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Semiconcave Functions with Power Moduli

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A function f is approximately convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) + R(\alpha, \|x-y\|),$$

for $x, y \in \mathrm{dom} f$, $\alpha \in [0,1]$ and for a respective perturbation term R.

If the above inequality is assumed only for $\alpha = \frac{1}{2}$, then the function f is called Jensen approximately convex.

The relation between Jensen approximate convexity and approximate convexity has been investigated in many papers, in particular for semiconcave functions in [1]. We improve an estimation involved in such relation from [1] and show that our result is sharp.

Keywords: Semiconcave function, paraconvex function, Jensen convexity, modulus of semiconcavity

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In the last fifty years there appeared several natural and strongly related versions of approximate convexity: ε-convexity [6, 7], (ε, p)-convexity [3, 4, 5], paraconvexity [11, 12] and semiconcavity [1]. One of the most important from the applications point of view is the notion of semiconcave functions [1, 2], it is a convenient tool in the study of Hamilton-Jacobi equations and optimal control problems.

For the convenience of the reader we recall the definitions of the semiconcave function [1] (we slightly adapt the notation). Let S be a subset of $\mathbb{R}^N$. By $[S]$ we denote the set of all pairs $(x, y) \in S \times S$ such that the line segment $[x, y] := \{tx + (1-t)y : t \in [0,1]\}$ is contained in S. For a function $u : S \to \mathbb{R}$ we define its *concave difference* $\mathcal{C}u : [S] \to \mathbb{R}$ by the formula

$$\mathcal{C}u(x, y; t) := tu(x) + (1-t)u(y) - u(tx + (1-t)y) \quad \text{for } (x, y) \in [S], t \in [0,1].$$

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By $\mathcal{M}$ we denote the set of all nondecreasing upper semi-continuous functions $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to 0^+} \omega(t) = 0$. Let $\omega \in \mathcal{M}$. We say that a function $u : S \to \mathbb{R}$ is *ω-semiconcave* if

$$\mathcal{C}u(x, y; t) \le t(1-t)\omega(\|x-y\|)\|x-y\| \quad \text{for } (x, y) \in [S], t \in [0,1]. \tag{1}$$

We call ω a modulus of semiconcavity for u in S. If (1) holds for $t = 1/2$, we say that u is Jensen ω-semiconcave. We say that u is *(Jensen) semiconcave* if it is (Jensen) ω-semiconcave with a certain modulus of (Jensen) semiconcavity $\omega \in \mathcal{M}$.

In general it is not trivial to verify if the given function is semiconcave. Since the Jensen semiconcavity is much easier to verify, the authors of [1] investigated the problem when Jensen semiconcavity implies semiconcavity.

Theorem CS (Theorem 2.1.20). *Let $\omega \in \mathcal{M}$ and let*

$$\tilde{\omega}(r) := \sum_{k=0}^{\infty} \omega(r/2^k).$$

If $\tilde{\omega}$ admits only finite values[1], *then $\tilde{\omega} \in \mathcal{M}$ and every continuous Jensen ω-semiconcave function is $\tilde{\omega}$-semiconcave.*

Moreover, if $\omega(r) = Cr$, then we can take $\tilde{\omega} = \omega$.

The most important and natural case, see [1, Remark 2.1.1], is when ω is the power function $\omega_p(r) = Cr^p$ for a certain $p \in (0,1]$. The case when $p > 1$ trivializes since by the result of Rolewicz [11] every ω_p-semiconcave function with a convex domain is concave. Approximately convex and concave functions with power form moduli were extensively studied by many authors [1, 2, 4, 5, 8, 9], [11]–[14], [16]. The terms: semiconcave, semiconvex, p-paraconvex, p-approximately convex function are in use.

From now on we assume that a function ω_p has the form

$$\omega_p(r) := Cr^p \quad \text{where } C > 0 \text{ and } p \in (0,1].$$

For $p \in (0,1]$ we define

$$A_p := \frac{1}{2 - 2^{1-p}}. \tag{2}$$

As a direct consequence of Theorem CS we get the following result.

Corollary CS. *Let $p \in (0,1]$ and let $u : S \to \mathbb{R}$ be a continuous Jensen ω_p-semiconcave function. Then u is $(2A_p\omega_p)$-semiconcave if $p < 1$, and $(A_1\omega_1)$-semiconcave if $p = 1$.*

One can show that for $p = 1$ the above result is sharp. There arises the question, see [1, Remark 2.1.1], if in the case $p \in (0,1)$ the estimation given in Corollary CS can be improved. We answer this question positively. We show that the constant from Corollary CS can be improved from $2A_p$ to A_p and that the constant A_p is optimal.

[1] One can easily verify that it is equivalent to the condition $\tilde{\omega}(1) < \infty$.

Now we begin our investigations. We will need the functions $d_k : \mathbb{R} \to \mathbb{R}$ defined as follows

$$d_k(x) := \frac{1}{2^k}\operatorname{dist}(2^k x; \mathbb{Z}) \quad \text{for } x \in \mathbb{R}.$$

By $B(\mathbb{R}, \mathbb{R})$ we denote the space of bounded functions with the supremum norm. In our investigation we will need the following reformulation of the de Rham's Theorem [10].

Theorem R. *Let $h \in B(\mathbb{R}, \mathbb{R})$, $a \in [0, 1)$, $b \in \mathbb{R}$. Let $R_h : B(\mathbb{R}, \mathbb{R}) \to B(\mathbb{R}, \mathbb{R})$ be an operator defined by*

$$(R_h f)(x) := h(x) + af(bx) \quad \textit{for } f \in B(\mathbb{R}, \mathbb{R}),\ x \in \mathbb{R}.$$

Then

(*i*) R_h *is a contraction which has a unique fixed point* f_h*;*
(*ii*) *if* $g \in B(\mathbb{R}, \mathbb{R})$ *is such that* $R_h g \le g$*, then* $f_h \le g$*.*

For $p \ge 0$ we need the Takagi-type function $T_p : \mathbb{R} \to \mathbb{R}_+$ [15] defined by the formula

$$T_p(x) = \sum_{k=0}^{\infty} \frac{1}{2^{kp}} d_k(x) \quad \text{for } x \in \mathbb{R}.$$

Proposition 1. *Let $p \in (0, 1]$. Then*

$$T_p(x) \le 2A_p x(1-x) \quad \textit{for } x \in [0, 1]. \tag{3}$$

Furthermore A_p is the minimal constant satisfying (3)*. Hence*

$$\sup_{x \in (0,1)} \frac{T_p(x)}{x(1-x)} = 2A_p. \tag{4}$$

Proof. One can easily notice that

$$d_k(1/2^n) = \begin{cases} 0 & \text{if } n \le k, \\ 1/2^n & \text{otherwise.} \end{cases}$$

This implies that

$$T_p(1/2^n) = \sum_{k=0}^{\infty} \frac{1}{2^{kp}} d_k(1/2^n) = \sum_{k=0}^{n-1} \frac{1}{2^{kp}} (1/2^n).$$

Thus $2^n T_p(1/2^n) \to \sum_{k=0}^{\infty} \frac{1}{2^{kp}} = 2A_p$ as $n \to \infty$. Consequently there is no constant less than A_p satisfying (3).

Using Theorem R, we show that inequality (3) holds. Consider the operator R_{d_0} : $B(\mathbb{R}, \mathbb{R}) \to B(\mathbb{R}, \mathbb{R})$,

$$(R_{d_0} f)(x) := d_0(x) + \frac{1}{2^{1+p}} f(2x) \quad \text{for } f \in B(\mathbb{R}, \mathbb{R}),\ x \in \mathbb{R}.$$

It is easy to check that the function T_p is a fixed point of R_{d_0}, that is

$$T_p(x) = d_0(x) + \frac{1}{2^{1+p}} T_p(2x) \quad \text{for } x \in \mathbb{R}.$$

We define the function $\psi_p : \mathbb{R} \to \mathbb{R}$ in the following way: ψ_p is 1-periodic and

$$\psi_p(x) = 2A_p x(1-x) \quad \text{for } x \in [0,1].$$

We are going to show that

$$T_p(x) \le \psi_p(x) \quad \text{for } x \in \mathbb{R}.$$

According to Theorem R (ii) it is sufficient to prove that

$$d_0(x) + \frac{1}{2^{1+p}} \psi_p(2x) \le \psi_p(x) \quad \text{for } x \in \mathbb{R}. \tag{5}$$

Since the functions d_0 and ψ_p are 1-periodic and symmetric with respect to $\frac{1}{2}$, it is enough to show the above inequality for $x \in [0, \frac{1}{2}]$. Then (5) takes the form

$$x + \frac{1}{2^{1+p}} 2A_p(2x)(1-2x) \le 2A_p x(1-x) \quad \text{for } x \in [0, 1/2].$$

The case $x = 0$ is trivial. If $x > 0$, then the above inequality reduces to

$$1 + \frac{4A_p}{2^{1+p}}(1-2x) \le 2A_p(1-x) \quad \text{for } x \in (0, 1/2].$$

One can easily verify that this inequality is valid for $x = 0$ and $x = \frac{1}{2}$. Since the left and right hand sides are affine functions, it holds for all $x \in [0, \frac{1}{2}]$. We have proved (3). Since $T_p(0) = T_p(1) = 0$, from just proved assertions we directly obtain (4). $\square$

To prove our main result we will need the following theorem.

Theorem T1 ([14, Proposition 4.1]). *Let $u : S \to \mathbb{R}$ be a continuous Jensen ω-semiconcave function. Then for every $(x, y) \in [S]$ we have*

$$\mathcal{C}u(x, y; t) \le \frac{1}{2} \left(\sum_{k=0}^{\infty} \omega(2^{-k}\|x-y\|) d_k(t) \right) \|x-y\| \quad \textit{for } t \in [0,1]. \tag{6}$$

Now we are ready to prove the main result of the paper, which generalizes Corollary CS.

Theorem 2. *Let $p \in (0,1]$ and let $u : S \to \mathbb{R}$ be continuous Jensen ω_p-semiconcave function. Then u is $(A_p w_p)$-semiconcave.*

Proof. Fix $(x,y) \in [S]$ and $t \in [0,1]$. From Theorem T1 and Proposition 1 we obtain

$$\begin{aligned}\mathcal{C}(x,y;t) &\le \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{d_k(t)}{2^{kp}}C\|x-y\|^p\right)\|x-y\| \\ &= \frac{1}{2}T_p(t)(C\|x-y\|^p)\|x-y\| \le A_p t(1-t)\omega_p(\|x-y\|)\|x-y\|.\end{aligned}$$

□

Now we discuss the question if the obtained result is optimal. We first present a direct consequence of Theorem 2.

Corollary 3. *Let $p \in (0,1]$ and $S \subset \mathbb{R}^N$ be given. We denote*

$$r_S := \sup_{(x,y)\in[S]} \|x-y\|.$$

Let

$$\tilde{\omega}_p^S(r) := A_p\omega_p(\min(r, r_S)) \quad \textit{for } r \in \mathbb{R}_+.$$

Then every continuous Jensen ω_p-semiconcave function $u : S \to \mathbb{R}$ is $\tilde{\omega}_p^S$-semiconcave.

To show that in general the estimation given in Corollary 3 cannot be improved, we will use:

Theorem T2 ([13, Corollary 2.1]). *For every $p \in (0,1]$ we have*

$$C(-T_p)(x,y;1/2) \le \frac{1}{2}|x-y|^{p+1} \quad \textit{for } x,y \in \mathbb{R}.$$

Theorem 4. *Let $p \in (0,1]$ and $S \subset \mathbb{R}^N$ be fixed, and let $\tilde{\omega} \in \mathcal{M}$ be such that for every continuous Jensen ω_p-semiconcave function $u : S \to \mathbb{R}$, the function u is $\tilde{\omega}$-semiconcave. Then*

$$\tilde{\omega} \ge \tilde{\omega}_p^S.$$

Proof. It is sufficient to show that $\tilde{\omega}(r) \ge \tilde{\omega}_p^S(r)$ for every $r < r_S$. We choose an arbitrary $r \in (0, r_S)$. By the definition of r_S there exists a $(\bar{x},\bar{y}) \in [S]$ such that $\|\bar{x}-\bar{y}\| = r$. By the Hahn-Banach Theorem we can find $\xi^* \in (\mathbb{R}^N)^*$ such that $\|\xi^*\| = 1$ and

$$\xi^*(\bar{y}-\bar{x}) = \|\bar{y}-\bar{x}\|.$$

We define the function $u : S \to \mathbb{R}$ by the formula

$$u(x) := -\frac{1}{2}r^{p+1}C(T_p)(\xi^*(x-\bar{x})/r) \quad \text{for } x \in S.$$

Clearly the above function is continuous. Making use of Theorem T2 one can easily verify that u is Jensen ω_p-semiconcave. Hence u is $\tilde{\omega}$-semiconcave. Since $u(\bar{x}) =$

$u(\bar{y}) = 0$, by semiconcavity of u we obtain

$$\tilde{\omega}(r) = \tilde{\omega}(\|\bar{y} - \bar{x}\|) \geq \sup_{t \in (0,1)} \frac{\mathcal{C}u(\bar{y}, \bar{x}; t)}{\|\bar{y} - \bar{x}\| t(1-t)} = \sup_{t \in (0,1)} \frac{-u(t\bar{y} + (1-t)\bar{x})}{rt(1-t)}$$
$$= \sup_{t \in (0,1)} \frac{\frac{1}{2} r^{p+1} C T_p(t)}{rt(1-t)}.$$

By (4) we conclude that $\tilde{\omega}(r) \geq A_p C r^p = \tilde{\omega}_p^S(r)$. $\square$

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