Journal of Convex Analysis
Volume 18 (2011), No. 3, 737–747

The Demyanov Metric for Convex, Bounded Sets and Existence of Lipschitzian Selectors*

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Received: December 6, 2009
Revised manuscript received: July 25, 2010

It is shown that an alternative formula for the Demyanov metric permits to extend this metric to the family of convex, bounded sets which need not be closed. Existence of Lipschitzian, linear selectors is discussed.

Keywords: Convex and bounded sets, the Demyanov metric, the Hausdorff metric, linear selectors

1991 Mathematics Subject Classification: 52A20, 54C65, 58C06, 54E35

1. Introduction

Apart well known and often used Hausdorff distance between subsets of metric spaces there are some other essentially different and more adequate in some situations – we have in mind here especially the Demyanov and Bartels-Pallaschke metrics defined in the family of convex, compact subsets of $\mathbb{R}^d$. They were applied in particular in relation to the quasi-differentiability of multivalued functions – a tool important in optimization. The basic information and further references can be found in [4], [5], [6], [9].

We show in the present paper that a slightly different formula for the Demyanov metric allows us to extend the scope of its validity to convex, bounded but not necessarily closed sets. In order to do this we introduce in that family of sets a relation of equivalence and define a metric in each class of equivalence. This metric is in fact that of uniform convergence in the set of mappings whose arguments are orthonormal systems of vectors in $\mathbb{R}^d$ and values are convex sets orthogonal to these vectors (this is strictly connected with the decomposition of convex sets into the relative interiors of extremal faces).

*This research was supported by a grant from the Faculty of Mathematics and Information Science of Warsaw University of Technology.

ISSN 0944-6532 / $ 2.50

This variant of the formula for the Demyanov metric allows us to compare it to a metric introduced in 1975 by Pliś in a paper on the problem of unicity of optimal controls [10]. It can be seen that the Pliś and Demyanov metrics coincide.

The final part of the paper is devoted to linear selectors. In [7], [11] one can find the proof of existence of linear selectors s defined on the family of compact, convex sets such that $s(A) \in \mathrm{ri}A$ (ri denotes the relative interior). These selectors are Lipschitzian with respect to the Hausdorff metric. Recently, in [3], a method of finding selectors for which $s(A)$ may also belong to the relative boundary of A has been studied. For these selectors the Lipschitz condition relative to the Demyanov metric is adequate (the Hausdorff metric being too weak). At the end of the paper we show the existence of selectors s defined for A convex, bounded, not necessarily closed. Moreover, for any fixed A_0 and $a_0 \in A_0$ we have s with $s(A_0) = a_0$. The Lipschitz condition is satisfied with respect to the variant of Demyanov metric introduced previously.

2. Basic notions and preliminaries

We keep as close as possible to the notations from [9] and [12].

By $\widetilde{\mathcal{K}}^d$ we denote the family of convex, bounded, nonempty subsets of $\mathbb{R}^d$ and by $\mathcal{K}^d$ its subfamily composed of compact sets. For any $A \in \widetilde{\mathcal{K}}^d$ and $u \in \mathbb{R}^d$

$$p_A(u) = \sup_{a\in A} \langle a, u\rangle, \qquad A(u) = \{a \in A : \ \langle a, u\rangle = p_A(u)\}$$

and by recurrence $A(u_1, \dots, u_i) = A(u_1, \dots, u_{i-1})(u_i)$. (It may happen that some of the sets $A(u_1, \dots, u_i)$ are empty if A is not closed.)

$\mathcal{E}$ will stand for the set of all orthonormal sequences $(e_1, \dots, e_k)$, $1 \le k \le d$. We shall often use a single, capital letter to denote elements of $\mathcal{E}$, like $E = (e_1, \dots, e_k)$.

The Hausdorff distance of $V, W \subset \mathbb{R}^d$ is defined by

$$\rho_H(V, W) = \max\{\epsilon(V, W), \epsilon(W, V)\}$$

where

$$\epsilon(V, W) = \sup_{v\in V} \mathrm{dist}(v, W) = \inf\{\varepsilon > 0 : \ V \subset W + \varepsilon\mathbb{U}\}$$

($\mathbb{U}$ is the open unit ball). This is a metric in the family of closed, bounded sets in $\mathbb{R}^d$.

Upper semicontinuity of a map ϕ defined on some metric space X whose values are subsets of $\mathbb{R}^d$ means that $\lim_{x\to x_0} \epsilon(\phi(x), \phi(x_0)) = 0$ for all $x_0 \in X$.

For each $A \in \mathcal{K}^d$ the map $u \mapsto A(u)$ is upper semicontinuous ([1], page 53, Theorem 6). Let us prove a lemma precising in some sense this property – it will be useful in the sequel.

Lemma 2.1. *Let $A \in \mathcal{K}^d$, $v, w \in \mathbb{R}^d$ be non-zero and not parallel. Then*

$$\lim_{\alpha\to 0+} \epsilon(A(v + \alpha w), A(v, w)) = 0$$

Proof. In view of the compactness of A it is enough to prove that for any $\alpha_j \to 0+$ and any convergent sequence $y_j \in A(v + \alpha_j w)$ its limit y_0 belongs to $A(v, w)$.

Remark first that due to the upper semicontinuity of the map $u \mapsto A(u)$ we have $y_0 \in A(v)$.

Let us choose some element x in $A(v, w)$. Then $\langle y_j, v + \alpha_j w\rangle \geq \langle x, v + \alpha_j w\rangle$, as $y_j \in A(v + \alpha_j w)$. Together with the inequalities $\langle y_j, v\rangle \leq \langle x, v\rangle$ this results in $\langle y_j, \alpha_j w\rangle \geq \langle x, \alpha_j w\rangle$. But $\alpha_j > 0$ so we get $\langle y_0, w\rangle \geq \langle x, w\rangle$ and this finally implies that $y_0 \in A(v, w)$ which was to be proved. $\square$

We use the notation $u \perp v$ to say that u, v are orthogonal vectors in $\mathbb{R}^d$. For $U, V \subset \mathbb{R}^d$ we write $U \perp V$ if $u \perp v$ for any $u \in U$, $v \in V$. $U^\perp$ denotes the subspace orthogonal to all elements of U.

3. The Demyanov difference and metric in $\mathcal{K}^d$.

We recall briefly how the Demyanov difference and metric are usually introduced (see [9] for example) and give an alternative formula for the Demyanov metric which permits further extensions to the case of convex but not necessarily closed sets. We recall a paper by Pliś [10] who in a different context and using other language defined a metric which occured to be equal to the Demyanov metric. At the end of the section, some metrics in $\mathcal{K}^d$ are proposed which are intermediary between the Hausdorff and Demyanov metrics.

Let T_A, for $A \in \mathcal{K}^d$, be the set of nonzero vectors $v \in \mathbb{R}^d$ for which $A(v)$ is a singleton. The set $\mathbb{R}^d \setminus T_A$ has always measure 0 and so for any two $A, B \in \mathcal{K}^d$ the complement of $T_A \cap T_B$ has also measure 0.

The Demyanov difference of $A, B \in \mathcal{K}^d$ is defined by

$$A \dot{-} B = \operatorname{clco}\{A(v) - B(v) : v \in T_A \cap T_B\} \tag{1}$$

where clco stands for the closed, convex hull. It is enough to take in (1) v from any $T \subset T_A \cap T_B$ such that the complement of T in $\mathbb{R}^d$ has measure 0 or even v from $T \cap S^{d-1}$.

The Demyanov distance of $A, B \in \mathcal{K}^d$ is defined as

$$\rho_D(A, B) = \max\{\|z\| : z \in A \dot{-} B\} \tag{2}$$

It is well known and easy to prove that $\rho_H(A, B) \leq \rho_D(A, B)$ but the metrics are not equivalent. We refer to [5], [9] for these and other properties of the metric space $(\mathcal{K}^d, \rho_D)$ and its applications. Here we recall and prove some properties which will serve to give a slightly different formula for ρ_D. This formula will be in some sense worse than that from the definition – it will use a redundant information – but will permit to extend the Demyanov metric to the family of convex, not necessarily closed sets.

3.1. A representation of the Demyanov metric.

The following property can be found in [9] – we give here a slightly different proof.

Lemma 3.1. *For $A, B \in \mathcal{K}^d$ and any $v \in \mathbb{R}^d$*

$$A(v) \div B(v) \subset A \div B \ . \tag{3}$$

Proof. This inclusion is obvious when $v = 0$ or $v \in T_A \cap T_B$. So it is enough to consider the case when at least one of sets $A(v)$, $B(v)$ is not a singleton and $v \neq 0$.

We show that for every $w \in T_{A(v)} \cap T_{B(v)}$ the difference $A(v,w) - B(v,w)$ belongs to $A \div B$ and this, in view of (1), proves the claim.

To this end it will be enough to find a sequence $u_n \in T_A \cap T_B$ such that $A(u_n) \to A(v,w)$ and $B(u_n) \to B(v,w)$.

Fix $n \in \mathbb{N}$. Due to Lemma 2.1 we may find $\alpha_n > 0$ sufficiently small to have

$$\epsilon(A(v+\alpha_n w), A(v,w)) < \frac{1}{2n} \quad \text{and} \quad \epsilon(B(v+\alpha_n w), B(v,w)) < \frac{1}{2n} \ .$$

There is $r_n > 0$ such that for $\|z - (v+\alpha_n w)\| < r_n$

$$\epsilon(A(z), A(v+\alpha_n w)) < \frac{1}{2n} \quad \text{and} \quad \epsilon(B(z), B(v+\alpha_n w)) < \frac{1}{2n} \ .$$

We take now $u_n \in T_A \cap T_B$ with $\|u_n - (v+\alpha_n w)\| < r_n$ to get

$$\|A(u_n) - A(v,w)\| < \frac{1}{n} \quad \text{and} \quad \|B(u_n) - B(v,w)\| < \frac{1}{n}$$

which provides the desired sequence. □

The lemma implies obviously that for $A, B \in \mathcal{K}^d$ and any $(e_1, \ldots, e_k) \in \mathcal{E}$

$$A(e_1, \ldots, e_k) \div B(e_1, \ldots, e_k) \subset A \div B \tag{4}$$

and

$$\rho_D(A(e_1, \ldots, e_k), B(e_1, \ldots, e_k)) \leq \rho_D(A, B) \tag{5}$$

We put $\mathcal{E}^k = \{(e_1, \ldots, e_j) \in \mathcal{E} : j \geq k\}$.

Corollary 3.2. *For all $A, B \in \mathcal{K}^d$ and $1 \leq k \leq d$*

$$\rho_D(A, B) = \sup_{E \in \mathcal{E}^k} \rho_H(A(E), B(E)) \tag{6}$$

Proof. The inequality "$\geq$" is a consequence of (5) and of the inequality $\rho_H(A(E), B(E)) \leq \rho_D(A(E), B(E))$. To justify "$\leq$" remark that if $u \in T_A \cap T_B$, with $\|u\| = 1$, then for any orthonormal system $(u, e_2, \ldots, e_k)$ we have $\|A(u) - B(u)\| = \|A(u, e_2, \ldots, e_k) - B(u, e_2, \ldots, e_k)\| = \rho_H(A(u, e_2, \ldots, e_k), B(u, e_2, \ldots, e_k))$ and (2) ends the proof. □

Corollary 3.2 may seem to be obvious and in fact it is – the right-hand side of (6) contains much redundant information. However, this expression for the Demyanov metric will permit to extend this metric to the family $\widetilde{\mathcal{K}}^d$ – it will be done in Section 4. It may be also considered to have some value in explaining the nature of the Demyanov metric.

For $k = 1$ we get in particular that

$$\rho_D(A, B) = \sup_{E \in \mathcal{E}} \rho_H(A(E), B(E))$$

and one can see that the Demyanov metric can be interpreted as the metric of uniform convergence of maps defined by $E \mapsto A(E)$ for each $A \in \mathcal{K}^d$. This, for example, implies instantly that the space $(\mathcal{K}^d, \rho_D)$ is complete – a well known fact.

3.2. A metric introduced by Pliś and its variants.

It is probably not widely known that Pliś [10] introduced a metric which can be proved to be equal to the Demyanov metric. He did it in the spirit of formula (6). Using our notation Pliś's definition can be writtten as

$$\eta(A, B) = \sup_{e \in S^{d-1}} \rho_H(A(e), B(e)) \tag{7}$$

The equality $\eta = \rho_D$ can be proved in exactly the same way as Corollary 3.2.

Let us remark that in order to define the Demyanov metric we may use not necessarily the whole sphere S^{d-1} but its dense subsets.

Proposition 3.3. *If V is a dense subset of S^{d-1} then for $A, B \in \mathcal{K}^d$*

$$\rho_D(A, B) = \sup_{v \in V} \rho_H(A(v), B(v)) \tag{8}$$

Proof. The inequality $\geq$ is obvious in view of (6). Let $e \in T_A \cap T_B$ and take any sequence $v_n \in V$ convergent to e. The mappings $u \to A(u)$, $u \to B(u)$ are upper semicontinuous, $A(e)$, $B(e)$ are singletons, so both sequences $\rho_H(A(v_n), A(e))$, $\rho_H(B(v_n), B(e))$ tend to 0. This implies

$$\rho_H(A(v_n), B(v_n)) \to \|A(e) - B(e)\|$$

and the opposite inequality follows which ends the proof. □

When V is not dense in S^{d-1} then the right-hand side of (8) does not define a metric. However, adding a term equal to the Hausdorff metric we may get a whole family of intermediary metrics stronger than the Hausdorff and weaker than the Demyanov metric. Namely, let $V \subset S^{d-1}$ be arbitrary and put

$$\eta_V(A, B) = \rho_H(A, B) + \sup_{v \in V} (\rho_H(A(v), B(v)) \tag{9}$$

Pliś compared in [10] the metric (7) introduced by himself to the Hausdorff metric. He noticed, of course, the inequality $\rho_H(A, B) \leq \eta(A, B)$ but discussed also their relation in the family of p-convex sets. A set $A \in \mathcal{K}^d$ is p-convex if for every $a \in \partial A$ and $v \in S^{d-1}$ such that $a \in A(v)$ the inequality

$$\langle x - a, v\rangle + p \cdot \|x - a\|^2 \leq 0$$

holds for all $x \in A$.

The following property is proved in [10]. Fix $p > 0$ and $M > 0$. There is then a constant $K > 0$ such that if $A, B \in \mathcal{K}^d$, $A, B \subset M\mathbb{U}$ and at least one of them is p-convex then

$$\eta(A, B)^2 \leq K\rho_H(A, B)$$

An obvious consequence of this inequality is the equivalence of ρ_D and ρ_H in the family of p-convex sets. In fact, it is known and easy to prove that this equivalence holds in the family of strictly convex sets.

4. The Demyanov metric in subspaces of $\widetilde{\mathcal{K}}^d$.

We put $\mathcal{E}_0 = \mathcal{E} \cup \{0\}$ and introduce in $\widetilde{\mathcal{K}}^d$ the following equivalence relation.

Definition 4.1. $A \sim B$ iff for every $E \in \mathcal{E}_0$ we have

$$A(E) \neq \emptyset \Leftrightarrow B(E) \neq \emptyset$$

Remark that if we fix an equivalence class then the set $\mathcal{U}$ composed of $E \in \mathcal{E}_0$ for which $A(E) \neq \emptyset$ is common for all the representants. It is thus possible to denote this equivalence class using $\mathcal{U}$ – we shall write $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$.

The sets $\mathcal{U} \subset \mathcal{E}_0$ corresponding to some equivalence class will be called admissible. Any admissible $\mathcal{U}$ satisfies the following two obvious conditions:

(i) $0 \in \mathcal{U}$

(ii) $(e_1, \ldots, e_k, e_{k+1}) \in \mathcal{U} \Rightarrow (e_1, \ldots, e_k) \in \mathcal{U}$

However, the conditions (i), (ii) are not sufficient for a set $\mathcal{U} \subset \mathcal{E}_0$ to be admissible which can be observed for $\mathcal{U}$ described in the following example.

Example 4.2. Let $d = 2$ and $\mathcal{U} = \{0\} \cup S^1$.

Suppose that $\widetilde{\mathcal{K}}^d_{\mathcal{U}} \neq \emptyset$ and take any $A \in \widetilde{\mathcal{K}}^d_{\mathcal{U}}$. Let e_1 be such that $\bar{A}(e_1)$ is an exposed point in $\bar{A}$ – the bar over a set A denotes its closure. Then $\bar{A}(e_1) \in A$ and for $e_2 \in S^1$ orthogonal to e_1 we have $A(e_1) = A(e_1, e_2) \neq \emptyset$ wheras $(e_1, e_2) \notin \mathcal{U}$ and so $\mathcal{U}$ is not admissible.

The formula (6) from Theorem 3.2 allows us to define in a natural way the Demyanov metric in $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$.

Fix $d \geq 1$ and a nonempty, admissible set $\mathcal{U} \subset \mathcal{E}_0$.

Definition 4.3. For $A, B \in \widetilde{\mathcal{K}}^d_{\mathcal{U}}$ we define

$$\rho_{\mathcal{U}}(A, B) = \sup_{E \in \mathcal{U}} \rho_H(A(E), B(E)) \tag{10}$$

Proposition 4.4. *$\rho_{\mathcal{U}}$ defines a metric in $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$.*

The symmetry and the triangle inequality are obvious as well as the implication $A = B \Rightarrow \rho_{\mathcal{U}}(A, B) = 0$. We check that $\rho_{\mathcal{U}}(A, B) = 0$ implies $A = B$. Suppose, on the contrary, that $\rho_{\mathcal{U}}(A, B) = 0$ but $A \setminus B \neq \emptyset$ and fix x in $A \setminus B$. For some $E \in \mathcal{U}$ this x must belong to $\operatorname{ri} A(E)$ – ri stands here for the relative interior of a convex set. The equality $\rho_{\mathcal{U}}(A, B) = 0$ implies $\rho_H(A(E), B(E)) = 0$ and so $\operatorname{ri} A(E) = \operatorname{ri} B(E)$. Thus $x \in B$ which is the seeked contradiction. $\square$

From now on we shall consider the space $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$ always equipped with the metric $\rho_{\mathcal{U}}$ if not stated otherwise.

The metric $\rho_{\mathcal{U}}$ provides a separable space only if $\mathcal{U}$ is finite (see Example 3.1 in [9]). The discussion of completeness, or rather noncompleteness, of spaces $(\widetilde{\mathcal{K}}^d_{\mathcal{U}}, \rho_{\mathcal{U}})$ and characterizing their complements is postponed to a paper in preparation. Let us mention here only that no such space is complete except for $d = 1$ and $\mathcal{U} = \{0, 1, -1\}$.

5. Selectors in $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$.

Much attention has been given to continuous, linear selectors (additive and positively homogeneous) in the family of closed, convex sets and many important results have been obtained. To mention just a few we refer to [3], [8], [11] and references therein. They all have their origins in the notion of Steiner point.

We consider any space $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$ and the problem of existence of linear selectors $s : \widetilde{\mathcal{K}}^d_{\mathcal{U}} \to \mathbb{R}^d$ satisfying the Lipschitz condition. For selectors we should have by definition $s(A) \in A$ but we may require in addition that

$$s(A_0) = a_0 \tag{11}$$

for any fixed $A_0 \in \widetilde{\mathcal{K}}^d_{\mathcal{U}}$ and $a_0 \in A_0$. We shall show also how the Lipschitz constant of s may depend on the nature of $\mathcal{U}$.

The solution of the problem of existence of selectors satisfying (11) when $a_0 \in \operatorname{int} A_0$ (or more generally $a_0 \in \operatorname{ri} A_0$) is covered, for example, in [11] so we consider the case $a_0 \in A_0 \setminus \operatorname{int} A_0$.

Let $E = (e_1, \ldots, e_k) \in \mathcal{U}$ be such that $a_0 \in A_0(E)$ but for no nonzero $v \perp E$ the set $A_0(e_1, \ldots, e_k, v)$ contains a_0 – this means that $A_0(E)$ is the minimal face containing a_0. We claim that $a_0 \in \operatorname{ri} A_0(e_1, \ldots, e_k)$ (recall that ri denotes the relative interior). In fact, when $k = d$ then $A_0(E)$ is a singleton and the condition holds. Suppose now that $k < d$ but $a_0 \notin \operatorname{ri} A_0(E)$. Then $A_0(E) \subset a_0 + E^{\perp}$, $a_0 \in \operatorname{rb} A_0(E)$, where rb stands for the relative boundary, and so there exists $e_{k+1} \perp E$ for which $a_0 \in A_0(E)(e_{k+1}) = A_0(e_1, \ldots, e_k, e_{k+1})$ contrary to the definition of E.

We define now a selector $s : \widetilde{\mathcal{K}}^d_{\mathcal{U}} \to \mathbb{R}^d$ for which $s(A_0) = a_0$. According to Theorem 4.1 and 4.2 in [11] (Section 4. The harmonic representation of convex sets) there is a probability measure α on S^{d-1} with strictly positive density with respect to the surface measure for which

$$a_0 = \int\limits_{S^{d-1} \cap T_{\overline{A(E)}}} \overline{A_0(E)}(e)\, d\alpha(e) \tag{12}$$

The complement of $S^{d-1} \cap T_{\overline{A(E)}}$ in S^{d-1} has the surface measure equal 0 and so, using the fact that α is absolutely continuous with respect to the surface measure, we have $\alpha(S^{d-1} \cap T_{\overline{A(E)}}) = \alpha(S^{d-1})$. In view of that we may integrate in integrals like (12) over whole S^{d-1} even if from the formal point of view it may be then the integral of a multivalued map. This multivalued map is, by the way, upper semicontinuous which implies the measurability of our integrand.

We define the needed selector by the formula

$$s(A) = \int_{S^{d-1}} \overline{A(E)}(e)\, d\alpha(e) \tag{13}$$

Let us check first that it is actually a selector – the equality (11) is obvious. In view of Theorem 4.2 in [11] we have $s(A) \in \operatorname{ri} A(E)$ and as $E \in \mathcal{U}$ so $\operatorname{ri} A(E) \subset A$.

The mapping $s(\cdot)$ is a composition

$$A \mapsto \overline{A(E)} \mapsto \int_{S^{d-1}} \overline{A(E)}(e)\, d\alpha(e)$$

which implies immediately that it is additive and positively homogenous as both composed mappings are.

We check now the Lipschitz condition. According to Section 4 in [11] there is a constant L such that for $A, B \in \widetilde{\mathcal{K}}^d_{\mathcal{U}}$ we have $\|s(A) - s(B)\| \le L\rho_H(\overline{A(E)}, \overline{B(E)})$ and in view of $\rho_H(\overline{A(E)}, \overline{B(E)}) = \rho_H(A(E), B(E)) \le \rho_{\mathcal{U}}(A, B)$ we get finally

$$\|s(A) - s(B)\| \le L\rho_{\mathcal{U}}(A, B)$$

The Lipschitz constant L depends on the position of a_0 in $A_0(E)$ and in general it cannot be bounded from above. However, in some cases it can be proved not to exceed 1 and this depends on a property of $\mathcal{U}$.

Suppose first that $k = d$. Then we put $s(A) = A(e_1, \ldots, e_d)$. We have, by definition, $s(A_0) = A_0(e_1, \ldots, e_d) = a_0$. Moreover, in view of Definition 4.3

$$\|s(A) - s(B)\| = \|A(e_1, \ldots, e_d) - B(e_1, \ldots, e_d)\| \le \rho_{\mathcal{U}}(A, B)$$

so s is Lipschitzian with constant 1.

Suppose now $k < d$ and assume that $A(E)(e) \ne \emptyset$ for all $e \in S^{d-1}$. If e is linearly independent from E then $A(E)(e) = A(E)(e')$ for some $e' \in S^{d-1}$ such that $(E, e') \in \mathcal{U}$. So $\rho_H(A(E, e), B(E, e)) = \rho_H(A(E, e'), B(E, e')) \le \rho_{\mathcal{U}}(A, B)$. The set of $e \in S^{d-1}$

which are linearly dependent from E has the surface measure equal 0 and due to the absolute continuity of α we get

$$\begin{aligned}\|s(A)-s(B)\| &\le \left\|\int_{S^{d-1}}(A(E)(e)-B(E)(e))\,d\alpha(e)\right\| \\ &\le \int_{S^{d-1}}\rho_H(A(E,e),B(E,e))\,d\alpha(e)\le \rho_{\mathcal{U}}(A,B)\end{aligned}$$

Note that the function $e\mapsto \rho_H(A(E,e),B(E,e))$ is continuous at every point e for which both $A(E,e)$ and $B(E,e)$ are singletons and so it is continuous almost everywhere on S^{d-1}. It is also bounded and thus integrable.

Let us consider yet the case when $\mathcal{U}$ is composed only of the zero vector which means that $\widetilde{\mathcal{K}}^d_{\mathcal{U}}$ coincides with the family of nonempty, convex, open subsets of $\mathbb{R}^d$. Here $\rho_{\mathcal{U}}$ coincides with the Hausdorff distance and Theorems 4.1 and 4.2 from [11] can be directly applied.

We summarize the above considerations in a theorem.

Theorem 5.1. *For every $A_0\in\widetilde{\mathcal{K}}^d_{\mathcal{U}}$ and $a_0\in A_0$ there is a linear selector $s:\widetilde{\mathcal{K}}^d_{\mathcal{U}}\to\mathbb{R}^d$ satisfying $s(A_0)=a_0$ and*

$$\|s(A)-s(B)\|\le L\rho_{\mathcal{U}}(A,B) \tag{14}$$

for some $L\ge 0$ and all $A,B\in\widetilde{\mathcal{K}}^d_{\mathcal{U}}$.

With an additional assumption we get another estimate for the Lipschitz constant with respect to the metric $\rho_{\mathcal{U}}$ than the one that can be derived from [11].

Theorem 5.2. *Let $E\in\mathcal{U}$ be such that $a_0\in A_0(E)$ and $a_0\notin A(E,v)$ for all $(E,v)\in\mathcal{U}$ with $v\in E^{\perp}\cap S^{d-1}$. If for all such v we have $(E,v)\in\mathcal{U}$ then we may put $L=1$ in* (14).

Such E as in the assumption above always exists and $A(E)$ is the minimal face containing a_0.

In particular, the assumption of Theorem 5.2 is always satisfied for $\mathcal{U}=\mathcal{E}_0$ and so we have

Corollary 5.3. *For every $A_0\in\widetilde{\mathcal{K}}^d_{\mathcal{E}_0}$ and $a_0\in A_0$ there is a linear selector $s:\widetilde{\mathcal{K}}^d_{\mathcal{E}_0}\to\mathbb{R}^d$ for which $s(A_0)=a_0$ and*

$$\|s(A)-s(B)\|\le\rho_{\mathcal{E}_0}(A,B)$$

for all $A,B\in\widetilde{\mathcal{K}}^d_{\mathcal{E}_0}$.

It may happen that E is composed only of the zero vector – this corresponds to the case when $a_0\in\operatorname{int}A_0$.

6. Some comments.

Baier and Farkhi define in [2] a metric very near to the Demyanov metric and investigate their relations. The difference between these two approaches can be explained using formula (6). For the Demyanov metric it contains terms $\rho_H(A(E), B(E))$ while in order to get the metric ρ_V from [2] one should replace them by the Hausdorff distance of projections $\pi(A(E))$ and $\pi(B(E))$ of $A(E)$ and $B(E)$ on $E^\perp$. So

$$\rho_V(A, B) = \sup_{E\in\mathcal{E}^k} \rho_H(\pi(A(E)), \pi(B(E)))$$

– true for each $k = 1, \dots, d$.

The Hausdorff distance of two sets can be expressed as the supremum of absolute values of differences of their support functions so we have

$$\rho_D(A, B) = \sup_{E\in\mathcal{E}^k} \sup_{v\in S^{d-1}} \left| p_{A(E)}(v) - p_{B(E)}(v) \right|$$

for the Demyanov metric and

$$\rho_V(A, B) = \sup_{E\in\mathcal{E}^k} \sup_{v\in S^{d-1}\cap E^\perp} \left| p_{A(E)}(v) - p_{B(E)}(v) \right|$$

for that of Baier and Farkhi.

A formula for ρ_V – analogous to the definition of Demyanov's metric given by formula (2) – can be written in the following way

$$\rho_V(A, B) = \sup_{v\in T_A\cap T_B} \sqrt{\|A(v) - B(v)\|^2 - \langle v, A(e) - B(e)\rangle^2}$$

Dentcheva defines in [7] generalized Steiner points of convex, compact sets in $\mathbb{R}^d$ applying the formula

$$\widetilde{\mathrm{St}}_\alpha(A) = \int_{\mathbb{U}} m(A(v))\, d\alpha(v)$$

where $m(U)$ stands for the element in $U \in K^d$ with minimal norm and α is a probability measure on $\mathbb{U}$ with density of class C^1. These generalized Steiner points provide linear selectors in $\mathcal{K}^d$ Lipschitzian with respect to the Hausdorff metric. Baier and Farkhi modify this formula to the shape

$$\mathrm{St}_\alpha(A) = \int_{\mathbb{U}} \mathrm{St}(A(v))\, d\alpha(v)$$

where $\mathrm{St}(U)$ denotes the usual Steiner point of the set U. They extend the family of probability measures using also convex combinations of Dirac measures concentrated on points of the sphere S^{d-1}. In that way the family of corresponding linear selectors is broader – it contains also selectors s for which $s(A_0) = \mathrm{St}(A_0(v_0))$ for any fixed $A_0 \in K^d$, $v_0 \in S^{d-1}$. These selectors are Lipshitzian with respect to the Demyanov metric.

Acknowledgements. The authors are grateful to Agnieszka Bogdewicz, Maria Moszyńska and Jerzy Grzybowski for valuable remarks and to the unknown referee for the remarks that helped us to improve the text.

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