Journal of Convex Analysis
Volume 18 (2011), No. 1, 139–152

A Weighted Steiner Minimal Tree for Convex Quadrilaterals on the Two-Dimensional K-Plane

Anastasios Zachos
University of Patras, Department of Mathematics,
26500 Rion, Greece
azachos@gmail.com

Received: August 18, 2009

We provide a method to find a weighted Steiner minimal tree for convex quadrilaterals on a two-dimensional hemisphere of radius $\frac{1}{\sqrt{K}}$, for $K > 0$ and the two dimensional hyperbolic plane of constant Gaussian Curvature K, for $K < 0$ by introducing a method of cyclical differentiation of the objective function with respect to four variable angles. By applying this method, we find a generalized solution to a problem posed by C. F. Gauss in the spirit of weighted Steiner trees.

Keywords: Steiner minimal tree, generalized convex quadrilaterals

1991 Mathematics Subject Classification: 51E12, 52A10, 52A55, 51E10

1. Introduction

In 1836, C. F. Gauss posed the following problem to the astronomer Schumacher (see [2], page 326, [4], Chapter 2): How to find a railway network of minimal total length which connects the four cities Bremen, Harburg (today part of the city of Hamburg), Hannover, and Braunschweig. In 1879, K. Bopp (see [3], [2], page 327) gave a complete solution to Gauss problem for any four given points in the two-dimensional Euclidean Space. He also gave a description of an experimental solution by applying the property of soapsuds to span a minimal surface between the given points and which have been developed later by R. Courant (see also [5], page 385–397). A formulation of the Steiner problem has been given in [5], page 360: "Given n points $A_1A_2...A_n$ to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system." In the classical paper of E. Gilbert and H. Pollak ([6], [2], page 328–329) the following fundamental result is proved:
A solution of the Steiner problem is a Steiner tree with at most $n-2$ Fermat-Torricelli (or Steiner) points which are vertices of the polygonal tree which do not belong in $\{A_1A_2...A_n\}$, where each Fermat-Torricelli point has valency 3, and the angle between any two edges incident with a Fermat-Torricelli point is of 120°.
The solution of the Steiner problem is not uniquely determined. For example, for the case that four given points forming a tetragon in the two-dimensional Euclidean Space, two equivalent solutions exist (see [5], page 361). Cases where a variant of

ISSN 0944-6532 / $ 2.50

Steiner problem is uniquely determined is given in [6], pages 6–7. Concerning the definition of the topologies of various Steiner trees on the two-dimensional Euclidean Space, you can consult [6].

In this paper, we provide a method to find a weighted (full) Steiner tree for convex quadrilaterals on the two-dimensional sphere, two dimensional hyperbolic plane by expressing the length of two geodesic arcs which are edges of the weighted Steiner tree as a function of the other two geodesic arcs (edges of the Steiner tree) and two angles which are formulated between these two geodesic arcs and two given sides of the quadrilateral, respectively. By applying a method of cyclical differentiation with respect to four angles, we derive that the two inner points which are located at the interior domain of the quadrilateral are two weighted Fermat-Torricelli points. A generalized solution of the Steiner problem for convex quadrilaterals (Gauss problem) is given on the two-dimensional Euclidean Space. We would like to note that we have excluded any degenerate minimal trees and we focus on the topology of a weighted Steiner minimal tree which is full and contains two points which are located at the interior domain of the convex quadrilateral. An open question is to derive a generalized condition of weighted inequalities such that a weighted Steiner tree exists and is unique. The answer to this question will generalize the floating and absorbed case of the generalized Fermat-Torricelli point which has been established by Y. Kupitz and H. Martini (see [2], page 250) in $\mathbb{R}^n$, on the two-dimensional sphere and the two-dimensional hyperbolic plane.

2. A weighted Steiner minimal tree for convex quadrilaterals on the K-plane.

We denote by K-plane ($\mathbb{S}^K$), the open hemisphere of radius $1/\sqrt{K}$ of the two-dimensional sphere S^2 if $K > 0$ and the Lobachevski plane (two-dimensional hyperboloid H^2) of curvature K if $K < 0$, the Euclidean plane $\mathbb{R}^2$ if $K = 0$ (see [1], page 2). Let $A_1A_2A_3A_4$ be a convex quadrilateral on the K-plane. Suppose that a positive number B_i(weight) corresponds to each vertex A_i, for $i = 1, 2, 3, 4$, respectively. We denote by d the length of the geodesic arc that connects A_0 with $A_{0'}$, a_{ij} the length of the geodesic arc that connects the vertex A_i with A_j, and α_{ijk}, the angle that is formulated between the geodesic arcs A_iA_j and A_jA_k, for $i, j, k \in \{0, 0', 1, 2, 3, 4\}$ and $i \neq j \neq k$. Furthermore, we denote by $a_{10} = a_1$, $a_{40} = a_4$, $a_{20'} = a_2$, and $a_{30'} = a_3$ (see Figure 2.1).

Theorem 2.1. *A weighted (full) Steiner minimal tree of $A_1A_2A_3A_4$ consists of two (weighted) Fermat-Torricelli points A_0, A_0' which are located at the interior convex domain with corresponding weights B_0=$B_{0'}$=B_5 and minimizes the objective function:*

$$B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + B_5d = \mathit{minimum}, \tag{1}$$

such that:

$$|B_i - B_j| < B_k < B_i + B_j \tag{2}$$

and

$$|B_l - B_m| < B_n < B_l + B_m \tag{3}$$

for $i, j, k \in \{1, 4, 5\}$, $l, m, n \in \{2, 3, 5\}$ and $i \neq j \neq k$, $l \neq m \neq n$.

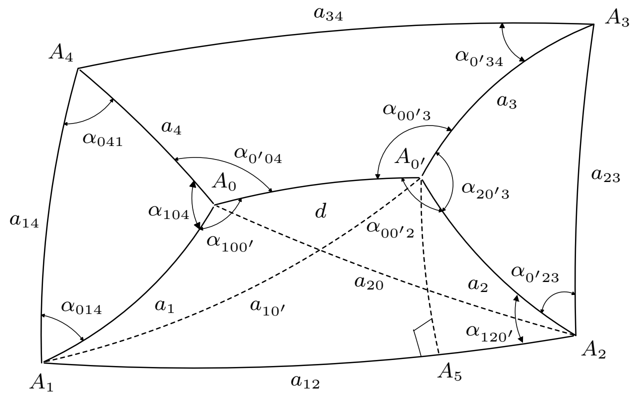

Figure 2.1

Proof of Theorem 2.1. (I) We study the case where $A_1A_2A_3A_4$ lies on a two-dimensional sphere S^2 of constant Gaussian curvature K or a two-dimensional hyperbolic plane H^2 of constant Gaussian curvature $-K$, for $K > 0$.

The length of geodesic arcs a_3, a_4, d can be expressed as functions of a_1, a_2, α_{014}, $\alpha_{120'}$, by applying the cosine law in a unified form (see [1], page 3) in the triangles $\nabla A_{0'}A_2A_3$, $\nabla A_0A_1A_4$, $\nabla A_0A_{0'}A_1$, respectively, on the K-plane:

$$\cos(\tilde{\kappa}a_3) = \cos(\tilde{\kappa}a_{23})\cos(\tilde{\kappa}a_2) + \sin(\tilde{\kappa}a_{23})\sin(\tilde{\kappa}a_2)\cos(\alpha_{123} - \alpha_{120'}), \tag{4}$$

$$\cos(\tilde{\kappa}a_4) = \cos(\tilde{\kappa}a_{14})\cos(\tilde{\kappa}a_1) + \sin(\tilde{\kappa}a_{14})\sin(\tilde{\kappa}a_1)\cos\alpha_{014}, \tag{5}$$

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_{10'})\cos(\tilde{\kappa}a_1) + \sin(\tilde{\kappa}a_{10'})\sin(\tilde{\kappa}a_1)\cos\alpha_{010'}\,, \tag{6}$$

or

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_{10'})\cos(\tilde{\kappa}a_1) + \sin(\tilde{\kappa}a_{10'})\sin(\tilde{\kappa}a_1)\cos(\alpha_{214} - \alpha_{014} - \alpha_{210'}) \tag{7}$$

where

$$\tilde{\kappa} = \begin{cases} \sqrt{K} & \text{if } K > 0, \\ i\sqrt{-K} & \text{if } K < 0. \end{cases}$$

From the cosine law in $\nabla A_1A_2A_{0'}$ we have:

$$\cos(\tilde{\kappa}a_{10'}) = \cos(\tilde{\kappa}a_{12})\cos(\tilde{\kappa}a_2) + \sin(\tilde{\kappa}a_{12})\sin(\tilde{\kappa}a_2)\cos\alpha_{120'}\,. \tag{8}$$

We show that the angle $\alpha_{210'}$ can be expressed as a function of a_2 and $\alpha_{120'}$.
We take a point A_5 that belongs in the geodesic arc A_1A_2 and the angle that is formulated between the geodesic arcs $A_{0'}A_5$ and A_1A_2 is $\frac{\pi}{2}$ and we denote by $a_{50'}$ the length of the geodesic arc $A_5A_{0'}$. From the sine law in the triangle $\nabla A_5A_2A_{0'}$ we have:

$$\frac{\sin(\tilde{\kappa}a_{50'})}{\sin\alpha_{120'}} = \frac{\sin(\tilde{\kappa}a_2)}{\sin\frac{\pi}{2}} \tag{9}$$

or

$$\sin(\tilde{\kappa}a_{50'}) = \sin\alpha_{120'}\sin(\tilde{\kappa}a_2). \tag{10}$$

From the sine law in the triangle $\nabla A_1A_5A_{0'}$ we have:

$$\frac{\sin(\tilde{\kappa}a_{50'})}{\sin\alpha_{210'}}=\frac{\sin(\tilde{\kappa}a_{10'})}{\sin\frac{\pi}{2}} \tag{11}$$

or

$$\sin(\alpha_{210'})=\frac{\sin(\tilde{\kappa}a_{50'})}{\sin(\tilde{\kappa}a_{10'})} \tag{12}$$

By replacing (10), (8) in (12), we derive that $\alpha_{210'}$ is a function of a_2 and $\alpha_{120'}$ and by replacing this result and (8) in (7), we derive that d depends on a_1, a_2, α_{014}, $\alpha_{120'}$.

By differentiating (1) with respect to a_1, a_2, α_{014}, $\alpha_{120'}$, respectively, we obtain:

$$B_1+B_3\frac{\partial a_3}{\partial a_1}+B_4\frac{\partial a_4}{\partial a_1}+B_5\frac{\partial d}{\partial a_1}=0, \tag{13}$$

$$B_2+B_3\frac{\partial a_3}{\partial a_1}+B_4\frac{\partial a_4}{\partial a_2}+B_5\frac{\partial d}{\partial a_2}=0, \tag{14}$$

$$B_3\frac{\partial a_3}{\partial \alpha_{014}}+B_4\frac{\partial a_4}{\partial \alpha_{014}}+B_5\frac{\partial d}{\partial \alpha_{014}}=0, \tag{15}$$

$$B_3\frac{\partial a_3}{\partial \alpha_{120'}}+B_4\frac{\partial a_4}{\partial \alpha_{120'}}+B_5\frac{\partial d}{\partial \alpha_{120'}}=0. \tag{16}$$

We calculate $\frac{\partial a_3}{\partial \alpha_{014}}$, $\frac{\partial a_4}{\partial \alpha_{014}}$, $\frac{\partial d}{\partial \alpha_{014}}$, in order to derive (15).

By differentiating (7) with respect to α_{014}, we have:

$$\tilde{\kappa}\frac{\partial d}{\partial \alpha_{014}}=-\frac{\sin(\tilde{\kappa}a_{10'})\sin(\tilde{\kappa}a_1)\sin(\alpha_{214}-\alpha_{014}-\alpha_{210'})}{\sin(\tilde{\kappa}d)}. \tag{17}$$

We apply the "sine law" in the triangle $\nabla A_1A_0A_{0'}$:

$$\frac{\sin(\tilde{\kappa}a_{10'})}{\sin(\tilde{\kappa}d)}=\frac{\sin(\alpha_{100'})}{\sin(\alpha_{214}-\alpha_{014}-\alpha_{210'})}. \tag{18}$$

By replacing (18) in (17), we get:

$$\tilde{\kappa}\frac{\partial d}{\partial \alpha_{014}}=-\sin(\tilde{\kappa}a_1)\sin\alpha_{100'}\,. \tag{19}$$

By differentiating (4) with respect to α_{014}, we derive that:

$$\frac{\partial a_3}{\partial \alpha_{014}}=0. \tag{20}$$

By differentiating (5) with respect to α_{014}, we derive that:

$$\frac{\partial a_4}{\partial \alpha_{014}}=\frac{\sin(\tilde{\kappa}a_{14})\sin(\tilde{\kappa}a_1)\sin\alpha_{014}}{\sin(\tilde{\kappa}a_4)}. \tag{21}$$

From the sine law in $\nabla A_0A_1A_4$, we have:

$$\frac{\sin(\tilde{\kappa}a_{14})}{\sin(\tilde{\kappa}a_4)} = \frac{\sin\alpha_{104}}{\sin\alpha_{014}}. \tag{22}$$

By replacing (22) in (21), we get:

$$\tilde{\kappa}\frac{\partial a_4}{\partial \alpha_{014}} = \sin(\tilde{\kappa}a_1)\sin\alpha_{104}. \tag{23}$$

By replacing (20), (23), (19) in (15), we have:

$$B_4\frac{\sin(\tilde{\kappa}a_1)}{\tilde{\kappa}}\sin\alpha_{104} - B_5\frac{\sin(\tilde{\kappa}a_1)}{\tilde{\kappa}}\sin\alpha_{100'} = 0 \tag{24}$$

or

$$\frac{B_4}{\sin\alpha_{100'}} = \frac{B_5}{\sin\alpha_{104}}. \tag{25}$$

From the cosine law in $\nabla A_0A_2A_{0'}$, the length of the geodesic arc can also be expressed as a function of a_1, a_2, α_{014}, $\alpha_{120'}$:

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_{20})\cos(\tilde{\kappa}a_2) + \sin(\tilde{\kappa}a_{20})\sin(\tilde{\kappa}a_2)\cos\alpha_{020'} \tag{26}$$

or

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_{20})\cos(\tilde{\kappa}a_2) + \sin(\tilde{\kappa}a_{20})\sin(\tilde{\kappa}a_2)\cos(\alpha_{120'} - \alpha_{120}). \tag{27}$$

From the cosine law in $\nabla A_0A_1A_2$, we have:

$$\cos(\tilde{\kappa}a_{20}) = \cos(\tilde{\kappa}a_1)\cos(\tilde{\kappa}a_{12}) + \sin(\tilde{\kappa}a_1)\sin(\tilde{\kappa}a_{12})\cos\alpha_{012} \tag{28}$$

or

$$\cos(\tilde{\kappa}a_{20}) = \cos(\tilde{\kappa}a_1)\cos(\tilde{\kappa}a_{12}) + \sin(\tilde{\kappa}a_1)\sin(\tilde{\kappa}a_{12})\cos(\alpha_{214} - \alpha_{014}). \tag{29}$$

We show that the angle α_{120} depends on a_1 and α_{014}.

We take a point $A_{5'}$ that belongs in the geodesic arc A_1A_2 and the angle that is formulated between the geodesic arcs $A_0A_{5'}$ and A_1A_2 is $\frac{\pi}{2}$ and we denote by $a_{5'0}$ the length of the geodesic arc $A_{5'}A_0$. From the sine law in the triangle $\nabla A_1A_{5'}A_0$, we have:

$$\frac{\sin(\tilde{\kappa}a_{5'0})}{\sin(\alpha_{214} - \alpha_{014})} = \frac{\sin(\tilde{\kappa}a_1)}{\sin\frac{\pi}{2}} \tag{30}$$

or

$$\sin(\tilde{\kappa}a_{5'0}) = \frac{\sin(\tilde{\kappa}a_1)}{\sin(\alpha_{214} - \alpha_{014})}. \tag{31}$$

From the sine law in the triangle $\nabla A_{5'}A_2A_0$ we have:

$$\frac{\sin(\tilde{\kappa}a_{5'0})}{\sin\alpha_{120}} = \frac{\sin(\tilde{\kappa}a_{20})}{\sin\frac{\pi}{2}} \tag{32}$$

or

$$\sin\alpha_{120} = \frac{\sin(\tilde{\kappa}a_{5'0})}{\sin(\tilde{\kappa}a_{20})}. \tag{33}$$

By replacing (31), (29) in (33), we derive that α_{120} is a function of a_1 and α_{014}.

By replacing (29) in (27) and by differentiating with respect to $\alpha_{120'}$, we have:

$$\tilde{\kappa}\frac{\partial d}{\partial \alpha_{120'}} = \frac{\sin(\tilde{\kappa}a_{20})\sin(\tilde{\kappa}a_2)\sin(\alpha_{120'} - \alpha_{120})}{\sin(\tilde{\kappa}d)}. \tag{34}$$

From the sine law in $\nabla A_0A_2A_{0'}$, we have:

$$\frac{\sin(\tilde{\kappa}a_{20})}{\sin(\tilde{\kappa}d)} = \frac{\sin\alpha_{00'2}}{\sin(\alpha_{120'} - \alpha_{120})}. \tag{35}$$

By replacing (35) in (34), we have:

$$\tilde{\kappa}\frac{\partial d}{\partial \alpha_{120'}} = \sin(\tilde{\kappa}a_2)\sin\alpha_{00'2}\,. \tag{36}$$

By differentiating (4) with respect to $\alpha_{120'}$, we have:

$$\tilde{\kappa}\frac{\partial a_3}{\partial \alpha_{120'}} = -\frac{\sin(\tilde{\kappa}a_2)\sin(\tilde{\kappa}a_{23})\sin(\alpha_{123} - \alpha_{120'})}{\sin(\tilde{\kappa}a_3)}. \tag{37}$$

From the sine law in $\nabla A_2A_3A_{0'}$ we have:

$$\frac{\sin(\tilde{\kappa}a_{23})}{\sin(\tilde{\kappa}a_3)} = \frac{\sin\alpha_{20'3}}{\sin(\alpha_{123} - \alpha_{120'})}. \tag{38}$$

By replacing (38) in (37), we have:

$$\tilde{\kappa}\frac{\partial a_3}{\alpha_{120'}} = -\sin(\tilde{\kappa}a_2)\sin\alpha_{20'3}. \tag{39}$$

By differentiating (5) with respect to $\alpha_{120'}$, we have:

$$\frac{\partial a_4}{\partial \alpha_{120'}} = 0. \tag{40}$$

By replacing (39), (36) and (40) in (16), we obtain:

$$-B_3\frac{\sin(\tilde{\kappa}a_2)}{\tilde{\kappa}}\sin\alpha_{20'3} + B_5\frac{\sin(\tilde{\kappa}a_2)}{\tilde{\kappa}}\sin\alpha_{00'2} = 0 \tag{41}$$

or

$$\frac{B_3}{\sin\alpha_{00'2}} = \frac{B_5}{\sin\alpha_{20'3}}. \tag{42}$$

We can also express a_1, a_2, d as a function of a_3, a_4, α_{041} and $\alpha_{0'34}$.

By applying the cosine law in the triangles $\nabla A_0A_1A_4$, $\nabla A_{0'}A_2A_3$, $\nabla A_0A_{0'}A_4$ and $\nabla A_0A_{0'}A_3$, respectively, we have:

$$\cos(\tilde{\kappa}a_1) = \cos(\tilde{\kappa}a_{14})\cos(\tilde{\kappa}a_4) + \sin(\tilde{\kappa}a_{14})\sin(\tilde{\kappa}a_4)\cos\alpha_{041}, \tag{43}$$

$$\cos(\tilde{\kappa}a_2) = \cos(\tilde{\kappa}a_{23})\cos(\tilde{\kappa}a_3) + \sin(\tilde{\kappa}a_{23})\sin(\tilde{\kappa}a_3)\cos(\alpha_{234} - \alpha_{0'34}), \tag{44}$$

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_4)\cos(\tilde{\kappa}a_{0'4}) + \sin(\tilde{\kappa}a_4)\sin(\tilde{\kappa}a_{0'4})\cos(\alpha_{0'41} - \alpha_{041}), \tag{45}$$

$$\cos(\tilde{\kappa}d) = \cos(\tilde{\kappa}a_3)\cos(\tilde{\kappa}a_{03}) + \sin(\tilde{\kappa}a_3)\sin(\tilde{\kappa}a_{03})\cos(\alpha_{0'34} - \alpha_{034}). \tag{46}$$

By differentiating (1) with respect to a_3, a_4, α_{041}, $\alpha_{0'34}$, we obtain, respectively:

$$B_1\frac{\partial a_1}{\partial a_3}+B_2\frac{\partial a_2}{\partial a_3}+B_3+B_5\frac{\partial d}{\partial a_3}=0, \tag{47}$$

$$B_1\frac{\partial a_1}{\partial a_4}+B_2\frac{\partial a_2}{\partial a_4}+B_4+B_5\frac{\partial d}{\partial a_4}=0, \tag{48}$$

$$B_1\frac{\partial a_1}{\partial \alpha_{041}}+B_2\frac{\partial a_2}{\partial \alpha_{041}}+B_5\frac{\partial d}{\partial \alpha_{041}}=0, \tag{49}$$

$$B_1\frac{\partial a_1}{\partial \alpha_{0'34}}+B_2\frac{\partial a_2}{\partial \alpha_{0'34}}+B_5\frac{\partial d}{\partial \alpha_{0'34}}=0. \tag{50}$$

Similarly, by working cyclically we can follow the previous process and by exchanging the indices $1\rightarrow 4$, $2\rightarrow 3$, $3\rightarrow 2$ and $1\rightarrow 4$, we derive two relations, respectively:

$$\frac{B_1}{\sin\alpha_{0'04}}=\frac{B_5}{\sin\alpha_{104}} \tag{51}$$

and

$$\frac{B_2}{\sin\alpha_{00'3}}=\frac{B_5}{\sin\alpha_{20'3}}. \tag{52}$$

From (25) and (51), we have:

$$\frac{B_1}{\sin\alpha_{0'04}}=\frac{B_4}{\sin\alpha_{100'}}=\frac{B_5}{\sin\alpha_{104}}=c. \tag{53}$$

The relationship between the angles $\alpha_{0'04}$, $\alpha_{100'}$ and α_{104} can be used, in order to show that c depends on B_1, B_4, B_5. Thus, we have:

$$\alpha_{0'04}+\alpha_{100'}+\alpha_{104}=2\pi \tag{54}$$

or

$$\sin\alpha_{0'04}=-\sin(\alpha_{100'}+\alpha_{104}). \tag{55}$$

From (55) and (53), we get:

$$c=\frac{2B_1B_4B_5}{(B_1+B_4+B_5)(B_1-B_4+B_5)(B_1+B_4-B_5)(-B_1+B_4+B_5)}. \tag{56}$$

By replacing (56) in (53), we have:

$$\cos\alpha_{100'}=\frac{B_4^2-B_1^2-B_5^2}{2B_1B_5}, \tag{57}$$

$$\cos\alpha_{0'04}=\frac{B_1^2-B_4^2-B_5^2}{2B_4B_5}, \tag{58}$$

$$\cos\alpha_{104}=\frac{B_5^2-B_1^2-B_4^2}{2B_1B_4}. \tag{59}$$

The relations (57), (58) and (59) show that A_0 is the weighted Fermat-Torricelli point of $\nabla A_1A_{0'}A_4$ which minimizes the function $f_1 = B_1a_1 + B_5d + B_4a_4$.

From (42) and (52), we have:

$$\frac{B_2}{\sin\alpha_{00'3}} = \frac{B_3}{\sin\alpha_{00'2}} = \frac{B_5}{\sin\alpha_{20'3}} = \mathfrak{c}. \tag{60}$$

The relationship between the angles $\alpha_{00'3}$, $\alpha_{00'2}$ and $\alpha_{20'3}$ can be used, in order to show that $\mathfrak{c}$ depends on B_2, B_3, B_5. Thus, we have:

$$\alpha_{00'3} + \alpha_{00'2} + \alpha_{20'3} = 2\pi \tag{61}$$

or

$$\sin\alpha_{00'3} = -\sin(\alpha_{00'2} + \alpha_{20'3}). \tag{62}$$

From (62) and (60), we get:

$$\mathfrak{c} = \frac{2B_2B_3B_5}{(B_2+B_3+B_5)(B_2-B_3+B_5)(B_2+B_3-B_5)(-B_2+B_3+B_5)}. \tag{63}$$

By replacing (63) in (60), we have:

$$\cos\alpha_{00'3} = \frac{B_2^2 - B_3^2 - B_5^2}{2B_3B_5}, \tag{64}$$

$$\cos\alpha_{00'2} = \frac{B_3^2 - B_5^2 - B_2^2}{2B_5B_2}, \tag{65}$$

$$\cos\alpha_{20'3} = \frac{B_5^2 - B_2^2 - B_3^2}{2B_2B_3}. \tag{66}$$

The relations (64), (65) and (66) show that $A_{0'}$ is the weighted Fermat-Torricelli point of $\nabla A_0A_2A_3$ which minimizes the function $f_2 = B_5d + B_2a_2 + B_3a_3$.

From (57), (58), (59), (64), (65) and (66) we derive (2) and (3).

(II) We study the case for $K = 0$, where $A_1A_2A_3A_4$ lies on the two-dimensional Euclidean Space R^2.
The length of the line segments arcs a_3, a_4, d can be expressed as functions of a_1, a_2, α_{014}, $\alpha_{120'}$, by applying the cosine law in the triangles $\nabla A_{0'}A_2A_3$, $\nabla A_0A_1A_4$, $\nabla A_0A_{0'}A_1$, respectively:

$$a_3^2 = a_{23}^2 + a_2^2 - 2a_{23}a_2\cos(\alpha_{123} - \alpha_{120'}), \tag{67}$$

$$a_4^2 = a_{14}^2 + a_1^2 - 2a_{14}a_1\cos\alpha_{014}\,, \tag{68}$$

$$d^2 = a_{10'}^2 + a_1^2 - 2a_{10'}a_1\cos\alpha_{010'} \tag{69}$$

or

$$d^2 = a_{10'}^2 + a_1^2 - 2a_{10'}a_1\cos(\alpha_{214} - \alpha_{014} - \alpha_{210'}). \tag{70}$$

From the cosine law in $\nabla A_1A_2A_{0'}$ we have:

$$a_{10'}^2 = a_{12}^2 + a_2^2 - 2a_{12}a_2\cos(\alpha_{120'}). \tag{71}$$

We show that the angle $\alpha_{210'}$ can be expressed as a function of a_2 and $\alpha_{120'}$.
We take the height $A_{0'}A_5$ of the triangle $\nabla A_1A_2A_{0'}$, where A_5 belongs in the line segment A_1A_2. From the triangles $\nabla A_1A_5A_{0'}$ and $\nabla A_2A_5A_{0'}$, we get:

$$\alpha_{210'} = \arctan \frac{a_2 \sin \alpha_{120'}}{a_{12} - a_2 \cos \alpha_{120'}}. \tag{72}$$

From (72), we derive that $\alpha_{210'}$ is a function of a_2 and $\alpha_{120'}$ and by replacing (71) and (72) in (70), we derive that d depends on a_1, a_2, α_{014}, $\alpha_{120'}$.

By differentiating (1) with respect to a_1, a_2, α_{014}, $\alpha_{120'}$, respectively, we obtain (13), (14), (15) and (16).

We calculate $\frac{\partial a_3}{\partial \alpha_{014}}$, $\frac{\partial a_4}{\partial \alpha_{014}}$, $\frac{\partial d}{\partial \alpha_{014}}$, in order to derive (15).

By differentiating (70) with respect to α_{014}, we have:

$$\frac{\partial d}{\partial \alpha_{014}} = -\frac{a_{10'}a_1 \sin(\alpha_{214} - \alpha_{014} - \alpha_{210'})}{d}. \tag{73}$$

We apply the "sine law" in the triangle $\nabla A_1A_0A_{0'}$:

$$\frac{a_{10'}}{d} = \frac{\sin(\alpha_{100'})}{\sin(\alpha_{214} - \alpha_{014} - \alpha_{210'})}. \tag{74}$$

By replacing (74) in (73), we get:

$$\frac{\partial d}{\partial \alpha_{014}} = -a_1 \sin \alpha_{100'} . \tag{75}$$

By differentiating (67) with respect to α_{014}, we derive that:

$$\frac{\partial a_3}{\partial \alpha_{014}} = 0. \tag{76}$$

By differentiating (68) with respect to α_{014}, we derive that:

$$\frac{\partial a_4}{\partial \alpha_{014}} = \frac{a_{14}a_1 \sin \alpha_{014}}{a_4}. \tag{77}$$

From the sine law in $\nabla A_0A_1A_4$, we have:

$$\frac{a_{14}}{a_4} = \frac{\sin \alpha_{104}}{\sin \alpha_{014}}. \tag{78}$$

By replacing (78) in (77), we get:

$$\frac{\partial a_4}{\alpha_{014}} = a_1 \sin \alpha_{104}. \tag{79}$$

By replacing (76), (79), (75) in (15), we obtain (25). From the cosine law in $\nabla A_0A_2A_{0'}$, the length d can also be expressed as a function of a_1, a_2, α_{014}, $\alpha_{120'}$:

$$d^2 = a_{20}^2 + a_2^2 - 2a_{20}a_2 \cos \alpha_{020'} \tag{80}$$

or

$$d^2 = a_{20}^2 + a_2^2 - 2a_{20}a_2 \cos(\alpha_{120'} - \alpha_{120}). \tag{81}$$

From the triangle $\triangledown A_1A_2A_0$, we have:

$$\tan \alpha_{120} = \frac{a_1 \sin(\alpha_{214} - \alpha_{014})}{a_{12} - a_1 \cos(\alpha_{214} - \alpha_{014})} \tag{82}$$

From the cosine law in $\triangledown A_0A_1A_2$, we have:

$$a_{20}^2 = a_1^2 + a_{12}^2 - 2a_1a_{12} \cos \alpha_{012} \tag{83}$$

or

$$a_{20}^2 = a_1^2 + a_{12}^2 - 2a_1a_{12} \cos(\alpha_{214} - \alpha_{014}). \tag{84}$$

By replacing (84) and (82) in (81) and by differentiating (81) with respect to $\alpha_{120'}$, we have:

$$\frac{\partial d}{\partial \alpha_{120'}} = \frac{a_{20}a_2 \sin(\alpha_{120'} - \alpha_{120})}{d}. \tag{85}$$

From the sine law in $\triangledown A_0A_2A_{0'}$, we have:

$$\frac{a_{20}}{d} = \frac{\sin \alpha_{00'2}}{\sin(\alpha_{120'} - \alpha_{120})}. \tag{86}$$

By replacing (86) in (85), we have:

$$\frac{\partial d}{\partial \alpha_{120'}} = a_2 \sin \alpha_{00'2}\,. \tag{87}$$

By differentiating (67) with respect to $\alpha_{120'}$, we have:

$$\frac{\partial a_3}{\partial \alpha_{120'}} = -\frac{a_2a_{23} \sin(\alpha_{123} - \alpha_{120'})}{a_3}. \tag{88}$$

From the sine law in $\triangledown A_2A_3A_{0'}$ we have:

$$\frac{a_{23}}{a_3} = \frac{\sin \alpha_{20'3}}{\sin(\alpha_{123} - \alpha_{120'})}. \tag{89}$$

By replacing (89) in (88), we have:

$$\frac{\partial a_3}{\partial \alpha_{120'}} = -a_2 \sin \alpha_{20'3}\,. \tag{90}$$

By differentiating (68) with respect to $\alpha_{120'}$, we get:

$$\frac{\partial a_4}{\partial \alpha_{120'}} = 0. \tag{91}$$

By replacing (90), (91) and (87), we derive (42).

We can also express a_1, a_2, d as a function of a_3, a_4, α_{041} and $\alpha_{0'34}$, by applying the cosine law in the triangles $\nabla A_0A_1A_4$, $A_{0'}A_2A_3$, $A_0A_{0'}A_4$ and $A_0A_{0'}A_3$, respectively.

By differentiating (1) with respect to a_3, a_4, α_{041}, $\alpha_{0'34}$, we obtain, (47), (48), (49) and (50), respectively.

Similarly, by following the process like in case (I)(S^2, H^2) and by working cyclically and exchanging the indices $1 \to 4$, $2 \to 3$, $3 \to 2$ and $1 \to 4$, we derive (51) and (52).

From (25) and (51), we obtain (53).

The relationship between the angles $\alpha_{0'04}$, $\alpha_{100'}$ and α_{104} can be used, in order to show that c depends on B_1, B_4, B_5. Thus, we have:

$$\alpha_{0'04} + \alpha_{100'} + \alpha_{104} = 2\pi \tag{92}$$

or

$$\sin \alpha_{0'04} = -\sin(\alpha_{100'} + \alpha_{104}). \tag{93}$$

From (93) and (53), we derive (56).

By replacing (56) in (53), we derive (57), (58) and (59).

The relations (57), (58) and (59) show that A_0 is the weighted Fermat-Torricelli point of $\nabla A_1A_{0'}A_4$ which minimizes the function $f_1 = B_1a_1 + B_5d + B_4a_4$.

From (42) and (52), we derive (60).

The relationship between the angles $\alpha_{00'3}$, $\alpha_{00'2}$ and $\alpha_{20'3}$ can be used, in order to show that $\mathfrak{c}$ depends on B_2, B_3, B_5. Thus, we have:

$$\alpha_{00'3} + \alpha_{00'2} + \alpha_{20'3} = 2\pi \tag{94}$$

or

$$\sin \alpha_{00'3} = -\sin(\alpha_{00'2} + \alpha_{20'3}). \tag{95}$$

From (95) and (60), we get (63).

By replacing (63) in (60), we obtain (64), (65) and (66).

The relations (64), (65) and (66) show that $A_{0'}$ is the weighted Fermat-Torricelli point of $\nabla A_0A_2A_3$ which minimizes the function $f_2 = B_5d + B_2a_2 + B_3a_3$. □

Remark 2.2. For given weights B_i, for $i = 1, 2, 3, 4, 5$, we calculate the angles $\alpha_{100'}$, α_{104} and $\alpha_{0'04}$, $\alpha_{00'2}$, $\alpha_{20'3}$ from (57), (58), (59), (64), (65) and (59). The calculation of A_0 and $A_{0'}$ is given by deriving a system of four equations that depend on a_1, a_2, α_{014} and $\alpha_{120'}$.

The length d of the line segment $A_0A_{0'}$ can be expressed as functions of a_1, a_2, α_{014} and $\alpha_{120'}$ in four different ways by applying the cosine law with respect to the triangles $\nabla A_1A_0A_{0'}$, $\nabla A_2A_0A_{0'}$, $\nabla A_3A_0A_{0'}$, $\nabla A_4A_0A_{0'}$.

By replacing (82) and (71) in (70), we have:

$$d^2 = a_1^2 + a_{12}^2 - 2a_1a_{12}\cos(\alpha_{214} - \alpha_{014}) + a_2^2 - 2\left(\sqrt{a_1^2 + a_{12}^2 - 2a_1a_{12}\cos(\alpha_{214} - \alpha_{014})}\right) a_2 \cos\left(\alpha_{120'} - \arctan\frac{a_1\sin(\alpha_{214} - \alpha_{014})}{a_{12} - a_1\cos(\alpha_{214} - \alpha_{014})}\right) \tag{96}$$

By replacing (71) and (72) in (70), we have:

$$d^2 = a_{12}^2 + a_2^2 - 2a_{12}a_2\cos(\alpha_{120'}) + a_1^2 - 2\left(\sqrt{a_{12}^2 + a_2^2 - 2a_{12}a_2\cos(\alpha_{120'})}\right) a_1 \cos\left(\alpha_{214} - \alpha_{014} - \arctan\frac{a_2\sin\alpha_{120'}}{a_{12} - a_2\cos\alpha_{120'}}\right). \tag{97}$$

From the sine law in $\nabla A_0A_1A_4$ and from (59), we have:

$$a_4 = \frac{a_{14}\sin\alpha_{014}}{\sin\left(\arccos\frac{B_5^2 - B_1^2 - B_4^2}{2B_1B_4}\right)}. \tag{98}$$

From the cosine law in $\nabla A_4A_0A_{0'}$, we have:

$$d^2 = a_{0'4}^2 + a_4^2 - 2a_4a_{0'4}\cos\left(\alpha_{143} - \alpha_{0'43} - \left(2\pi - \alpha_{014} - \arccos\frac{B_5^2 - B_1^2 - B_4^2}{2B_1B_4}\right)\right) \tag{99}$$

From the cosine law in $\nabla A_3A_4A_{0'}$, we have:

$$a_{0'4}^2 = a_3^2 + a_{34}^2 - 2a_3a_{34}\cos(\alpha_{234} - \alpha_{230'}) \tag{100}$$

or

$$a_{0'4}^2 = a_3^2 + a_{34}^2 - 2a_3a_{34}\cos\left(\alpha_{234} - \left(2\pi - (\alpha_{123} - \alpha_{120'}) - \arccos\left(\frac{B_5^2 - B_2 - B_3^2}{2B_2B_3}\right)\right)\right). \tag{101}$$

From the sine law in $\nabla A_{0'}A_2A_3$, we have:

$$a_3 = \sin(\alpha_{123} - \alpha_{120'})a_{23}\sin\left(\arccos\frac{B_5^2 - B_2^2 - B_3^2}{2B_2B_3}\right). \tag{102}$$

From the triangle $\nabla A_{0'}A_4A_3$, we have:

$$\tan(\alpha_{0'43}) = \frac{a_3\sin\left(\alpha_{234} - \left(2\pi - (\alpha_{123} - \alpha_{120'}) - \arccos\left(\frac{B_5^2 - B_2 - B_3^2}{2B_2B_3}\right)\right)\right)}{a_{34} - a_3\cos\left(\alpha_{234} - \left(2\pi - (\alpha_{123} - \alpha_{120'}) - \arccos\left(\frac{B_5^2 - B_2 - B_3^2}{2B_2B_3}\right)\right)\right)}. \tag{103}$$

By replacing (102) in (103) and (101) in (99), we derive a third relation that expresses d as a function of a_1, a_2, α_{014} and $\alpha_{120'}$.

From the cosine law in $\nabla A_{0'}A_0A_3$, we have:

$$d^2 = a_{03}^2 + a_3^2 - 2a_{03}a_3\cos(\alpha_{234} - \alpha_{034} - \alpha_{230'}) \tag{104}$$

or

$$\begin{aligned} d^2 = a_{03}^2 + a_3^2 - 2a_{03}a_3\cos\Bigg(\alpha_{234} \\ - \left(2\pi - (\alpha_{123} - \alpha_{120'}) - \arccos\left(\frac{B_5^2 - B_2 - B_3^2}{2B_2B_3}\right) - \alpha_{034}\right)\Bigg). \end{aligned} \tag{105}$$

From the cosine law in $\nabla A_0A_3A_4$, we have:

$$a_{03}^2 = a_4^2 + a_{34}^2 - 2a_4a_{34}\cos(\alpha_{340}) \tag{106}$$

or

$$\begin{aligned} a_{03}^2 = a_4^2 + a_{34}^2 - 2a_4a_{34}\cos\Bigg(\alpha_{341} \\ - \left(2\pi - \alpha_{014} - \arccos\frac{B_5^2 - B_1^2 - B_4^2}{2B_1B_4}\right)\Bigg). \end{aligned} \tag{107}$$

From the triangle $\nabla A_0A_3A_4$, we have:

$$\alpha_{034} = \arctan\frac{a_4\sin\left(\alpha_{143} - \left(2\pi - \alpha_{014} - \arccos\frac{B_5^2-B_1^2-B_4^2}{2B_1B_4}\right)\right)}{a_{34} - \cos\left(\alpha_{143} - \left(2\pi - \alpha_{014} - \arccos\frac{B_5^2-B_1^2-B_4^2}{2B_1B_4}\right)\right)}. \tag{108}$$

By replacing (108), (107) and (102) in (105), we obtain a fourth relation which expresses d as a function of a_1, a_2, α_{014} and $\alpha_{120'}$.

By subtracting (96) from (97), (96) from (99), (96) from (105) and (97) from (105), we obtain a system of four equations that depends on a_1, a_2, α_{014} and $\alpha_{120'}$. The numerical solution of this system of equations gives the location of A_0 and A_0'.

Corollary 2.3. *A (full) Steiner minimal tree of $A_1A_2A_3A_4$ consists of two Fermat-Torricelli points A_0, A_0' which are located at its interior domain and minimizes the objective function:*

$$B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + B_5d = minimum. \tag{109}$$

Proof of Corollary 2.3. By following the same proof of Theorem 2.1 and by setting $B_1 = B_2 = B_3 = B_4 = B_5$ in (57), (58), (59), (64), (65) and (66), we derive that:

$$\alpha_{0'04} = \alpha_{100'} = \alpha_{104} = 120^\circ$$

and

$$\alpha_{00'3} = \alpha_{00'2} = \alpha_{20'3} = 120^\circ.$$

□

Corollary 2.4 ([7], Theorem 2.2). *A degenerate weighted (full) Steiner minimal tree of the convex quadrilateral $A_1A_2A_3A_4$ consists of one weighted Fermat-Torricelli point A_0 which is located at the interior of the triangle $A_jA_kA_l$, if there is one and only i such that: $B_i = 0$ and*

$$|B_j - B_l| < B_k < B_j + B_l,$$

for $i, j, k, l \in \{1, 2, 3, 4\}$ and $j, k, l \neq i$.

Remark 2.5. The case II of Theorem 2.1 corresponds to the weighted (full) Steiner minimal tree of a convex quadrilateral in $\mathbb{R}^2$ and provides a solution to the generalized (weighted) Gauss problem (restricted Steiner problem) by considering two Fermat-Torricelli points A_0, $A_{0'}$ which are located at the convex domain of $A_1A_2A_3A_4$ having two equal positive weights $B_0 = B_{0'} = B_5$.

Remark 2.6. A weighted Steiner minimal tree may not be unique, since we can have two possible (full) topologies of Steiner trees for convex quadrilaterals in $\mathbb{R}^2$. In this case, we need to apply the same method by considering two interior points A_0° and $A_{0'}^\circ$ and the objective function which will be minimized is:

$$B_1a_1^\circ + B_2a_2^\circ + B_3a_3^\circ + B_4a_4^\circ + B_5d^\circ$$

where A_0° may be located inside the quadrilateral $A_1A_2A_{0'}A_0$ and $A_{0'}^\circ$ may be located inside the quadrilateral $A_4A_0A_{0'}A_3$.

Remark 2.7. A future work will be to study weighted Steiner trees on the K-plane.

References

[1] I. D. Berg, I. G. Nikolaev: On an extremal property of quadrilaterals in an Aleksandrov space of curvature $\leq K$, in: The Interaction of Analysis and Geometry (Novosibirsk, 2004), V. I. Burenkov et al. (ed.), Contemp. Math. 424, AMS, Providence (2007) 1–15.

[2] V. Boltyanski, H. Martini, V. Soltan: Geometric Methods and Optimization Problems, Kluwer, Dordrecht (1999).

[3] K. Bopp: Über das kürzeste Verbindungssystem zwischen vier Punkten, Dissertation, Universität Göttingen, Göttingen (1879).

[4] D. Cieslik: Shortest Connectivity. An Introduction with Applications in Phylogeny, Springer, New York (2005).

[5] R. Courant, H. Robbins: What is Mathematics?, Oxford University Press, New York (1951).

[6] E. N. Gilbert, H. O. Pollak: Steiner minimal trees, SIAM J. Appl. Math. 16 (1968) 1–29.

[7] A. N. Zachos, G. Zouzoulas: The weighted Fermat-Torricelli problem and an "inverse" problem, J. Convex. Analysis 15(1) (2008) 55–62.