ denotes the set of minimal integral vectors of a, i.e.,

$$S(a) = \{\, x \in \mathbb{Z}^n \setminus \{0\} \mid {}^t x a x = m(a) \,\}.$$

For any $y \in \mathbb{R}^n$, φ_y denotes the linear form $v \mapsto {}^t y v y$ on V_n.

Definition. Let $a \in P_n$. We fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t bb$. An element a is said to be perfect if the linear forms φ_{bx} $(x \in S(a))$ span the dual space V_n^* of V_n. An element a is said to be eutactic if there exist $\rho_x > 0$ $(x \in S(a))$ such that

$$\mathrm{tr} = \sum_{x \in S(a)} \rho_x \varphi_{bx}. \tag{3}$$

These definitions of perfection and eutaxy are independent of a choice of b. If $\{\varphi_{bx}\}_{x\in S(a)}$ spans V_n^*, then so does $\{\varphi_{hbx}\}_{x\in S(a)}$ for any orthogonal matrix h. If tr is represented as (3), then we have

$$\mathrm{tr} = \sum_{x \in S(a)} \rho_x \varphi_{hbx}$$

for any orthogonal matrix h. The coefficients ρ_x are independent of h. Voronoï's theorem can be stated that the Hermite invariant γ attains a local maximum on $a \in P_n$ if and only if a is perfect and eutactic.

Let ϕ be an arbitrary type one function. It follows from (TO_1) and (TO_2) that ϕ is log-concave, i.e,

$$\log\phi((1-\theta)a_1+\theta a_2)) \geq (1-\theta)\log\phi(a_1)+\theta\log\phi(a_2)$$

holds for all $a_1, a_2 \in P_n$ and $0<\theta<1$. We say ϕ is strictly log-concave if this inequality is strict for $a_1 \neq a_2$. Assume ϕ is differentiable on P_n. Then

$$(\partial\log\phi)_b(v) = \lim_{t\to 0}\frac{\log\phi({}^tb(I_n+tv)b)-\log\phi({}^tbb)}{t}$$

exists for $b \in GL_n(\mathbb{R})$ and $v \in V_n$. We define ϕ-eutaxy as follows:

Definition. Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^tbb$. An element a is said to be ϕ-eutactic if there exist $\rho_x > 0$ ($x \in S(a)$) such that $(\partial\log\phi)_b = \sum_{x\in S(a)}\rho_x\varphi_{bx}$.

In a similar fashion as eutaxy, this definition is independent of a choice of b. An element $a \in P_n$ is said to be ϕ-extreme (resp. strictly ϕ-extreme) if F_ϕ attains a local maximum (resp. a strict local maximum) on a up to the multiplication by an element of $\mathbb{R}_{>0}$. A goal of this section is to prove the following theorem:

Theorem 3.1. *Let ϕ be a strictly log-concave and differentiable type one function. Then, $a \in P_n$ is ϕ-extreme if and only if a is perfect and ϕ-eutactic. Moreover, any ϕ-extreme point is strictly ϕ-extreme.*

To prove this theorem, we follow the same line as the proof of [3, Theorems 3.4.5 and 3.4.6]. The next is the same as [3, Lemmas 3.4.2 and 3.4.3].

Lemma 3.2. *Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^tbb$.*

(*1*) *There exists a neighborhood $\mathcal{U}$ of I_n in $GL_n(\mathbb{R})$ such that $S({}^tb{}^tuub) \subset S(a)$ for any $u \in \mathcal{U}$.*

(*2*) *There exists a neighborhood $\mathcal{V}$ of 0 in V_n such that the equivalence*

$$m({}^tb(I_n+v)b) = m(a) \Leftrightarrow \min_{x\in S(a)}\varphi_{bx}(v) = 0$$

holds for any $v \in \mathcal{V}$.

Next is a generalization of [3, Lemma 3.4.4].

Lemma 3.3. *Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^tbb$.*

(*1*) *There exists a neighborhood $\mathcal{V} \subset V_n$ of 0 such that either $v = 0$ or $\phi({}^tb(I_n+v)b) < \phi(a)$ holds for any $v \in \mathcal{V}$ with $(\partial\log\phi)_b(v) \leq 0$ and $I_n + v \in P_n$.*

(*2*) *Let $\mathcal{C}$ be a closed cone in V_n such that $(\partial\log\phi)_b(v) > 0$ for all $v \in \mathcal{C}\setminus\{0\}$. Then there exists $\alpha > 0$ such that $\phi({}^tb(I_n+v)b) > \phi(a)$ holds for any $v \in \mathcal{C}$ with $0 < \sigma(v) < \alpha$.*

Proof. (*1*) We set $\mathcal{V} = \{v \in V_n \mid I_n + v \in P_n\}$, and fix $v \in \mathcal{V}\setminus\{0\}$ so that

$(\partial \log \phi)_b(v) \le 0$. By the strict log-concavity, we obtain

$$\begin{aligned}
&\frac{\log \phi({}^t b(I_n+\theta v)b) - \log \phi(a)}{\theta} \\
&> \frac{\{(1-\theta)\log \phi(a) + \theta \log \phi({}^t b(I_n+v)b)\} - \log \phi(a)}{\theta} \\
&= -\log \phi(a) + \log \phi({}^t b(I_n+v)b)
\end{aligned}$$

for all $0<\theta<1$. As $\theta \downarrow 0$, we have

$$(\partial \log \phi)_b(v) \ge -\log \phi(a) + \log \phi({}^t b(I_n+v)b).$$

From $(\partial \log \phi)_b(v) \le 0$, it follows $\log \phi({}^t b(I_n+v)b) \le \log \phi(a)$. We show that the equality $\log \phi({}^t b(I_n+v)b) = \log \phi(a)$ leads us to a contradiction. Assume this equality holds. We fix a constant θ_0 with $0<\theta_0<1$. By using the strict log-concavity again, we have

$$\begin{aligned}
\log \phi({}^t b(I_n+\theta_0 v)b) &> (1-\theta_0)\log \phi(a) + \theta_0 \log \phi({}^t b(I_n+v)b) \\
&= (1-\theta_0)\log \phi(a) + \theta_0 \log \phi(a) \\
&= \log \phi(a),
\end{aligned}$$

and hence

$$\begin{aligned}
(\partial \log \phi)_b(\theta_0 v) &= \lim_{t\to 0} \frac{\log \phi({}^t b(I_n+t\theta_0 v)b) - \log \phi(a)}{t} \\
&= \lim_{t\to +0} \frac{\log \phi({}^t b(I_n+t\theta_0 v)b) - \log \phi(a)}{t} \\
&\ge \lim_{t\to +0} \frac{\{(1-t)\log \phi(a) + t\log \phi({}^t b(I_n+\theta_0 v)b)\} - \log \phi(a)}{t} \\
&= \lim_{t\to +0} \frac{-t\log \phi(a) + t\log \phi({}^t b(I_n+\theta_0 v)b)}{t} \\
&= -\log \phi(a) + \log \phi({}^t b(I_n+\theta_0 v)b) \\
&> 0\,.
\end{aligned}$$

This contradicts $(\partial \log \phi)_b(\theta_0 v) = \theta_0(\partial \log \phi)_b(v) \le 0$. Therefore, $\log \phi({}^t b(I_n+v)b)$ is less than $\log \phi(a)$, i.e., $\phi({}^t b(I_n+v)b) < \phi(a)$.

(2) Let $\Sigma = \{\, u \in V_n \mid \sigma(u) = 1 \,\}$ be the unit sphere of V_n and let $u \in \mathcal{C} \cap \Sigma$. The function $g_u(t) = \log \phi(I_n + tu)$ in $t \in \mathbb{R}$ is defined on a sufficiently small neighborhood of 0. Since $(\partial \log \phi)_b(u) > 0$, there exists $0 < t_u < 1$ such that $\log \phi({}^t b(I_n+tu)b) > \log \phi(a)$ for all $t \in (0, t_u]$. By the continuity of ϕ, there exists an open neighborhood $\mathcal{O}(u)$ of u in $\mathcal{C} \cap \Sigma$ such that $\log \phi({}^t b(I_n+t_u u')b) > \log \phi(a)$ for all $u' \in \mathcal{O}(u)$. From this and the strict log-concavity, it follows

$$\begin{aligned}
\log \phi({}^t b(I_n+\theta t_u u')b) &> (1-\theta)\log \phi(a) + \theta \log \phi({}^t b(I_n+t_u u')b) \\
&> (1-\theta)\log \phi(a) + \theta \log \phi(a) \\
&= \log \phi(a)
\end{aligned}$$

for any $0 < \theta < 1$. Hence we deduce $\phi({}^t b(I_n + tu')b) > \phi(a)$ for all $t \in (0, t_u]$. Since $\mathcal{O}(u)$ $(u \in \mathcal{C} \cap \Sigma)$ is an open covering of the compact set $\mathcal{C} \cap \Sigma$, there exists a finite covering

$$\mathcal{C} \cap \Sigma \subset \bigcup_{i=1}^{r} \mathcal{O}(u_i)$$

with $u_1, \ldots, u_r \in \mathcal{C} \cap \Sigma$. Put $\alpha = \min\{t_{u_1}, \ldots, t_{u_r}\}$. For $v \in \mathcal{C}$ with $0 < \sigma(v) < \alpha$, put $u = v/\sigma(v) \in \mathcal{C} \cap \Sigma$. There exists i such that $u \in \mathcal{O}(u_i)$. Then we have $\phi({}^t b(I_n + tu)b) > \phi(a)$ for $t \in (0, \alpha] \subset (0, t_{u_i}]$. At $t = \sigma(v)$, this yields $\phi({}^t b(I_n + v)b) > \phi(a)$. □

Now the proof of Theorem 3.1 follows from the same argument as in the proof of [3, Theorem 3.4.6] by using Lemmas 3.2 and 3.3.

4. Hermite like constants

In this section, we assume that ϕ is an arbitrary type one class function. Let $\partial K_1(m)$ be the boundary of the kernel $K_1(m)$ and $\partial^0 K_1(m)$ the set of all vertices of $\partial K_1(m)$. It is known that $\mathbb{R}_{>0} \cdot \partial^0 K_1(m)$ coincides with the set of all perfect elements of P_n and $K_1(m)$ is the convex hull of $\mathbb{R}_{\geq 1} \cdot \partial^0 K_1(m)$, (cf. [8, §3.1]). The unimodular group $GL_n(\mathbb{Z})$ acts on $\partial^0 K_1(m)$ as $(a, g) \mapsto {}^t gag$ for $a \in \partial^0 K_1(m)$ and $g \in GL_n(\mathbb{Z})$. Let S_p denote a complete set of representatives for $\partial^0 K_1(m)/GL_n(\mathbb{Z})$. This S_p is a finite set. From $P_n \subset \mathbb{R}_{>0} \cdot K_1(m)$, it follows

$$\sup_{a \in P_n} F_\phi(a) = \sup_{a \in K_1(m)} F_\phi(a) = \sup_{a \in \partial K_1(m)} \frac{1}{\phi(a)} .$$

Any $a \in K_1(m)$ is represented as $a = \lambda_1 a_1 + \cdots + \lambda_r a_r$ by some $a_1, \cdots, a_r \in \partial^0 K_1(m)$ and $\lambda_1, \cdots, \lambda_r \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_r \geq 1$. Then, since $\phi(a) \geq \min\{\phi(a_1), \cdots, \phi(a_r)\}$ and ϕ is a class function, one has

$$\sup_{a \in \partial K_1(m)} \frac{1}{\phi(a)} = \sup_{a \in \partial^0 K_1(m)} \frac{1}{\phi(a)} = \max_{a \in S_p} \frac{1}{\phi(a)} .$$

The constant $\delta_\phi = \sup_{a \in P_n} F_\phi(a)$ is regarded as an analog of Hermite's constant γ_n. We show that "the dual constant" of δ_ϕ also exists.

Proposition 4.1. *If ϕ is a type one class function, then the superior*

$$\widehat{\delta}_\phi = \sup_{a \in P_n} \frac{\phi(a)}{w(a)}$$

is finite.

Proof. There exists an $\alpha > 0$ such that $\phi(\alpha a) \geq 1$ for all $a \in S_p$. Since $K_1(m)$ is the convex hull of $\mathbb{R}_{\geq 1} \cdot \partial^0 K_1(m)$ and ϕ is a type one class function, we have

$$\alpha \cdot K_1(m) \subset K_1(\phi) .$$

By taking the duals of both sides,

$$K_1(\phi^\circ) \subset \frac{1}{\alpha} \cdot K_1(w).$$

By replacing ϕ with ϕ° and using the relation $K_1(\phi) \subset K_1(\phi^{\circ\circ})$, we have

$$\alpha \cdot K_1(\phi) \subset K_1(w).$$

Since $(\alpha/\phi(a))a \in \alpha \cdot K_1(\phi) \subset K_1(w)$ for any $a \in P_n$, ϕ/w is bounded by α on P_n. □

Proposition 4.2. *For any type one class function ϕ, $\delta_\phi = \widehat{\delta}_{\phi^\circ}$.*

Proof. Since $m(a) \leq \delta_\phi \phi(a)$ for any $a \in P_n$, we have $\delta_\phi \cdot K_1(m) \subset K_1(\phi)$. By passing to the dual, we have $\delta_\phi \cdot K_1(\phi^\circ) \subset K_1(w)$. This leads us to $\widehat{\delta}_{\phi^\circ} \leq \delta_\phi$. In a similar fashion, we obtain

$$\sup_{a \in P_n} \frac{m(a)}{\phi^{\circ\circ}(a)} \leq \widehat{\delta}_{\phi^\circ}.$$

Since $\phi^{\circ\circ} = \phi$ on P_n by Proposition 2.9, we have $\delta_\phi \leq \widehat{\delta}_{\phi^\circ}$. □

For a type one class function ϕ, we set $\xi_\phi = \delta_\phi \cdot \delta_{\phi^\circ}$.

Proposition 4.3. *The inequality $\xi_w \leq \xi_\phi$ holds for any type one class function ϕ.*

Proof. As in the proof of Proposition 4.2, we have $m(a) \leq \delta_\phi \cdot \phi(a)$ and $\delta_\phi \cdot w(a) \geq \phi^\circ(a)$ for all $a \in P_n$. By replacing ϕ with ϕ°, we get $\delta_{\phi^\circ} \cdot w(a) \geq \phi^{\circ\circ}(a) = \phi(a)$ for $a \in P_n$. This implies $\delta_w \leq \delta_\phi \cdot \delta_{\phi^\circ}$. □

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