Journal of Convex Analysis
Volume 18 (2011), No. 2, 447–454

On Approximately h-Convex Functions*

Pál Burai
*Department of Applied Mathematics and Probability Theory,
University of Debrecen, 4010 Debrecen, Pf. 12, Hungary
burai@inf.unideb.hu*

Attila Házy
*Department of Applied Mathematics, University of Miskolc,
3515 Miskolc-Egyetemváros, Hungary
matha@uni-miskolc.hu*

Received: August 17, 2009
Revised manuscript received: April 5, 2010

A real valued function $f : D \to \mathbb{R}$ defined on an open convex subset D of a normed space X is called *rationally* (h, d)*-convex* if it satisfies

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) + d(x, y)$$

for all $x, y \in D$ and $t \in \mathbb{Q} \cap [0, 1]$, where $d : X \times X \to \mathbb{R}$ and $h : [0, 1] \to \mathbb{R}$ are given functions.

Our main result is of Bernstein-Doetsch type. Namely, we prove that if f is locally bounded from above at a point of D and rationally (h, d)-convex then it is continuous and (h, d)-convex.

Keywords: Convexity, approximate convexity, h-convexity, s-convexity, Bernstein–Doetsch theorem, regularity properties of generalized convex functions

26A51, 26B25, 39B62 Mathematics Subject Classification:

1. Introduction

Convexity and its generalizations are very important both in pure mathematics and in applications. It is wildly known that in applications we can not state convexity (generalized convexity) on the examined function, but we know some convexity like (generalized convexity like) behavior on the function. This means, we do not know whether the function in question is convex (generalized convex) or not, but we know that it is close to a convex (generalized convex) function. So, examining approximate convexity and approximate generalized convexity is an important task mainly in terms of optimization theory.

On the other hand we can not know regularity on the unknown function in general. Improving regularity of an unknown function is also an important and useful topic not only in applications but in pure mathematics too. Improving regularity means getting a stronger regularity property from a weaker one.

*This research has been supported by the Hausdorff Research Institute for Mathematics, Bonn, and the Hungarian Scientific Research Fund (OTKA) Grants K-62316, NK-68040, NK-81402.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Several authors have dealt with the above mentioned questions. The first approximate convexity result is due to Hyers and Ulam [10], where the authors used a constant error term. Another possibility to make a mathematical model of "being close to a convex function" is to use a function error term, which depends on the distance of the variables (see e.g. [12] or [15]).

Probably the most significant result in the early history of regularity theory of convex functions was developed by Bernstein and Doetsch [1]. They proved that the local boundedness from above at a point of a Jensen convex function implies its continuity and convexity as well on the whole domain.

The starting point of this work is a very recent generalization of convexity, namely h convexity. The sophisticated definition runs as follows: a real valued function $f : D \to \mathbb{R}$ is called *h-convex*, if

$$f\left(tx + (1-t)y\right) \leq h(t)f(x) + h(1-t)f(y) \tag{1}$$

for every $x, y \in D$, $t \in [0,1]$, where $h : [0,1] \to \mathbb{R}$ is a given function, and D is a nonempty, open, convex subset of a normed space $(X, \|\cdot\|)$. It is natural to assume that h is nonnegative, furthermore $h(t)$ and $h(1-t)$ is not equal to zero at the same time. If it is, then we get the class of bounded functions from above with zero. The concept of h convexity appeared first in [19] defined by Varošanec. This is a far generalization not only of convexity ($h(t) = t$) but e.g. of s-convexity ($h(t) = t^s$, $0 < s \leq 1$) due to Breckner [3], and other classes of functions (see [19] again) too.

We examine such functions which are "close to" an h-convex function in some sense. A function, on which some natural requirements are made, measures the error. More precisely: let $d : X \times X \to \mathbb{R}$ be given. A function $f : D \to \mathbb{R}$ is said to be *(h, d)-convex* if

$$f\left(tx + (1-t)y\right) \leq h(t)f(x) + h(1-t)f(y) + d(x, y) \tag{2}$$

for all $x, y \in D$ and $t \in [0,1]$. We are using two other concepts, *rational (h, d)-convexity* and *(t, h, d)-convexity*, which mean f fulfills (2) only for all rational t or a fixed t $(0 < t < 1)$ respectively.

In the earlier investigations, $\|x - y\|^p$ $(p > 0)$ or a linear combination with positive coefficients of such expressions were applied as an error term (see e.g. [12], [15], [8]). These define metrics on X, where the distance of two points does not change if we translate them, and they possess some kind of homogeneity property. Thus, generalizing and summing of the previous attributes simultaneously, we assume that d is a *ψ-subhomogeneous, translation invariant semimetric*, namely

$$\begin{array}{ll}
(i) & d(x,y) \geq 0,\\
(ii) & d(x,y) = d(y,x),\\
(iii) & d(x,y) \leq d(x,z) + d(z,y),\\
(iv) & d(x+z, y+z) = d(x,y),\\
(v) & d(ux, uy) \leq \psi(u) d(x,y)
\end{array}$$

for all $x, y, z \in X$ and $u > 0$, where the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded. The first three property declare that d is a semimetric on X, (iv) states the translation invariance of d, and (v) is the subhomogeneity of d with respect to ψ.

In the sequel d always denotes a continuous, ψ-subhomogeneous, translation invariant semimetric.

2. Main results

We have two main results. The first one is a regularity theorem on rationally (h, d)-convex, and (h, d)-convex functions. The second one is on (h, d)-convexity of rationally (h, d)-convex functions.

It is very important to note, that we suppose some technical conditions on h, namely we assume

$$\lim_{t\to 0} h(t) = 0, \qquad \lim_{t\to 1} h(t) = 1, \qquad h(t) \text{ and } h(1-t) \text{ not zero simultaneously} \tag{3}$$

henceforth. We draw the reader's attention to the fact, that the special functions mentioned in the introduction fulfill these conditions.

Theorem 2.1. *Assume that $d(x, x) = 0$, and $f : D \to \mathbb{R}$ is a rationally (h, d)-convex or (h, d)-convex function. If f is locally bounded from above at a point of D, then it is continuous on D.*

Theorem 2.2. *Assume that $d(x, x) = 0$, and $f : D \to \mathbb{R}$ is a ratonally (h, d)-convex function. If f is locally bounded from above at a a point of D, then it is (h, d)-convex.*

It is clear that we cannot expect the continuity of f without continuity assumption on d. Keeping in mind the nonnegativity of d, it seems natural at once to assume $d(x, x) = 0$.

It is worthy to mention the fact, that these theorems do not remain true with (t, h, d)-convexity assumption on f. On the other hand, it is an open problem, to find the smallest (in some sense, e.g. measure, category etc.) subset of the unit interval, such that the theorems remain true if f fulfills (2) only for all t from this subset. We know that any dense subset of $[0, 1]$ is enough.

3. Proofs of the main theorems

We begin with two lemmas. In the first one we deal with boundedness of (t, h, d)-convex functions. We recall that a function $f : D \to \mathbb{R}$ is called locally bounded from above at a point of D (or locally upper bounded at a point of D), if there exists a neighborhood U of this point such that f is locally bounded from above on U, and f is locally bounded from above on D, if it is bounded from above at every point of D. One can define local lower boundedness, and boundedness at a point or on the whole domain in a similar way.

Lemma 3.1. *If a (t, h, d)-convex function is locally bounded from above at a point of its domain, then it is locally bounded on the whole domain.*

Proof. Let $f : D \to \mathbb{R}$ be (t, h, d)-convex with a fixed $t \in]0, 1[$, and locally bounded from above at $w \in D$. First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0 := \{w\}, \qquad D_{n+1} := tD_n + (1-t)D.$$

Then, it follows by induction that

$$D_n = t^n w + (1-t^n)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption f is locally upper bounded at $w \in D_0$. Assume that f is locally upper bounded at each point of D_n. For an arbitrary $x \in D_{n+1}$, there exist $x_0 \in D_n$ and $y_0 \in D$ such that $x = tx_0 + (1-t)y_0$. By the inductive assumption, there exist constants $r > 0$ and $M_0 \geq 0$, such that $f(x') \leq M_0$, and $d(x', x_0) \leq M_0$ for all $x' \in B(x_0, r)$, where $B(x_0, r)$ denotes the open ball centered at x_0 with radius r. Because of the continuity of d, we can choose r such a way, that the previous will be true. Then, by the (t, h, d)-convexity of f, and the properties of d, we have

$$\begin{aligned} f(tx' + (1-t)y_0) &\leq h(t)f(x') + h(1-t)f(y_0) + d(x', y_0) \\ &\leq h(t)M_0 + h(1-t)f(y_0) + d(x', x_0) + d(x_0, y_0) \\ &\leq h(t)M_0 + h(1-t)f(y_0) + M_0 + d(x_0, y_0) =: M. \end{aligned}$$

Therefore, for $y \in U := tB(x_0, r) + (1-t)y_0 = B(tx_0 + (1-t)y_0, tr) = B(x, tr)$, we get that $f(y) \leq M$. Thus, f is locally bounded from above at $x \in D_{n+1}$, so f is locally bounded from above on D_{n+1} .

On the other hand, one can easily see that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Indeed, for fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - t^n w}{1 - t^n}.$$

Therefore $x_n \to x$ if $n \to \infty$. Being the set open there exists an $n_0 \in \mathbb{N}$, such that $x_n \in D$ if $n \geq n_0$. Therefore

$$x = t^n w + (1-t^n)x_n \in t^n w + (1-t^n)D = D_n.$$

Thus f is locally bounded from above on D.

We prove now that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\varrho > 0$ and $M > 0$ such that $f(x) \leq M$ and $d(x, q) \leq M$ if $x \in B(q, \varrho)$. (Just like in the first part of the proof, we can find such ϱ, using the continuity of d.) Let $x \in B(q, (1-t)\varrho)$ and $y := \frac{1}{1-t}q - \frac{t}{1-t}x$. Then y is in $B(q, t\varrho) \subset B(q, \varrho)$. By (t, h, d)-convexity of f, and by ψ-subhomogeneity and translation invariance of d, we get

$$\begin{aligned} f(q) = f(tx + (1-t)y) &\leq h(t)f(x) + h(1-t)f(y) + d(x, y) \\ &\leq h(t)f(x) + h(1-t)f(y) + \psi\left(\frac{1}{1-t}\right) d(x, q), \end{aligned}$$

which implies

$$\begin{aligned} f(x) &\geq \frac{1}{h(t)} f(q) - \frac{h(1-t)}{h(t)} f(y) - \frac{1}{h(t)} \psi\left(\frac{1}{1-t}\right) d(x,q) \\ &\geq \frac{1}{h(t)} f(q) - \frac{h(1-t)}{h(t)} M - \frac{1}{h(t)} \psi\left(\frac{1}{1-t}\right) M =: M^*. \end{aligned}$$

Therefore f is locally bounded from below at any point of D. □

Remark 3.2. We did not use (3) in the previous proof.

The next result states that the local upper boundedness of a rationally (h,d)-convex function at a point of D yields its continuity at this point as well.

Lemma 3.3. *Let $d(x,x) = 0$, and $f : D \to \mathbb{R}$ be a rationally (h,d)-convex function. If f is locally bounded from above at a point of its domain, then it is continuous at the same point.*

Proof. Let f be locally bounded from above at $w \in D$, then there exist constants $r > 0$ and $K \geq 0$ such that $f(x) \leq K$ for every $x \in B(w,r)$. Let ε be an arbitrary positive real number. Then there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$ is an arbitrarily fixed positive integer, the following three inequalities hold at the same time

$$h\left(\frac{1}{n}\right) K + \left[h\left(\frac{n-1}{n}\right) - 1\right] f(w) < \frac{\varepsilon}{4}, \tag{4}$$

$$\left(\frac{h\left(\frac{1}{n}\right)}{h\left(\frac{n-1}{n}\right)}\right) K + \left[1 - \frac{1}{h\left(\frac{n-1}{n}\right)}\right] f(w) < \frac{\varepsilon}{4} \tag{5}$$

and

$$\frac{2}{h\left(\frac{n-1}{n}\right)} < 3. \tag{6}$$

Let $r_1 = \min\{r, \frac{\varepsilon}{4}\}$. Using the continuity of d and the equality $d(w,w) = 0$, there exists $r_1' < r_1$ such that $d(x,w) < r_1$ if $x \in B(w, r_1')$, and let $\delta < \frac{r_1'}{n}$. We prove that

$$|f(x) - f(w)| < \varepsilon \quad (x \in B(w,\delta)).$$

For $x \in B(w,\delta)$ there exist $y, z \in B(w, r_1')$ such that

$$\begin{aligned} x &= \frac{1}{n} y + \frac{n-1}{n} w, \quad \text{so } y = nx - (n-1)w \\ w &= \frac{1}{n} z + \frac{n-1}{n} x, \quad \text{so } z = nw - (n-1)x. \end{aligned}$$

Indeed,

$$\|y - w\| = \|nx - nw\| < n\delta < r_1',$$

and similarly

$$\|z - w\| = \|(n-1)(x-w)\| < (n-1)\delta < r_1';$$

that is $y, z \in B(w, r_1')$.

According to rational (h, d)-convexity of f,

$$f(x) \le h\left(\frac{1}{n}\right) f(y) + h\left(\frac{n-1}{n}\right) f(w) + d(y, w) \qquad (7)$$
$$\le h\left(\frac{1}{n}\right) K + h\left(\frac{n-1}{n}\right) f(w) + r_1,$$

and

$$f(w) \le h\left(\frac{1}{n}\right) f(z) + h\left(\frac{n-1}{n}\right) f(x) + d(z, x)$$
$$\le h\left(\frac{1}{n}\right) K + h\left(\frac{n-1}{n}\right) f(x) + d(z, w) + d(w, x) \qquad (8)$$
$$\le h\left(\frac{1}{n}\right) K + h\left(\frac{n-1}{n}\right) f(x) + 2r_1.$$

Using (7) and (4) we get

$$f(x) - f(w) \le h\left(\frac{1}{n}\right) K + \left[h\left(\frac{n-1}{n}\right) - 1\right] f(w) + r_1 \qquad (9)$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$

and using the inequality (8) we get

$$f(x) \ge \frac{f(w) - h\left(\frac{1}{n}\right) K - 2r_1}{h(\frac{n-1}{n})},$$

which together with (5) imply that

$$f(x) - f(w) \ge \left[\frac{1}{h(\frac{n-1}{n})} - 1\right] f(w) - \left(\frac{h(\frac{1}{n})}{h(\frac{n-1}{n})}\right) K - \frac{2r_1}{h(\frac{n-1}{n})}$$
$$> -\left(\frac{\varepsilon}{4} + \frac{2}{h(\frac{n-1}{n})} \frac{\varepsilon}{4}\right).$$

According to (6) we get

$$f(x) - f(w) > -\left(\frac{\varepsilon}{4} + 3\frac{\varepsilon}{4}\right) = -\varepsilon. \qquad (10)$$

The inequalities (9) and (10) show that $|f(x) - f(w)| < \varepsilon$, that is f is continuous at w, so the proof is complete. □

Proof of Theorem 2.1. According to Lemma 3.1, f is locally bounded at every point of D. So, we can use Lemma 3.3, which implies the continuity of f at every point of D. □

Proof of Theorem 2.2. We prove that the function f is (t, h, d)-convex for all $t \in [0, 1]$. Let $t \in [0, 1]$ arbitrary. Then there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \in$

$\mathbb{Q} \cap]0,1[$ and $t_n \to t$ (when n tends to ∞). Applying rational (h,d)-convexity of f, we get

$$f\left(t_n x + (1-t_n)y\right) \leq h(t_n)f(x) + h(1-t_n)f(y) + d(x,y). \tag{11}$$

The local upper boundedness of f implies the continuity of f (according to Lemma 2.1). Therefore, taking the limit $n \to \infty$ in (11), we get

$$f\left(tx + (1-t)y\right) \leq h(t)f(x) + h(1-t)f(y) + d(x,y),$$

which proves the (h,d)-convexity of f. □

4. Corollaries and applications

In this section we show some applications of the previous results. We begin with a corollary, which is an immediate consequence of Lemma 3.1. We call f (h,d)-convex with respect to $S \subset]0,1[$, if f fulfills (2) only for all $t \in S$.

Corollary 4.1. *Let $f : D \to \mathbb{R}$ be a (h,d)-convex function with respect to $S \subset]0,1[$. If f is locally bounded from above at a point of D, then it is locally bounded on D.*

For the readers convenience we recall the following two theorems:

Theorem 4.2 (Steinhaus, [18]). *Let $A, B \subset \mathbb{R}^n$ be arbitrary sets with positive measure. Then* $\operatorname{int}(A+B) \neq \emptyset$.

Theorem 4.3 (Piccard, [16]). *Let $(X;+)$ be a topological group, and $A, B \subset X$ two sets of the second category with the Baire property. Then* $\operatorname{int}(A+B) \neq \emptyset$.

If the underlying space is of finite dimension, the local boundedness from above assumption can be weakened. Using Theorem 2.1, Theorem 2.2 and Steinhaus' and Piccard's theorems (cf. [18], [16]), we get the following two corollaries.

Corollary 4.4. *Let D be an open, convex subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a rationally (h,d)-convex function, and $d(x,x) = 0$. If there exists a set $S \subset D$ of positive Lebesgue measure, such that f is bounded from above on S, then f is continuous, and (h,d)-convex on D.*

Corollary 4.5. *Let D be an open, convex subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a rationally (h,d)-convex function, and $d(x,x) = 0$. If there exists a Baire-measurable set $S \subset D$ of second category, such that f is bounded from above on S, then f is continuous, and (h,d)-convex on D.*

Theorem 2.1, Theorem 2.2, Lemma 3.1, and Lemma 3.3 are far generalizations of earlier theorems.

We get [2, Theorem 2] from Lemma 3.1 with $d \equiv 0$, and $h(\lambda) = \lambda^s$ ($\lambda \in]0,1[$ is fixed here).

With $d \equiv 0$, and $h(t) = t^s$, we get [3, Satz 2.1] from Lemma 3.3. With the same casting one can deduce [3, Satz 2.2 and Korollar 2.3] from Lemma 3.1, and Theorem 2.1 with Theorem 2.2 respectively.

We have [11, Theorem 6.2.1, Theorem 6.2.2, Theorem 6.2.3 148 p.] from Lemma 3.1 (taking into account Remark 3.2) with $h(t) = \frac{1}{2}$ and $d \equiv 0$.

References

[1] F. Bernstein, G. Doetsch: Zur Theorie der konvexen Funktionen, Math. Ann. 76 (1915) 514–526.

[2] P. Burai, A. Házy, T. Juhász: Bernstein-Doetsch type results for s-convex functions, Publ. Math. 75(1–2) (2009) 23–31.

[3] W. W. Breckner: Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen, Publ. Inst. Math., Nouv. Sér. 23 (1978) 13–20.

[4] W. W. Breckner, G. Orbán: Continuity Properties of Rationally s-Convex Mappings with Values in Ordered Topological Linear Space, "Babes-Bolyai" University, Kolozsvár (1978).

[5] A. Házy: On approximate t-convexity, Math. Inequal. Appl. 8(3) (2005) 389–402.

[6] A. Házy: On the stability of t-convex functions, Aequationes Math. 74 (2007) 210–218.

[7] A. Házy, Zs. Páles: Approximately midconvex functions, Bull. Lond. Math. Soc. 36 (2004) 339–350.

[8] A. Házy, Zs. Páles: On approximately t-convex functions, Publ. Math. 66(3–4) (2005) 489–501.

[9] H. Hudzik, L. Maligranda: Some remarks on s_i-convex functions, Aequationes Math. 48 (1994) 100–111.

[10] D. H. Hyers, S. M. Ulam: Approximately convex functions, Proc. Amer. Math. Soc. 3 (1952) 821–828.

[11] M. Kuczma: An Introduction to the Theory of Functional Equations and Inequalities, Państwowe Wydawnictwo Naukowe, Uniwersytet Śląski, Warszawa (1985).

[12] D. T. Luc, H. V. Ngai, M. Théra: Approximate convex functions, J. Nonlinear Convex Anal. 1(2) (2000) 155–176.

[13] C. T. Ng, K. Nikodem: On approximately convex functions, Proc. Amer. Math. Soc. 118(1) (1993) 103–108.

[14] Zs. Páles: Bernstein-Doetsch-type results for general functional inequalities, Rocz. Nauk.-Dydakt., Pr. Mat. 17 (2000) 197–206.

[15] Zs. Páles: On approximately convex functions, Proc. Amer. Math. Soc. 131 (2003) 243–252.

[16] S. Piccard: Sur des ensembles parfaits, Mém. Univ. Neuchâtel 16, Secrétariat de l'Université, Neuchâtel (1942).

[17] M. Pycia: A direct proof of the s-Hölder continuity of Breckner s-convex functions, Aequationes Math. 61 (2001) 128–130.

[18] H. Steinhaus: Sur les distances des points des ensembles de mesure positive, Fundam. Math. 1 (1920) 93-104.

[19] S. Varošanec: On h-convexity, J. Math. Anal. Appl. 32 (2007) 303–311.