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On Multiple Solutions for Multivalued Elliptic Equations under Navier Boundary Conditions*

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We employ variational methods for non-smooth functionals to show existence of multiple solutions for multivalued fourth order elliptic equations under Navier boundary conditions. Our main result extends similar ones known for the Laplacian.

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1. Introduction

We deal with existence and multiplicity of solutions of the problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u \in \partial\Psi(u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\alpha \geq 0$, $-\infty < \beta < \alpha\lambda_1$, ($\lambda_k$ is the k^{th} eigenvalue of $(-\Delta, H_0^1(\Omega))$), and the principal λ_1-eigenfunction is ϕ_1, normalized such that $\int_\Omega \phi_1^2 dx = 1$,

$$\Delta^2 u = \sum_{i,j=1}^{N} \frac{\partial^4 u}{\partial^2 x_i \partial^2 x_j}.$$

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The boundary condition $\mathcal{B}u = 0$ on $\partial\Omega$ means that

$$u = 0 \text{ on } \partial\Omega \text{ if } \alpha = 0 \quad \text{and} \quad u = \Delta u = 0 \text{ on } \partial\Omega \text{ if } \alpha > 0, \text{ (trace sense).}$$

For each function $u \in L^2(\Omega)$, we set

$$F(x,s) = \int_0^s f(x,t)dt \quad \text{and} \quad \Psi(u) = \int_\Omega F(x,u)dx,$$

where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a suitable measurable function.

The functional $\Psi(u)$ is locally Lipschitz continuous and its subdifferential is denoted by $\partial\Psi(u)$, (cf. Sections 3, 5 for details).

By a solution of (1) we mean an element $u \in H := H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\alpha\Delta^2 u + \beta\Delta u \in \partial\Psi(u) \quad \text{and} \quad \mathcal{B}u = 0 \text{ on } \partial\Omega.$$

Our aim is to find multiple solutions of (1), under the condition

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{t} = \mu_1, \quad x \in \Omega,$$

where $\mu_1 := \lambda_1(\alpha\lambda_1 - \beta)$ is the first eigenvalue of the eigenvalue problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

(cf. Section 4 for a recall). In order to establish our main result we need some notations and definitions which, for the reader's convenience will be recalled on Sections 3, 4, 5.

At first consider the space H endowed with the inner product

$$\langle u,v\rangle_H = \alpha \int_\Omega \Delta u \Delta v - \beta \int_\Omega \nabla u \nabla v, \quad u,v \in H$$

and corresponding norm

$$\|u\|^2 = \langle u,u\rangle_H.$$

As a consequence of the inequality below

$$\int_\Omega |\Delta u|^2 \geq \lambda_1 \int_\Omega |\nabla u|^2, \tag{3}$$

H is a Hilbert space, details in Section 4.

It will be shown that the solutions of (1) are the critical points (in a suitable sense) of the energy functional

$$I(u) = \frac{1}{2}\left(\alpha \int_\Omega |\Delta u|^2 - \beta \int_\Omega |\nabla u|^2\right) - \int_\Omega F(x,u), \quad u \in H.$$

In order to establish our main result we set

$$\underline{f}(x,t)=\liminf_{s\to t} f(x,s), \qquad \overline{f}(x,t)=\limsup_{s\to t} f(x,s)$$

and we shall assume that there are functions $\tau\in L^2(\Omega)$, $F_\infty\in L^1(\Omega)$ with $F_\infty\geq 0$ and $\widehat{H}\in L^1(\Omega)$ satisfying the following basic conditions:

$$(f_1)\quad \begin{aligned}&\text{(i)}\quad \max\{\left|\underline{f}(x,t)-\mu_1 t\right|,\left|\overline{f}(x,t)-\mu_1 t\right|\}\stackrel{|t|\to\infty}{\longrightarrow} 0 \ \text{ a.e. } x\in\Omega,\\ &\text{(ii)}\quad |f(x,t)-\mu_1 t|\leq \tau(x) \ \text{ a.e. } x\in\Omega,\end{aligned}$$

$$(f_2)\quad \begin{aligned}&\text{(i)}\quad \left(F(x,t)-\tfrac{\mu_1}{2}t^2\right)\stackrel{|t|\to\infty}{\longrightarrow} F_\infty(x) \ \text{ a.e. } x\in\Omega,\\ &\text{(ii)}\quad |F(x,t)|\leq \tfrac{\mu_1}{2}t^2+\widehat{H}(x) \ \text{ a.e. } x\in\Omega.\end{aligned}$$

Our main result is

Theorem 1.1. *Assume that $f:\Omega\times\mathbf{R}\to\mathbf{R}$ is measurable, $f(x,0)=0$ a.e. $x\in\Omega$ and satisfies (f_1)(i)(ii), (f_2)(i)(ii). Assume, in addition, the conditions:*

There exist $m\in L^\infty(\Omega)$, $\delta>0$, $0\leq m<\mu_1$, $m\not\equiv 0$ such that,

$$(f_3)\quad \begin{aligned}&(i)\quad F(x,t)\leq \frac{\mu_2}{2}t^2, \quad \text{a.e. } x\in\Omega,\ t\in\mathbf{R},\ \text{where } \mu_2=\lambda_2(\alpha\lambda_2-\beta),\\ &(ii)\quad F(x,t)\leq \frac{m(x)t^2}{2}, \quad \text{a.e. } x\in\Omega,\ |t|\leq\delta.\end{aligned}$$

There exist numbers $t_\pm\in\mathbf{R}$ with $t_-<0<t_+$ such that

$$(f_4)\qquad \int_\Omega (F(x,t_\pm\phi_1)-F_\infty(x))>\mu_1\frac{(t_\pm)^2}{2}.$$

Then (1) admits at least three non-trivial solutions, say $u_-,u_+,u_0\in H$ satisfying

$$\begin{cases}\alpha\Delta^2 u+\beta\Delta u\in[\underline{f}(x,u(x)),\overline{f}(x,u(x))] & a.e.\ x\in\Omega,\\ \mathcal{B}u=0 & on\ \partial\Omega\ (trace\ sense),\end{cases}$$

$$I(u_+)=\min\left\{I(v)\mid v\in H,\ \int_\Omega v\phi_1>0\right\}<0,$$

$$I(u_-)=\min\left\{I(v)\mid v\in H,\ \int_\Omega v\phi_1<0\right\}<0,$$

and

$$I(u_0)=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t))>0,$$

where

$$\Gamma=\{\gamma\in C([0,1],H)\mid \gamma(0)=0,\ \gamma(1)=t_+\phi_1\}.$$

2. Background

In [1] Benci, Bartolo and Fortunato proved that the problem

$$\begin{cases} -\Delta u - \lambda_k u + g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_k, $(k \geq 1)$ is an eigenvalue of $(-\Delta, H_0^1(\Omega))$, admits a solution if g is a smooth function such that

$$g(t) \overset{|t|\to\infty}{\longrightarrow} 0, \qquad G(t) \leq G_\infty, \quad t \in \mathbf{R}, \qquad g'(0) = \sup_{t\in\mathbf{R}} g'(t).$$

In [5], Goncalves and Miyagaki improved the result above by requiring g to be continuous and to satisfy the conditions

$$g(t) \overset{|t|\to\infty}{\longrightarrow} 0, \qquad G(t) \overset{|t|\to\infty}{\longrightarrow} G_\infty \in \mathbf{R},$$

and one of the following sets of conditions,

$$m < 0,\ 2G(t) \leq mt^2,\ t \in \mathbf{R},\ G_\infty \leq 0,$$

$$m > 0,\ 2G(t) \leq mt^2,\ t \in \mathbf{R},\ G_\infty \geq 0,$$

for some number m.

Goncalves and Miyagaki in [6] and Costa and Silva in [3] showed results on existence of two solutions.

Later on, in [7], Goncalves and Miyagaki proved that under the set of conditions

$$g(t) \overset{|t|\to\infty}{\longrightarrow} 0, \qquad G(t) \overset{|t|\to\infty}{\longrightarrow} 0$$

$$2G(t) \geq mt^2,\ |t| \leq \delta \ \text{ for some } \delta > 0 \text{ and } m \in (0, \lambda_1),$$

$$2G(t) \geq (\lambda_1 - \lambda_2)t^2, \quad t \in \mathbf{R},$$

$$\int_\Omega (G(t_\pm \phi_1) - G_\infty) < 0, \quad \text{for some } t_- < 0 < t_+.$$

the problem

$$-\Delta u - \lambda_1 u + g(u) = 0 \ \text{ in } \Omega,\ u \in H_0^1(\Omega)$$

admits at least three non-trivial solutions.

Such result was extended for multivalued quasilinear equations by Kourogenis and Papageorgiou [11].

We refer the reader to Filipakis and Papageorgiou [13], Kyritsi and Papageorgiou [14], Halidias and Naniewicz [15], Fiacca, Matzakos, Papageorgiou and Servadei [16], Liu and Guo [17] and their references for further related results.

3. Abstract Framework

In this section we recall, for the reader's convenience, some definitions and bassic results on the critical point theory of locally Lipschitz continuous functionals as developed by Clarke [4, 9], Chang [10].

Let X be a real Banach space. A functional $I : X \to \mathbf{R}$ is locally Lipschitz continuous, $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbf{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ and some constant $K = K_V > 0$ such that

$$|I(v_2) - I(v_1)| \le K \|v_2 - v_1\|, \quad v_i \in V, \; i = 1, 2.$$

The directional derivative of I at u in the direction of $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h \to 0, \; \lambda \downarrow 0} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}.$$

One shows that $I^0(u; .)$ is subadditive and positively homogeneous in the sense that

$$I^0(u; v_1 + v_2) \le I^0(u; v_1) + I^0(u; v_2) \quad \text{and} \quad I^0(u; \lambda v) = \lambda I^0(u; v),$$

for $u, v, v_1, v_2 \in X$ and $\lambda > 0$.

Using those facts there is some $K = K_u > 0$ such that

$$\begin{aligned} \left| I^0(u; v_1) - I^0(u; v_2) \right| &\le I^0(u; v_1 - v_2) \\ &\le K \|v_1 - v_2\|. \end{aligned}$$

Hence $I^0(u; .)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$\partial I^0(u; z) = \left\{ \mu \in X^*; I^0(u; v) \ge I^0(u; z) + \langle \mu, v - z \rangle, \; v \in X \right\},$$

where $\langle ., . \rangle$ is the duality pairing between X^* and X. The generalized gradient of I at u is the set

$$\partial I(u) = \left\{ \mu \in X^*; \langle \mu, v \rangle \le I^0(u; v), \; v \in X \right\}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$.

A few definitions and properties will be recalled below.

$$\partial I(u) \subset X^* \text{ is convex, non-empty and weak*-compact},$$

$$m(u) = \min \left\{ \|\mu\|_{X^*}; \mu \in \partial I(u) \right\},$$

and

$$\partial I(u) = \{ I'(u) \}, \quad \text{if } I \in C^1(X, \mathbf{R}).$$

A critical point of I is an element $u_0 \in X$ such that $0 \in \partial I(u_0)$ and a critical value of I is a real number c such that $I(u_0) = c$ for some critical point $u_0 \in X$.

If an element $u_0 \in X$ is a local minimum of $I \in \mathrm{Lip}_{\mathrm{loc}}(X; \mathbf{R})$ then it is a critical point of I.

For each $u, v \in X$,

$$I^0(u; v) = \max\{\langle \mu, v\rangle \mid \mu \in \partial I(u)\}.$$

Let $C \subset X$ be non-empty set. The support of C at $\xi \in X^*$ is defined by

$$\sigma(C, \xi) = \sup\{\langle \xi, x\rangle \mid x \in C\}.$$

If X is a reflexive space and $\Sigma \subset X^*$, the support of Σ at $v \in X$ can be defined of the following way

$$\sigma(\Sigma, v) = \sup\{\langle \xi, v\rangle \mid \xi \in \Sigma.\}, \quad v \in X.$$

The support function σ defined enjoys the following properties:

S_1) For each $x_0 \in X$

$$\sigma(\{x_0\}, \xi) = \langle \xi, x_0\rangle, \quad \xi \in X^*.$$

S_2) If $B \subset X$, $B^* \subset X^*$ are the unit balls, then

$$\sigma(B, \xi) = ||\xi||_{X^*}, \quad \sigma(B^*, v) = ||v||_X, \quad \xi \in X^*, \ v \in X.$$

S_3) If $C, D \subset X$ are non-empty, closed and convex and $\Sigma, \Delta \subset X^*$ are non-empty, weak*-closed and convex then

(i) $$C \subset D \Leftrightarrow \sigma(C, \xi) \leq \sigma(D, \xi), \quad \xi \in X^*,$$

(ii) $$\Sigma \subset \Delta \Leftrightarrow \sigma(\Sigma, v) \leq \sigma(\Delta, v), \quad v \in X.$$

S_4) Given $\xi \in X^*$ and $w \in X$,
(i) $\sigma(C_1 + C_2, \xi) = \sigma(C_1, \xi) + \sigma(C_2, \xi)$;
(ii) $\sigma(\Sigma_1 + \Sigma_2, w) = \sigma(\Sigma_1, w) + \sigma(\Sigma_2, w)$;
(iii) $\sigma(\lambda C, \xi) = \lambda\sigma(C, \xi)$, $\lambda > 0$;
(iv) $\sigma(\lambda\Sigma, w) = \lambda\sigma(\Sigma, w)$, $\lambda > 0$,

where $C_i \subset X$ and $\Sigma_i \subset X^*$.

S_5) If X is reflexive, $I^0(x; v)$ can be viewed as the support function of $\partial I(x) \subset X^*$.

Some definitions and critical point theorems will be recalled below.

Let $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbf{R})$ and assume that $C \subset X$ is convex c is a real number. The non-smooth functional I satisfies the $(PS)_{c,C}$ condition if any sequence $(u_n) \subset C$ such that

$$I(u_n) \overset{n\to\infty}{\longrightarrow} c \quad \text{and} \quad m(u_n) \overset{n\to\infty}{\longrightarrow} 0,$$

admits a subsequence which converges to some point of C.

The theorem below improves results by Mizoguchi [18], Goncalves & Miyagaki [7].

Theorem 3.1. *Let $I : X \to \mathbf{R}$ be locally Lipschitz continuous, bounded from below. Assume that X is reflexive and $C \subset X$ is a convex, closed set such that* $\mathrm{int}(C) \neq \emptyset$. *Set*

$$c = \inf_{u \in C} I(u).$$

Assume in addition that

$$I(\widetilde{u}) < \inf_{\partial C} I(u) \quad \textit{for some } \widetilde{u} \in \operatorname{int}(C). \tag{4}$$

Then I admits a local minimum $u \in \operatorname{int}(C)$ if

$$I \textit{ satisfies } (PS)_{c,C}.$$

Proof. Let

$$d(x,y) = \|x-y\|, \quad x,y \in C.$$

Then $C = (C,d)$ is a complete metric space. Let $\epsilon > 0$. By Ekeland's Variational Principle, (cf. Ekeland [21]), there is $u_\epsilon \in C$ such that both

$$I(u_\epsilon) < \inf_{u\in C} I(u) + \epsilon \tag{5}$$

and

$$I(u_\epsilon) < I(u) + \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon, \ u \in C. \tag{6}$$

By (4), there is $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial C} I(u) - \inf_{u\in \operatorname{int}(C)} I(u). \tag{7}$$

By (5) and (7) we have

$$\begin{aligned} I(u_\epsilon) &< \inf_{u\in C} I(u) + \epsilon \\ &\leq \inf_{u\in \operatorname{int}(C)} I(u) + \epsilon \\ &< \inf_{u\in\partial C} I(u), \end{aligned}$$

showing that $u_\epsilon \in \operatorname{int}(C)$. By (6) the functional

$$J(u) = I(u) + \epsilon \|u - u_\epsilon\|, \quad u \in C,$$

has a local minimum say $u_\epsilon \in \operatorname{int}(C)$. Thus,

$$(J(u_\epsilon + \lambda v) - J(u_\epsilon))/\lambda \geq 0, \quad v \in X, \ \lambda > 0, \ \text{small}.$$

Hence,

$$(I(u_\epsilon + \lambda v) - I(u_\epsilon))/\lambda + \epsilon \|v\| \geq 0,$$

so that

$$\limsup_{\lambda\to 0^+} (I(u_\epsilon + \lambda v) - I(u_\epsilon))/\lambda + \epsilon \|v\| \geq 0.$$

Thus

$$I^0(u_\epsilon; v) + \epsilon \|v\| \geq 0, \quad v \in X. \tag{8}$$

Using the reflexivity again,

$$I^0(u_\epsilon; v) + \epsilon \|v\| = \sigma(\partial I(u_\epsilon), v) + \epsilon\sigma(B^*, v), \quad v \in X,$$

where

$$B^* = \{\xi \in X^* \mid \|\xi\|_{X^*} \leq 1\}.$$

It follows by S_4(ii)–(iv) that

$$I^0(u_\epsilon; v) + \epsilon \|v\| = \sigma(\partial I(u_\epsilon) + \epsilon B^*, v), \quad v \in X,$$

and by (8),

$$\sigma(\partial I(u_\epsilon) + \epsilon B^*, v) \geq \sigma(\{0\}, v), \quad v \in X.$$

Since $\partial I(u_\epsilon) + \epsilon B^*$ and $\{0\}$ are convex, non-empty, weak*-closed subsets of X^* we have by S_3,

$$\{0\} \subset \partial I(u_\epsilon) + \epsilon B^*.$$

So there are $\mu_\epsilon \in \partial I(u_\epsilon)$ and $\eta \in B^*$ such that

$$\mu_\epsilon + \epsilon\eta = 0$$

and so

$$m(u_\epsilon) = \min\{\|\mu\|_{X^*}; \mu \in \partial I(u_\epsilon)\} \leq \|\mu_\epsilon\|_{H^*} \leq \epsilon.$$

Thus

$$I(u_\epsilon) \stackrel{\epsilon \to 0}{\longrightarrow} c.$$

Setting $\epsilon = \frac{1}{n}$, we have

$$m(u_n) \stackrel{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad I(u_n) \stackrel{n\to\infty}{\longrightarrow} c.$$

If I satisfies the $(PS)_{c,C}$ condition and C is closed and convex, we have by eventually passing to a subsequence,

$$u_n \stackrel{n\to\infty}{\longrightarrow} u \text{ in } X,$$

for some $u \in \text{int}(C)$.

So

$$I(u_n) \stackrel{n\to\infty}{\longrightarrow} I(u) \quad \text{and} \quad I(u) = \min_{v \in C} I(v).$$

□

We state below the Mountain Pass Theorem for locally Lipschitz continuous functionals, (cf. Ambrosetti and Rabinowitz [20], Chang [19]).

Theorem 3.2. *Let $I \in \text{Lip}_{\text{loc}}(X, \mathbf{R})$ be such that $I(0) = 0$. Assume that there exist $\rho, r > 0$ and $e \in X$ with $\|e\| > r$ such that*

$$I(e) \leq 0 \quad \text{and} \quad \inf_{\|u\|=r} I(u) \geq \rho.$$

Set

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Then $c \geq \rho$ and there is a sequence $(u_n) \subset X$ such that

$$I(u_n) \stackrel{n\to\infty}{\longrightarrow} c \quad \text{and} \quad m(u_n) \stackrel{n\to\infty}{\longrightarrow} 0.$$

4. Some properties of the operator $Lu = \alpha\Delta^2 u + \beta\Delta u$

At first let us show (3). Indeed, by the generalized Green's Identity,

$$\int_\Omega \nabla u \nabla v = -\int_\Omega u\Delta v, \quad u, v \in H,$$

which gives

$$\int_\Omega |\nabla u|^2 \leq \|u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)} .$$

Applying the standard inequality below (cf. Gupta and Kwong [8, p. 474]),

$$\lambda_1{}^2 \int_\Omega u^2 \leq \int_\Omega |\Delta u|^2$$

we get (3).

Using (3) and $-\infty < \beta < \alpha\lambda_1$, it follows easily that

$$\langle u, v\rangle_H = \alpha \int_\Omega \Delta u \Delta v - \beta \int_\Omega \nabla u \nabla v, \quad u, v \in H$$

defines an inner product in H and

$$\|u\|^2 = \langle u, u\rangle_H$$

is its corresponding norm. Now, standard arguments can be applied to show that H is a Hilbert space.

By minimization technique one shows easily that for each $\xi \in L^2(\Omega)$ the problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = \xi & \text{in } \Omega \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

admits a solution in H, which is in fact unique. A solution operator

$$\begin{array}{rcl} S : L^2(\Omega) & \rightarrow & H \\ \xi & \mapsto & S(\xi) = u \end{array}$$

is well defined and satisfies

$$\|S(\xi)\| \leq c\,\|\xi\|_{L^2(\Omega)}\,, \quad \xi \in L^2(\Omega), \text{ for some } c > 0,$$

with $S : L^2(\Omega) \rightarrow L^2(\Omega)$ being compact and symmetric operator.

As a consequence, the eigenvalues of S form a sequence labeled $(\overline{\mu}_n)$ and actually

$$\overline{\mu}_n \overset{n\to\infty}{\longrightarrow} 0, \ \ \overline{\mu}_n > 0.$$

It is an easy matter to check that the eigenvalues of (2) are given by $\mu_n = 1/\overline{\mu}_n$ and the following properties hold true,

$$\mu_n = \lambda_n(\alpha\lambda_n - \beta), \tag{10}$$

where the corresponding eigenfunctions are the $(-\Delta, H_0^1(\Omega))$ eigenfunctions ϕ_n. From definitions of μ_1 and μ_2, a direct computation leads to the following inequalities

$$\mu_1 \int_\Omega |v|^2 \leq \alpha \int_\Omega |\Delta v|^2 - \beta \int_\Omega |\nabla v|^2, \quad v \in H, \tag{11}$$

and

$$\mu_2 \int_\Omega |w|^2 \leq \alpha \int_\Omega |\Delta w|^2 - \beta \int_\Omega |\nabla w|^2, \quad w \in H, \quad \int_\Omega w\phi_1 = 0. \tag{12}$$

5. $\mathrm{Lip}_{\mathrm{loc}}$ Functionals and Results on Multivalued Equations

The result below will be used in the sequence and the reader is referred to Chang [10], Costa and Goncalves [2] for further details.

Theorem 5.1. *Assume that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable satisfying (f_1)(ii) and $\underline{f}, \overline{f} : \Omega \times \mathbf{R} \to \mathbf{R}$ are N-measurable, that is, for each $u \in L^2(\Omega)$, we have*

$$x \mapsto \underline{f}(x, u(x)) \text{ and } x \mapsto \overline{f}(x, u(x)) \text{ are Lebesgue measurable.} \tag{f_5}$$

If

$$\Psi(u) = \int_\Omega F(x, u), \quad u \in L^2(\Omega)$$

then $\Psi : L^2(\Omega) \to \mathbf{R}$ is $\mathrm{Lip}_{\mathrm{loc}}$ and

$$\partial\Psi(u) \subset [\underline{f}(x, u(x)), \overline{f}(x, u(x))], \quad a.e.\ x \in \Omega.$$

Moreover, setting $\widehat{\Psi} \equiv \Psi\,|_H$ we have

$$\partial\widehat{\Psi}(u) \subset \partial\Psi(u), \quad u \in H.$$

Proposition 5.2. *Assume (f_1)(ii) and (f_5) and set*

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u), \quad u \in H.$$

If $u_0 \in H$ is a critical point of Φ, then $u_0 \in H^4(\Omega)$ and

$$\begin{cases} \alpha\Delta^2 u_0(x) + \beta\Delta u_0(x) \in [\underline{f}(x, u_0(x)), \overline{f}(x, u_0(x))] & a.e.\ x \in \Omega \\ \mathcal{B}u_0 = 0 & on\ \partial\Omega,\ (trace\ sense). \end{cases}$$

Proof. Let

$$Q(u) = \frac{1}{2}\|u\|^2, \quad u \in H.$$

Then

$$\Phi(u) = Q(u) - \widehat{\Psi}(u), \quad u \in H.$$

Since Q is a C^1-functional, we derive

$$0 \in \{Q'(u_0)\} - \partial\widehat{\Psi}(u_0)$$

and so

$$0 \equiv Q'(u_0) - \mu_0,$$

for some

$$\mu_0 \in \partial\widehat{\Psi}(u_0).$$

It follows that

$$\langle 0, v\rangle = \langle Q'(u_0), v\rangle_H - \langle \mu_0, v\rangle_H, \quad v \in H.$$

By Theorem 5.1, $\mu_0 = \xi$ for some $\xi \in L^2(\Omega)$,

$$0 = \alpha \int_\Omega \Delta u_0 \Delta v - \beta \int_\Omega \nabla u_0 \nabla v - \int_\Omega \xi v, \quad v \in H \tag{13}$$

and

$$\xi(x) \in [\underline{f}(x, u_0(x)), \overline{f}(x, u_0(x))] \ \text{ a.e. } x \in \Omega.$$

By the elliptic regularity theory $u_0 \in H^4(\Omega)$ and

$$\alpha\Delta^2 u_0 + \beta\Delta u_0 = \xi \ \text{ a.e. in } \Omega,$$

so that

$$\alpha\Delta^2 u_0(x) + \beta\Delta u_0(x) \in [\underline{f}(x, u_0(x)), \overline{f}(x, u_0(x))] \ \text{ a.e. } x \in \Omega.$$

In order to show that

$$\mathcal{B}u_0 = 0 \ \text{ on } \partial\Omega \text{ in sense of trace},$$

we assume that $\alpha > 0$, (the other case is standard).

Since

$$\alpha \int_\Omega \Delta u_0 \Delta v - \beta \int_\Omega \nabla u_0 \nabla v = \int_\Omega \xi v, \quad v \in H,$$

and there is an only $w \in H$ such that

$$\Delta w = \xi \ \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega.$$

From the Generalized Green Identity, we derive

$$\alpha \int_\Omega \Delta u_0 \Delta v + \beta \int_\Omega u_0 \Delta v = \int_\Omega w \Delta v.$$

Thus

$$\int_\Omega \left(\Delta u_0 - \frac{1}{\alpha} w + \frac{\beta}{\alpha} u_0 \right) \Delta v = 0.$$

Since for each $h \in L^2(\Omega)$, there is an only $v \in H$ such that

$$\Delta v = h \ \text{ in } \Omega, \qquad v = 0 \ \text{ on } \partial\Omega$$

we have

$$\int_\Omega \left(\Delta u_0 - \frac{1}{\alpha}w + \frac{\beta}{\alpha}u_0\right) h = 0, \quad h \in L^2(\Omega).$$

Thus,

$$\Delta u_0 = \frac{1}{\alpha}w - \frac{\beta}{\alpha}u_0 \in H_0^1(\Omega),$$

showing that

$$\Delta u_0 = 0 \;\text{ on } \partial\Omega, \;\text{(trace sense)}.$$

□

6. Proof of Theorem 1.1

Consider the energy functional associated to (1),

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u(x))dx, \quad u \in H.$$

By Theorem 5.1 and Proposition 5.2, $I \in \mathrm{Lip}_{\mathrm{loc}}(H)$ and its critical points are solutions of (1).

In the sequel, we shall establish a few technical lemmas involving the functional I.

Lemma 6.1. *Let $C \subset H$ be closed, convex and non-empty. Assume (f_1)(i)(ii) and (f_2)(i)(ii). If, in addition, $c \neq -\int_\Omega F_\infty$ then*

$$I \textit{ satisfies } (PS)_{c,C}.$$

Proof. Let $(u_n) \subset C$ be a sequence such that

$$I(u_n) \overset{n\to+\infty}{\longrightarrow} c \quad \text{and} \quad m(u_n) \overset{n\to\infty}{\longrightarrow} 0,$$

where

$$m(u_n) = \min\{\|\mu\|_{H'} \;|\; \mu \in \partial I(u_n)\}.$$

Hereafter, we denotes by $\mu_n \in \partial I(u_n)$ the linear functional that verifies the equality $m(u_n) = \|\mu_n\|_{H'}$. Repeating the same arguments used in the proof of Proposition 5.2, there is $v_n \in \partial\Psi(u_n)$ such that

$$\langle \mu_n, \phi\rangle = \alpha\int_\Omega \Delta u_n \Delta\phi - \beta\int_\Omega \nabla u_n \nabla\phi - \int_\Omega v_n\phi, \quad \forall \phi \in H$$

and

$$v_n(x) \in [\underline{f}(x,u(x)), \overline{f}(x,u(x))], \quad \text{a.e. } x \in \Omega.$$

Combining these information with (f_1)(ii), we reach

$$|v_n(x)| \leq \mu_1|u_n(x)| + \tau(x) \;\text{ a.e. } x \in \Omega. \tag{14}$$

Claim 1. $\{u_n\}$ is bounded in H.

Assume for while *Claim 1* has been shown, we can suppose without loss of generality that there is $u \in C$ such that

$$u_n \rightharpoonup u \ \text{ in } H.$$

The Sobolev embedding combined with (14) implies that

$$\int_\Omega v_n(u_n - u) \to 0 \ \text{ as } n \to +\infty.$$

From this, we can conclude that

$$\alpha \int_\Omega \Delta u_n \Delta(u_n - u) - \beta \int_\Omega \nabla u_n \nabla(u_n - u) \to 0.$$

Then

$$\left(\alpha \int_\Omega |\Delta u_n|^2 - \beta \int_\Omega |\nabla u_n|^2\right) \to \left(\alpha \int_\Omega |\Delta u|^2 - \beta \int_\Omega |\nabla u|^2\right),$$

which shows that

$$u_n \to u \ \text{ in } H.$$

Thus $\{u_n\}$ satisfies $(PS)_{c,C}$.

Verification of *Claim 1*. Set $u_n = t_n\phi_1 + w_n$, where $\int_\Omega w_n\phi_1 = 0$.

Let $\epsilon > 0$. Using the notations in the proof of Proposition 5.2,

$$\langle \mu_n, w_n \rangle = \|w_n\|^2 - \int_\Omega v_n w_n,$$

we have, by (11) and (12),

$$\begin{aligned}
\epsilon \|w_n\| \geq \|w_n\|^2 - \int_\Omega v_n w_n, & \\
= \ \|w_n\|^2 - \int_\Omega (v_n w_n - \mu_1 u_n w_n) - \mu_1 \int_\Omega u_n w_n & \\
\geq \ \|w_n\|^2 - C\|\tau\|_{L^2(\Omega)} \|w_n\| - \frac{\mu_1}{\mu_2}\|w_n\|^2 & \\
= \ \left(1 - \frac{\mu_1}{\mu_2}\right)\|w_n\|^2 - C\|\tau\|_{L^2(\Omega)}\|w_n\|. &
\end{aligned}$$

for some constant $C > 0$ and n large enough, showing that w_n is bounded.

Now, assume by the way of contradiction that

$$\|u_n\| \longrightarrow \infty.$$

From

$$\frac{u_n}{\|u_n\|} = \frac{t_n\phi_1}{\|u_n\|} + \frac{w_n}{\|u_n\|},$$

we have

$$1 = \left(\frac{t_n^2 \|\phi_1\|^2 + \|w_n\|^2}{\|u_n\|^2} \right)^{\frac{1}{2}},$$

and passing to the limit, we obtain

$$\frac{\|t_n \phi_1\|}{\|u_n\|} \longrightarrow 1.$$

Notice that

$$\frac{t_n \phi_1}{\|u_n\|} = -\frac{w_n}{\|u_n\|} + \frac{u_n}{\|u_n\|} \rightharpoonup \widehat{u}$$

for some $\widehat{u} \in H$. Thus

$$\frac{t_n \phi_1}{\|u_n\|} \to \widehat{u} \ \text{ in } H.$$

Notice that

$$\widehat{u} = t_0 \phi_1 \ \text{ for some } t_0 \in \mathbf{R}.$$

Hence

$$\frac{u_n}{\|u_n\|} \longrightarrow t_0 \phi_1,$$

and so

$$|u_n(x)| \to \infty \ \text{ a.e. } x \in \Omega.$$

For n large enough and $\epsilon > 0$, we have

$$\begin{aligned} \epsilon \|w_n\| &\geq \left(1 - \frac{\mu_1}{\mu_2}\right) \|w_n\|^2 - \int_\Omega (v_n - \mu_1 u_n) w_n \\ &\geq \left(1 - \frac{\mu_1}{\mu_2}\right) \|w_n\|^2 - \left(\int_\Omega |v_n - \mu_1 u_n|^2 \right)^{\frac{1}{2}} \|w_n\|_{L^2(\Omega)}, \end{aligned}$$

from where it follows that

$$\left(\epsilon + C \left(\int_\Omega |v_n - \mu_1 u_n|^2 \right)^{\frac{1}{2}} \right) \|w_n\| \geq \left(1 - \frac{\mu_1}{\mu_2}\right) \|w_n\|^2. \tag{15}$$

By Theorem 5.1,

$$v_n(x) \in [\underline{f}(x, u_n(x)), \overline{f}(x, u_n(x))] \ \text{ a.e. } x \in \Omega$$

which gives

$$\underline{f}(x, u_n(x)) - \mu_1 u_n(x) \leq v_n(x) - \mu_1 u_n(x) \leq \overline{f}(x, u_n(x)) - \mu_1 u_n(x) \ \text{ a.e. } x \in \Omega$$

and consequently

$$|v_n(x) - \mu_1 u_n(x)| \leq \max\{ \left| \underline{f}(x, u_n(x)) - \mu_1 u_n(x) \right|, \left| \overline{f}(x, u_n(x)) - \mu_1 u_n(x) \right| \}$$

By (f_1)(i)(ii), there is $\hat{h} \in L^2(\Omega)$ such that

$$|v_n(x) - \mu_1 u_n(x)|^2 \overset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad |v_n(x) - \mu_1 u_n(x)|^2 \leq \tau^2(x) \ \text{ a.e. } x \in \Omega.$$

Passing to the limit in (15) we have

$$2\epsilon \|w_n\| \geq \left(1 - \frac{\mu_1}{\mu_2}\right) \|w_n\|^2$$

so that $w_n \to 0$ in H. Now, a straightforward computation yields

$$I(u_n) = \frac{1}{2}\|w_n\|^2 - \int_\Omega \left(F(x,u_n) - \frac{\mu_1}{2}|u_n|^2\right) - \frac{\mu_1}{2}\int_\Omega |u_n|^2.$$

Hence, Passing to the limit in the above sentence, we get $c = -\int_\Omega F_\infty(x)$, which is a contradiction. This way, the verification of *Claim 1* is finished, and so, the Lemma 6.1 is proved. □

Lemma 6.2. *Assume that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable. If the conditions (f_2)(ii) and*

(f_6) *There exist two not identically zero functions $m, F_\sigma \in L^\infty(\Omega)$ with $\|m\|_\infty < \mu_1$, $\sigma \in (2, \frac{2N}{N-2})$ if $N \geq 3$ and $\sigma \in (2, \infty)$ if $N \leq 2$, satisfying*

$$F(x,t) \leq \min\left\{\frac{\mu_2}{2}, \frac{m(x)}{2} + F_\sigma(x)|t|^{\sigma-2}\right\} t^2, \quad a.e.\ x \in \Omega,\ t \in \mathbf{R}$$

hold, then there are $a, b > 0$ such that

$$I(u) \geq a\|u\|^2 - b\|u\|^\sigma, \quad u \in H.$$

Proof. From (f_6), it follows that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_\Omega m u^2 - \int_\Omega F_\sigma |u|^\sigma \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\|m\|_\infty \int_\Omega u^2 - \|F_\sigma\|_\infty \int_\Omega |u|^\sigma, \end{aligned}$$

which gives

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\|m\|_\infty}{\mu_1}\right)\|u\|^2 - C\|F_\sigma\|_\infty \|u\|^\sigma, \quad \forall u \in H.$$

The Lemma is proved by setting

$$a = \frac{1}{2}\left(1 - \frac{\|m\|_\infty}{\mu_1}\right) \quad \text{and} \quad b = C\|F_\sigma\|_\infty.$$

□

Existence of u_+ and u_-. Consider the closed, convex subsets of H,

$$\mathcal{C}^+ = \left\{ t\phi_1 + w \in H \mid t \geq 0, \ \int_\Omega w\phi_1 = 0 \right\}$$

and

$$\mathcal{C}^- = \left\{ t\phi_1 + w \in H \mid t \leq 0, \ \int_\Omega w\phi_1 = 0 \right\}.$$

Notice that

$$\operatorname{int}(\mathcal{C}^+), \qquad \operatorname{int}(\mathcal{C}^-) \neq \emptyset$$

and

$$\partial\mathcal{C}^+ = \partial\mathcal{C}^- = \left\{ w \in H \mid \int_\Omega w\phi_1 = 0 \right\}.$$

Using (f_3) we have for each $w \in \partial\mathcal{C}^+$,

$$\begin{aligned} I(w) &= \frac{1}{2}\|w\|^2 - \int_\Omega F(x,w) \\ &\geq \frac{1}{2}\|w\|^2 - \frac{\mu_2}{2}\int_\Omega w^2 \geq 0. \end{aligned}$$

Using (f_4), setting

$$u_+ = t_+\phi_1 \in \mathcal{C}^+,$$

where t_+ is given by (f_4), we have

$$\begin{aligned} I(u_+) &= \frac{1}{2}\|u_+\|^2 - \int_\Omega F(x,u_+) \\ &< \frac{1}{2}\|u_+\|^2 - \int_\Omega F_\infty - \frac{\mu_1}{2}\int_\Omega (u_+)^2 \\ &= -\int_\Omega F_\infty \leq 0. \end{aligned}$$

Hence,

$$I(u_+) < 0 \leq \inf_{\partial\mathcal{C}^+} I(u).$$

On the other hand, we get by (f_2)(ii)

$$\begin{aligned} I(t\phi_1 + w) &= \frac{1}{2}t^2\|\phi_1\|^2 + \frac{1}{2}\|w\|^2 - \int_\Omega F(x, t\phi_1 + w), \\ &\geq \frac{1}{2}t^2\|\phi_1\|^2 + \frac{1}{2}\|w\|^2 - \frac{\mu_1 t^2}{2}\int_\Omega \phi_1^2 - \frac{\mu_1}{2}\int_\Omega w^2 - \left\|\widehat{H}\right\|_{L^1(\Omega)} \\ &\geq \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right)\|w\|^2 - \left\|\widehat{H}\right\|_{L^1(\Omega)}. \end{aligned}$$

for $u \in \mathcal{C}^+$ and $t \geq 0$.

As a consequence,

$$I \text{ is bounded from below in } \mathcal{C}^+,$$

and so

$$c^+ = \inf_{\mathcal{C}^+} I > +\infty.$$

Since

$$I(u^+) < -\int_\Omega F_\infty(x)dx$$

we have

$$c^+ < \int_\Omega F_\infty(x)dx,$$

and therefore, it follows by Lemma 6.1, that

$$I \text{ satisfies } (PS)_{c^+,\mathcal{C}^+}.$$

By Theorem 3.1 I admits a local minimum $u_+ \in \operatorname{int}(\mathcal{C}^+)$ and in fact

$$I(u_+) = c_+ < 0.$$

In a similar way one shows that there is a local minimum of I, $u_- \in \operatorname{int}(\mathcal{C}^-)$ such that

$$I(u_-) = c_- < 0.$$

Existence of u_0. Setting $\|u\| = \rho,\ \rho > 0$, we have by Lemma 6.2,

$$I(u) \geq a\rho^2 - b\rho^\sigma, \quad u \in H.$$

Now, pick $\alpha > 0$ such that

$$I(u) \geq a\rho^2 - b\rho^\sigma \geq \alpha > 0,$$

for $\rho > 0$ small enough.

Using $I(0) = 0$ and (f_4),

$$I(t_+\phi_1) < -\int_\Omega F_\infty dx \leq 0.$$

Setting $e_+ = t_+\phi_1$ we have

$$\max\{I(0), I(e_+)\} \leq 0 < \alpha < \inf_{\|u\|=\rho} I(u).$$

Since

$$c_0 := \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)) > \alpha \geq 0 \geq -\int_\Omega F_\infty,$$

I satisfies $(PS)_{c_0,H}$, (cf. Lemma 6.1).

By Theorem 3.2, c_0 is a critical value of I and as a consequence there is a critical point $u_0 \in H$ such that

$$I(u_0) = c_0.$$

This ends the proof of Theorem 1.1. □

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