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An Evolutionary Structure of Pyramids in the Three-Dimensional Euclidean Space

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We solve the problem of evolution for some classes of pentahedra (pyramids) in the three dimensional Euclidean space by applying the inverse weighted Fermat-Torricelli problem of 5 rays that meet at the weighted Fermat-Torricelli point A_0 and the invariance property of the weighted Fermat-Torricelli point. The main result is the three dimensional property of plasticity which states that: If we decrease the weights that correspond to the first, third and fourth ray which passes from the apex of the pyramid, then the weights that correspond to the second and fifth ray increase. Finally, we introduce the notion of the generalized plasticity for weighted pyramids via a specific discretization of the five weights along the five given prescribed rays.

Keywords: Fermat-Torricelli point, inverse Fermat-Torricelli problem, plasticity property, weighted pyramids

1991 Mathematics Subject Classification: 51E10, 52A15, 52B10

1. Introduction

An evolutionary structure of convex quadrilaterals in $\mathbb{R}^2$ has been studied in [4] and an evolutionary structure of tetrahedra in $\mathbb{R}^3$ has been studied in [5].

In this paper, we derive an evolutionary structure of some classes of pentahedra (pyramids) $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$. We apply the following ideas:

(1) Obtain a method of cyclical differentiation with respect to some specific angles by calculating the weighted Fermat-Torricelli point for pyramids.
(2) Apply the invariance property of the weighted Fermat-Torricelli point and the inverse weighted Fermat-Torricelli problem of five rays that meet at the weighted Fermat-Torricelli point at A_0.
(3) Derive some evolutionary equations that point out the plasticity of the weighted pyramids by viewing them as a dynamical system of the weights in $\mathbb{R}^3$ by using specific symbolic computations that deal with the decomposition of weights of pyramid to some specific weighted tetrahedra.

The main result of this paper is the derivation of a plasticity property of pyramids.

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Furthermore, we introduce the notion of generalized plasticity for pyramids via an evolution due to the discretization of the weights that correspond to the weighted Fermat-Torricelli point along the five given prescribed rays where the floating case occurs.

At the appendix, we mention an elementary proof of the invariance property of the weighted Fermat-Torricelli point in $\mathbb{R}^3$ and all the necessary results (floating case, absorbed case, existence uniqueness) that we need to complete our study.

2. The weighted "Fermat-Torricelli" problem for pyramids

We start by stating the problem for a pyramid $A_1A_2A_3A_4A_5$ in $\mathbb{R}^3$.

Problem 2.1. *Let $A_1A_2A_3A_4A_5$ be a pyramid and $A_1A_2A_3A_5$ be the base of the pyramid. Suppose that a non-negative number (weight) B_i, corresponds to each vertex A_i for $i = 1, 2, 3, 4, 5$, respectively. Find the weighted Fermat-Torricelli point A_0 of $A_1A_2A_3A_4A_5$ which minimizes the sum of the lengths of the line segments a_i that connect every vertex with A_0 multiplied by the positive weight B_i:*

$$B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + B_5a_5 = minimum. \tag{1}$$

Solution of Problem 2.1. The independent variables a_1, a_2, α will be used, in order to find A_0, where α is the dihedral angle between the planes $A_0A_1A_2$ and $A_3A_1A_2$ (see Figure 2.1). The variables a_3, a_4, a_5 can be expressed as functions of a_1, a_2 and α:

$$a_3 = a_3(a_1, a_2, \alpha), \qquad a_4 = a_4(a_1, a_2, \alpha), \qquad a_5 = a_5(a_1, a_2, \alpha). \tag{2}$$

From (1) and (2) the following equation is obtained:

$$B_1a_1 + B_2a_2 + B_3a_3(a_1, a_2, \alpha) + B_4a_4(a_1, a_2, \alpha) + B_5a_5(a_1, a_2, \alpha) = \text{minimum}. \tag{3}$$

By differentiation of (3) with respect to the variables a_1, a_2 and α we get

$$B_1 + B_3\frac{\partial a_3}{\partial a_1} + B_4\frac{\partial a_4}{\partial a_1} + B_5\frac{\partial a_5}{\partial a_1} = 0, \tag{4}$$

$$B_2 + B_3\frac{\partial a_3}{\partial a_2} + B_4\frac{\partial a_4}{\partial a_2} + B_5\frac{\partial a_5}{\partial a_2} = 0, \tag{5}$$

$$B_3\frac{\partial a_3}{\partial \alpha} + B_4\frac{\partial a_4}{\partial \alpha} + B_5\frac{\partial a_5}{\partial \alpha} = 0. \tag{6}$$

We proceed by calculating equation (6).

We express a_3 as a function of a_1, a_2 and α by using the following equations:

$$\cos(\alpha_{102}) = \frac{a_1^2 + a_2^2 - a_{12}^2}{2a_1a_2}, \tag{7}$$

$$h_{0,12} = \frac{a_1a_2\sin(\alpha_{102})}{a_{12}}, \tag{8}$$

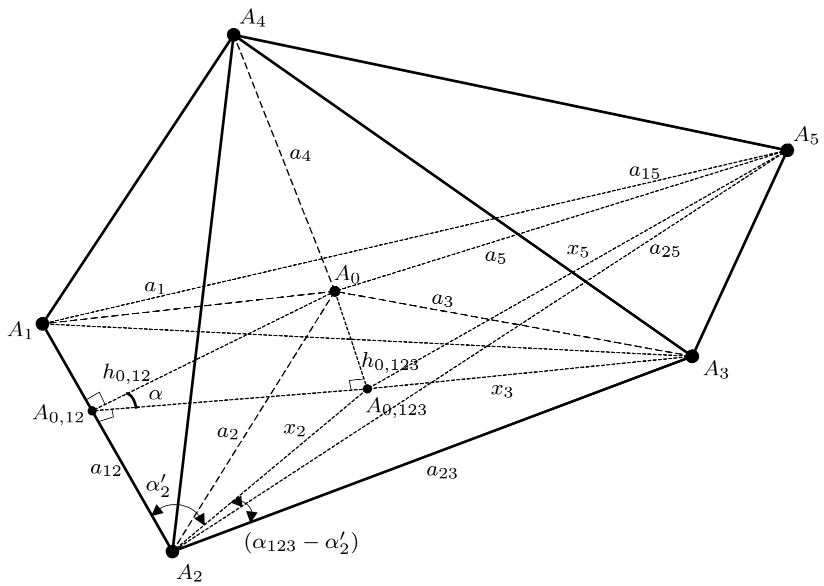

Figure 2.1.

$$h_{0,123} = h_{0,12} \sin(\alpha), \tag{9}$$

$$x_2^2 = a_2^2 - h_{0,123}^2, \tag{10}$$

$$\sin(\alpha_2') = \frac{h_{0,12} \cos(\alpha)}{x_2}, \tag{11}$$

$$x_3^2 = x_2^2 + a_{23}^2 - 2x_2 a_{23}, \cos(\alpha_{123} - \alpha_2'), \tag{12}$$

$$a_3^2 = x_3^2 + h_{0,123}^2, \tag{13}$$

or

$$a_3^2 = a_2^2 + a_{23}^2 - 2a_{23} \left[\sqrt{a_2^2 - h_{0,12}^2} \cos(\alpha_{123}) + h_{0,12} \sin(\alpha_{123}) \cos(\alpha) \right], \tag{14}$$

where $h_{0,12}$ is the height of the triangle $\nabla A_0 A_1 A_2$ from A_0 to $A_1 A_2$ and $h_{0,123}$ is the distance from A_0 to the plane $A_1 A_2 A_3$ (see Figure 2.1). We express a_4 as a function of a_1, a_2 and α by using the following equations:

$$h_{0,124} = h_{0,12} \sin(\alpha_g - \alpha), \tag{15}$$

$$x_2'^2 = a_2^2 - h_{0,124}^2, \tag{16}$$

$$\sin(\alpha_2'') = \frac{h_{0,12} \cos(\alpha_g - \alpha)}{x_2'}, \tag{17}$$

$$x_4^2 = x_2'^2 + a_{24}^2 - 2x_2' a_{24} \cos(\alpha_{124} - \alpha_2''), \tag{18}$$

$$a_4^2 = x_4^2 + h_{0,124}^2, \tag{19}$$

or

$$a_4^2 = a_2^2 + a_{24}^2 - 2a_{24} \left[\sqrt{a_2^2 - h_{0,12}^2} \cos(\alpha_{124}) + h_{0,12} \sin(\alpha_{124}) \cos(\alpha_g - \alpha) \right], \tag{20}$$

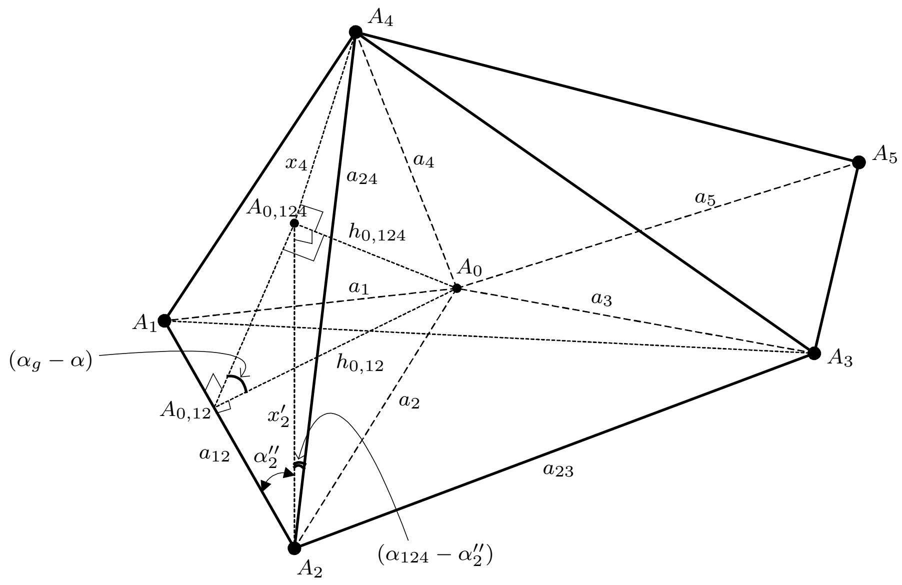

Figure 2.2.

where α_g is the given dihedral angle between the planes $A_3A_1A_2A_5$ and $A_4A_1A_2$ and $h_{0,124}$ is the distance from A_0 to the plane $A_1A_2A_4$ (see Figure 2.2).
We express a_5 as a function of a_1, a_2 and α by using the following equations:

$$a_5^2 = x_5^2 + h_{0,123}^2 \tag{21}$$

$$x_5^2 = x_2^2 + a_{25}^2 - 2x_2a_{25}\cos(\alpha_{125} - \alpha_2'), \tag{22}$$

$$a_5^2 = a_2^2 + a_{25}^2 - 2x_2a_{25}\cos(\alpha_{125} - \alpha_2') \tag{23}$$

or

$$a_5^2 = a_2^2 + a_{25}^2 - 2a_{25}\left[\sqrt{a_2^2 - h_{0,12}^2}\cos(\alpha_{125}) + h_{0,12}\sin(\alpha_{125})\cos(\alpha)\right], \tag{24}$$

We differentiate (14), (20), (24) with respect to α and we obtain (25), (26) and (27), respectively.

$$a_3\frac{\partial a_3}{\partial \alpha} = +a_{23}h_{0,12}\sin(\alpha_{123})\sin(\alpha), \tag{25}$$

$$a_4\frac{\partial a_4}{\partial \alpha} = -a_{24}h_{0,12}\sin(\alpha_{124})\sin(\alpha_g - \alpha), \tag{26}$$

$$a_5\frac{\partial a_5}{\partial \alpha} = a_{25}h_{0,12}\sin(\alpha_{125})\sin(\alpha). \tag{27}$$

By replacing (25), (26), (27) in (6) and by multiplying both members of (6) by $\frac{1}{3}a_{12}$, we get:

$$\frac{B_3}{a_3}VOL(A_0A_1A_2A_3) - \frac{B_4}{a_4}VOL(A_0A_1A_2A_4) + \frac{B_5}{a_5}VOL(A_0A_1A_2A_5) = 0 \tag{28}$$

or

$$\frac{B_3}{B_4}\frac{a_4VOL(A_0A_1A_2A_3)}{a_3VOL(A_0A_1A_2A_4)} + \frac{B_5}{B_4}\frac{a_4VOL(A_0A_1A_2A_5)}{a_5VOL(A_0A_1A_2A_4)} = 1. \tag{29}$$

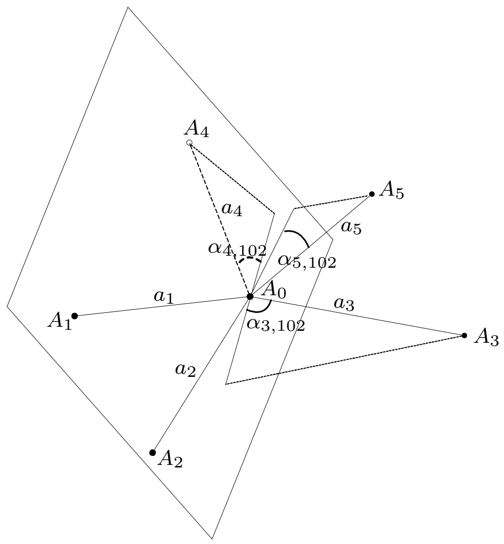

Figure 2.3.

The volume formula for $A_0A_1A_2A_4$, $A_0A_1A_2A_3$, $A_0A_1A_2A_5$ are used by applying the orthogonal projection of a_3, a_4 and a_5 on the plane $A_0A_1A_2$ (see Figure 2.3):

$$VOL(A_0A_1A_2A_4) = \frac{1}{6}a_1a_2a_4\sin(\alpha_{102})\sin(\alpha_{4,102}),$$

$$VOL(A_0A_1A_2A_3) = \frac{1}{6}a_1a_2a_3\sin(\alpha_{102})\sin(\alpha_{3,102}),$$

$$VOL(A_0A_1A_2A_5) = \frac{1}{6}a_1a_2a_5\sin(\alpha_{102})\sin(\alpha_{5,102}).$$

and we place them into (29), which gives

$$\frac{B_3}{B_4}\frac{\sin(\alpha_{3,102})}{\sin(\alpha_{4,102})} + \frac{B_5}{B_4}\frac{\sin(\alpha_{5,102})}{\sin(\alpha_{4,102})} = 1. \tag{30}$$

We denote by $\alpha_{i,j0k}$ the angle that is formulated by the line segment that connects A_0 with the trace of the orthogonal projection of A_i to the plane $A_jA_0A_k$ with a_i, for $i, j, k, l = 1, 2, 3, 4, 5$, $i \neq j \neq k \neq i$.

Similarly, by differentiating cyclically with respect to a_2, a_3, α', where α' is the dihedral angle between the base of the pyramid which is $A_1A_2A_3A_5$ and the plane $A_0A_2A_3$, we obtain the following equation:

$$\frac{B_1}{a_1}VOL(A_0A_1A_2A_3) - \frac{B_4}{a_4}VOL(A_0A_2A_3A_4) + \frac{B_5}{a_5}VOL(A_0A_2A_3A_5) = 0 \tag{31}$$

or

$$\frac{B_1}{B_4}\frac{a_4VOL(A_0A_1A_2A_3)}{a_1VOL(A_0A_2A_3A_4)} + \frac{B_5}{B_4}\frac{a_4VOL(A_0A_2A_3A_5)}{a_5VOL(A_0A_2A_3A_4)} = 1$$

or

$$\frac{B_1}{B_4}\frac{\sin(\alpha_{1,203})}{\sin(\alpha_{4,203})}+\frac{B_5}{B_4}\frac{\sin(\alpha_{5,203})}{\sin(\alpha_{4,203})}=1. \tag{32}$$

Similarly, by differentiating cyclically with respect to a_3, a_5, α'', where α'' is the dihedral angle between the base of the pyramid which is $A_1A_2A_3A_5$ and the plane $A_0A_3A_5$, we derive the following equation:

$$\frac{B_1}{a_1}VOL(A_0A_1A_3A_5)+\frac{B_2}{a_2}VOL(A_0A_2A_3A_5)-\frac{B_4}{a_4}VOL(A_0A_3A_4A_5)=0 \tag{33}$$

or

$$\frac{B_1}{B_4}\frac{a_4VOL(A_0A_1A_3A_5)}{a_1VOL(A_0A_3A_4A_5)}+\frac{B_2}{B_4}\frac{a_4VOL(A_0A_2A_3A_5)}{a_2VOL(A_0A_3A_4A_5)}=1$$

or

$$\frac{B_1}{B_4}\frac{\sin(\alpha_{1,305})}{\sin(\alpha_{4,305})}+\frac{B_2}{B_4}\frac{\sin(\alpha_{2,305})}{\sin(\alpha_{4,305})}=1. \tag{34}$$

Similarly, by differentiating cyclically with respect to a_1, a_5, α''', where α''' is the dihedral angle between the base of the pyramid which is $A_1A_2A_3A_5$ and the plane $A_0A_1A_5$, we get the following equation:

$$\frac{B_2}{a_2}VOL(A_0A_1A_2A_5)+\frac{B_3}{a_3}VOL(A_0A_1A_3A_5)-\frac{B_4}{a_4}VOL(A_0A_1A_4A_5)=0 \tag{35}$$

or

$$\frac{B_2}{B_4}\frac{a_4VOL(A_0A_1A_2A_5)}{a_2VOL(A_0A_1A_4A_5)}+\frac{B_3}{B_4}\frac{a_4VOL(A_0A_1A_3A_5)}{a_3VOL(A_0A_1A_4A_5)}=1$$

or

$$\frac{B_2}{B_4}\frac{\sin(\alpha_{2,105})}{\sin(\alpha_{4,105})}+\frac{B_3}{B_4}\frac{\sin(\alpha_{3,105})}{\sin(\alpha_{4,105})}=1. \tag{36}$$

By adding (28), (31), (33), (35), we obtain:

$$\frac{B_4}{a_4}VOL(A_1A_2A_3A_4A_5)=\left(\frac{B_1}{a_1}+\frac{B_2}{a_2}+\frac{B_3}{a_3}+\frac{B_4}{a_4}+\frac{B_5}{a_5})VOL(A_0A_1A_2A_3A_5\right) \tag{37}$$

or

$$\frac{B_4}{a_4}\left(\frac{H_{4,1235}}{h_{0,1235}}-1\right)=\frac{B_1}{a_1}+\frac{B_2}{a_2}+\frac{B_3}{a_3}+\frac{B_5}{a_5}, \tag{38}$$

where $H_{4,1235}$ is the distance from the vertex A_4 to the plane $A_1A_2A_3A_5$ and $h_{0,1235}=h_{0,123}$. We continue with the calculation of A_0 which is clarified by a_1, a_2, α. We replace $a_3=a_3(a_1,a_2,\alpha)$, $a_4=a_4(a_1,a_2,\alpha)$, $a_5=a_5(a_1,a_2,\alpha)$ by (14), (20), (24) in (38) and square both parts, in order to obtain:

$$\left(\frac{B_4}{a_4(a_1,a_2,\alpha)}\left(\frac{H_{4,1235}}{h_{0,12}\sin(\alpha)}-1\right)\right)^2=\left(\frac{B_1}{a_1}+\frac{B_2}{a_2}+\frac{B_3}{a_3(a_1,a_2,\alpha)}+\frac{B_5}{a_5(a_1,a_2,\alpha)}\right)^2. \tag{39}$$

By replacing (14), (20), (24) in (28), we get:

$$\frac{B_3}{a_3(a_1,a_2,\alpha)}h_{3,12}+\frac{B_5}{a_5(a_1,a_2,\alpha)}h_{5,12}=\frac{B_4}{a_4(a_1,a_2,\alpha)}h_{4,12}\frac{\sin(a_g-\alpha)}{\sin(\alpha)}, \tag{40}$$

or

$$\left(\frac{B_3}{a_3(a_1,a_2,\alpha)}h_{3,12}+\frac{B_5}{a_5(a_1,a_2,\alpha)}h_{5,12}\right)^2=\left(\frac{B_4}{a_4(a_1,a_2,\alpha)}h_{4,12}\right)^2\left(\frac{\sin(a_g-\alpha)}{\sin(\alpha)}\right)^2, \tag{41}$$

where $h_{i,12}$ is the distance from the vertex A_i to the line defined by the vertices A_1, A_2, for $i=3,4,5$.

Two equations (39), (41) are derived with the independent variables a_1, a_2, α and the third equation will be found from (4) . We continue by replacing $\frac{\partial a_3}{\partial a_1}$, $\frac{\partial a_4}{\partial a_1}$, $\frac{\partial a_5}{\partial a_1}$ in (4):

$$\begin{aligned}\left(\frac{B_1}{a_1}\right)+\left(\frac{B_3}{a_3}\right)&\left[\frac{a_{23}}{a_{12}}\cos(\alpha_{123})-\sin(\alpha_{123})\cos(\alpha)\frac{a_{23}}{2a_{12}^2h_{0,12}}(-a_1^2+a_2^2+a_{12}^2)\right]\\+\frac{B_4}{a_4}&\left[\frac{a_{24}}{a_{12}}\cos(\alpha_{124})-\sin(\alpha_{124})\cos(\alpha_g-\alpha)\frac{a_{24}}{2a_{12}^2h_{0,12}}(-a_1^2+a_2^2+a_{12}^2)\right]\\+\left(\frac{B_5}{a_5}\right)&\left[\frac{a_{25}}{a_{12}}\cos(\alpha_{125})-\sin(\alpha_{125})\cos(\alpha)\frac{a_{25}}{2a_{12}^2h_{0,12}}(-a_1^2+a_2^2+a_{12}^2)\right]=0.\end{aligned} \tag{42}$$

By replacing $a_3=a_3(a_1,a_2,\alpha)$, $a_4=a_4(a_1,a_2,\alpha)$, $a_5=a_5(a_1,a_2,\alpha)$ by (14), (20), (24) in (42) and by taking into account that $h_{0,12}$ is a function with respect to a_1, a_2 (replace (7) in (8)), we derive the third equation that depends on a_1, a_2, α.

The three equations (39), (41), (42) depend on a_1, a_2 and α and can be solved numerically. □

Remark 2.2. We assumed that the weighted Fermat-Torricelli point A_0 is an interior point of $A_1A_2A_3A_4A_5$ (Floating Case of (I) at Appendix A) and the corresponding weights satisfy some weighted inequalities. For extreme cases, we refer at Appendix A (Absorbed Case of (I) of Appendix A).

Example 2.3. Given a pyramid $A_1A_2A_3A_4A_5$ with vertices $A_1=(-2,0,0)$, $A_2=(0,-2,0)$, $A_3=(3,0,0)$, $A_4=(0,0,5)$, $A_5=(0,3,0)$ which gives $a_{12}=2\sqrt{2}$, $a_{23}=\sqrt{13}$, $a_{24}=\sqrt{29}$, $a_{25}=5$, $\cos(\alpha_{123})=-\frac{1}{\sqrt{26}}$, $\cos(\alpha_{124})=\sqrt{\frac{2}{29}}$, $\cos(\alpha_{125})=\frac{1}{\sqrt{2}}$, $a_g=1.29515$ rad, $h_{3,12}=3.53553$, $h_{5,12}=3.53553$, $h_{4,12}=5.19653$, $H_{4,1235}=5$ and weights that correspond to the vertices $B_1=1$, $B_2=0.7$, $B_3=0.5$, $B_4=1.5$, $B_5=0.3$, respectively, which gives $\sum_{i=1}^{5}B_i=4$. By taking into consideration the three equations (39), (41), (42) which depend on a_1, a_2, α and by choosing three starting values for instance $a_1^{\circ}=0.9$, $a_2^{\circ}=2.9$ and $\alpha=0.6$ rad, the Newton method gives $a_1=2.30471$, $a_2=2.38021$ and $\alpha=1.00316$ rad which gives the coordinates of $A_0(-0.33452,-0.24607,1.57396)$. This result coincides with the result derived by the Weiszfeld algorithm ([3]) that approximates $A_0=(-0.33452,-0.24607,1.57396)$ with 5-digit precision.

3. The plasticity property for pyramids in the three-dimensional Euclidean Space.

Problem 3.1. *Given the Fermat-Torricelli point A_0 for some classes of hexahedra with their vertices lie on five prescribed rays that meet at A_0, find the ratios between the non negative weights $\frac{B_i}{B_j}$, $i, j = 1, 2, 3, 4, 5$, such that:*

$$\sum_{i=1}^{5} B_i = constant. \tag{43}$$

Solution of the inverse weighted Fermat-Torricelli problem: The invariance property of the weighted Fermat-Torricelli point A_0 (see Appendix A) gives us the possibility to consider a pyramid $A_1A_2A_3A_4A_5$ as a subset of a closed hexahedron with the vertices lying on five prescribed rays that meet at A_0. From the calculation of A_0 (see solution of Problem 2.1) the equations (30), (32), (34), (36) and (43) give a solution to the inverse weighted Fermat-Torricelli problem. □

We show that the solution of the inverse weighted Fermat-Torricelli problem for hexahedra is obtained by (30), (32), (34), (36) which have been derived from the calculation of the weighted Fermat-Torricelli point of pyramids because these equations depend only on the angles α_{i0j} (see Figure 3.1), for $i, j = 1, 2, 3, 4, 5$, $i \neq j$ and not on the shape of the pyramids and by considering the invariance property of the weighted Fermat-Torricelli point (see Appendix A) we derive an evolutionary structure for some classes of closed hexahedra in $\mathbb{R}^3$.

Proposition 3.2. *The following equations point out the plasticity of weighted pyramids with respect to the non-negative variable weights $(B_i)_{12345}$ in $\mathbb{R}^3$:*

$$\left(\frac{B_3}{B_4}\right)_{12345} = \left(\frac{B_3}{B_4}\right)_{1234} \left(1 - \left(\frac{B_5}{B_4}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{1245}\right), \tag{44}$$

$$\left(\frac{B_1}{B_4}\right)_{12345} = \left(\frac{B_1}{B_4}\right)_{1234} \left(1 - \left(\frac{B_5}{B_4}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{2345}\right), \tag{45}$$

$$\left(\frac{B_2}{B_4}\right)_{12345} = \left(\frac{B_2}{B_4}\right)_{1234} \left(1 + \left(\frac{B_5}{B_4}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{1345}\right), \tag{46}$$

where the weight $(B_i)_{12345}$ corresponds to the vertex that lies in the ray A_0A_i, for $i = 1, 2, 3, 4, 5$, and the weight $(B_j)_{jklm}$ corresponds to the vertex A_j that lies in the ray A_0A_j regarding the tetrahedron $A_jA_kA_lA_m$, for $j, k, l, m = 1, 2, 3, 4, 5$ and $j \neq k \neq l \neq m$.

Proof of Proposition 3.2. By taking into consideration (30), we obtain:

$$\left(\frac{B_3}{B_4}\right)_{12345} = \frac{\sin(\alpha_{4,102})}{\sin(\alpha_{3,102})} \left(1 - \left(\frac{B_5}{B_4}\right)_{12345} \frac{\sin(\alpha_{5,102})}{\sin(\alpha_{4,102})}\right). \tag{47}$$

We make use of the solution of the inverse weighted Fermat-Torricelli problem for the tetrahedra $A_1A_2A_3A_4$ and $A_1A_2A_4A_5$ (see [5]):

$$\left(\frac{B_3}{B_4}\right)_{1234} = \frac{\sin(\alpha_{4,102})}{\sin(\alpha_{3,102})} \tag{48}$$

and

$$\left(\frac{B_4}{B_5}\right)_{1245} = \frac{\sin(\alpha_{5,102})}{\sin(\alpha_{4,102})}. \tag{49}$$

By replacing (48), (49) in (47), we obtain (44).
Similarly, by taking into consideration (32), we obtain:

$$\left(\frac{B_1}{B_4}\right)_{12345} = \frac{\sin(\alpha_{4,203})}{\sin(\alpha_{1,203})}\left(1 - \left(\frac{B_5}{B_4}\right)_{12345}\frac{\sin(\alpha_{5,203})}{\sin(\alpha_{4,203})}\right). \tag{50}$$

We make use of the solution of the inverse weighted Fermat-Torricelli problem for the tetrahedra $A_1A_2A_3A_4$ and $A_2A_3A_4A_5$ (see [5]):

$$\left(\frac{B_1}{B_4}\right)_{1234} = \frac{\sin(\alpha_{4,203})}{\sin(\alpha_{1,203})} \tag{51}$$

and

$$\left(\frac{B_4}{B_5}\right)_{2345} = \frac{\sin(\alpha_{5,203})}{\sin(\alpha_{4,203})}. \tag{52}$$

By replacing (51), (52) in (50), we obtain (45).
We proceed by differentiating (1) with respect to a_1, a_3, α'''' where α'''' is the dihedral angle between the base of the pyramid which is $A_1A_2A_3A_5$ and the plane $A_0A_1A_3$ and one of the derived equations we obtain with respect to α'''' is:

$$\frac{B_2}{a_2}VOL(A_0A_1A_2A_3) - \frac{B_4}{a_4}VOL(A_0A_1A_3A_4) - \frac{B_5}{a_5}VOL(A_0A_1A_3A_5) = 0 \tag{53}$$

or

$$\left(\frac{B_2}{B_4}\right)_{12345}\frac{a_4VOL(A_0A_1A_3A_2)}{a_2VOL(A_0A_1A_3A_4)} - \left(\frac{B_5}{B_4}\right)_{12345}\frac{a_4VOL(A_0A_1A_3A_5)}{a_5VOL(A_0A_1A_3A_4)} = 1$$

or

$$\left(\frac{B_2}{B_4}\right)_{12345}\frac{\sin(\alpha_{2,103})}{\sin(\alpha_{4,103})} - \left(\frac{B_5}{B_4}\right)_{12345}\frac{\sin(\alpha_{5,103})}{\sin(\alpha_{4,103})} = 1$$

or

$$\left(\frac{B_2}{B_4}\right)_{12345} = \frac{\sin(\alpha_{4,103})}{\sin(\alpha_{2,103})}\left(1 + \left(\frac{B_5}{B_4}\right)_{12345}\frac{\sin(\alpha_{5,103})}{\sin(\alpha_{4,103})}\right). \tag{54}$$

We make use of the solution of the inverse weighted Fermat-Torricelli problem for the tetrahedra $A_1A_2A_3A_4$, $A_1A_3A_4A_5$, we get, respectively (see [5]):

$$\left(\frac{B_2}{B_4}\right)_{1234} = \frac{\sin(\alpha_{4,103})}{\sin(\alpha_{2,103})}, \tag{55}$$

$$\left(\frac{B_4}{B_5}\right)_{1345} = \frac{\sin(\alpha_{5,103})}{\sin(\alpha_{4,103})}. \tag{56}$$

By replacing (55) and (56) in (54), we derive (46).

Given the angles α_{i0j} that are formulated between the rays A_0A_i and A_0A_j (see Figure 3.1) of the weighted Fermat-Torricelli point A_0, we will calculate the angles $a_{i,j0k}$, for $i,j,k = 1,2,3,4,5$, $i \neq j \neq k \neq i$. We express the unit vectors $\vec{a_i}$ for $i = 1,2,3,4,5$ in a parametric form:

$$\vec{a_1} = (1,0,0), \tag{57}$$

$$\vec{a_2} = (\cos(\alpha_{102}), \sin(\alpha_{102}), 0), \tag{58}$$

$$\vec{a_3} = (\cos(\alpha_{3,102})\cos(\omega_{3,102}), \cos(\alpha_{3,102})\sin(\omega_{3,102}), \sin(\alpha_{3,102})), \tag{59}$$

$$\vec{a_4} = (\cos(\alpha_{4,102})\cos(\omega_{4,102}), \cos(\alpha_{4,102})\sin(\omega_{4,102}), \sin(\alpha_{4,102})), \tag{60}$$

$$\vec{a_5} = (\cos(\alpha_{5,102})\cos(\omega_{5,102}), \cos(\alpha_{5,102})\sin(\omega_{5,102}), \sin(\alpha_{5,102})), \tag{61}$$

such that: $|\vec{a_i}| = 1$. The inner product of $\vec{a_i}$, $\vec{a_j}$ is:

$$\vec{a_i} \cdot \vec{a_j} = \cos(\alpha_{i0j}). \tag{62}$$

We take into consideration (62), for $i,j = 1,2,3,4,5$, in order to find the angles $\alpha_{3,102}$, $\alpha_{4,102}$, $\alpha_{5,102}$. The angles $\alpha_{i,j0k}$ can be derived by working cyclically with $\vec{a_i}$ and choosing similar parametrization with respect to (57)–(61), regarding the plane $A_jA_0A_k$ for $i,j,k = 1,2,3,4,5$, $i \neq j \neq k \neq i$.

We consider the following inner products:

$$\vec{a_1} \cdot \vec{a_3} = \cos(\alpha_{103}) = \cos(\alpha_{3,102})\cos(\omega_{3,102}), \tag{63}$$

$$\begin{aligned} \vec{a_2} \cdot \vec{a_3} &= \cos(\alpha_{203}) \\ &= \cos(\alpha_{102})\cos(\alpha_{3,102})\cos(\omega_{3,102}) + \sin(\alpha_{102})\cos(\alpha_{3,102})\sin(\omega_{3,102}). \end{aligned} \tag{64}$$

From (63), we replace $\cos(\omega_{3,102})$ and $\sin(\omega_{3,102})$ in (64) and obtain the equation:

$$\cos^2(\alpha_{3,102}) = \frac{\cos^2(\alpha_{203}) + \cos^2(\alpha_{103}) - 2\cos(\alpha_{203})\cos(\alpha_{103})\cos(\alpha_{102})}{\sin^2(\alpha_{102})}. \tag{65}$$

Similarly, we obtain the equation for $\alpha_{4,102}$:

$$\cos^2(\alpha_{4,102}) = \frac{\cos^2(\alpha_{204}) + \cos^2(\alpha_{104}) - 2\cos(\alpha_{204})\cos(\alpha_{104})\cos(\alpha_{102})}{\sin^2(\alpha_{102})}. \tag{66}$$

Similarly, we obtain the equation for $\alpha_{5,102}$:

$$\cos^2(\alpha_{5,102}) = \frac{\cos^2(\alpha_{205}) + \cos^2(\alpha_{105}) - 2\cos(\alpha_{205})\cos(\alpha_{105})\cos(\alpha_{102})}{\sin^2(\alpha_{102})}. \tag{67}$$

The ratio $\left(\frac{B_j}{B_i}\right)_{ijkm}$, referring to the tetrahedron $A_iA_jA_kA_m$ is given by the relation (see also in [5]):

$$\left(\frac{B_j}{B_i}\right)_{ijkm}^2 = \frac{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0i}) - \cos^2(\alpha_{k0i}) + 2\cos(\alpha_{m0i})\cos(\alpha_{k0i})\cos(\alpha_{k0m})}{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0j}) - \cos^2(\alpha_{k0j}) + 2\cos(\alpha_{m0j})\cos(\alpha_{k0j})\cos(\alpha_{k0m})} \tag{68}$$

The ratio $\left(\frac{B_j}{B_i}\right)_{ijkm}$ depends on five given angles a_{l0n} for $l, n \in \{i, j, k, m\}$, $l \neq n, \{i, j, k, m\} \in \{1, 2, 3, 4, 5\}$ and from this result it can be derived that the ratios $\left(\frac{B_j}{B_4}\right)_{12345}$ of the pyramid $A_1A_2A_3A_4A_5$ depend on seven given angles a_{n0p} for $n, p \in \{1, 2, 3, 4, 5\}$ and $n \neq p$.

By replacing (44), (45), (46) in (43) we obtain a linear dynamical system with respect to $(B_i)_{12345}$ that depends on $(B_5)_{12345}$ for $i = 1, 2, 3, 4$.

The equation (43) is used to decrease the independent variables of the initial dynamical system with respect to $(B_i)_{12345}$ for $i = 1, 2, 3, 4, 5$ from two independent variables to one independent variable, for instance $(B_5)_{12345}$. □

The following corollary shows the decomposition of the weights $(B_i)_{12345}$ to the weights B_{ijk4} that correspond to evolutionary tetrahedra $A_iA_jA_kA_4$, for $i, j, k = 1, 2, 3, 4, 5$, $i \neq j \neq k$ and deduces the qualitative behavior of the dynamical system in $\mathbb{R}^3$ with respect to the variable weights $(B_i)_{12345}$.

Corollary 3.3. *Set* $\sum_{12345} B := (B_4)_{12345}\left(\frac{B_1}{B_4} + \frac{B_2}{B_4} + \frac{B_3}{B_4} + 1 + \frac{B_5}{B_4}\right)_{12345}$. *If* $\sum_{12345} B = \sum_{1234} B = \sum_{1245} B = \sum_{2345} B$, *then*

$$(B_i)_{12345} = x_i(B_5)_{12345} + (B_i)_{1234}, \quad i = 1, 2, 3, 4 :$$

$$x_1 = x_4 \left(\frac{B_1}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{2345} \left(\frac{B_1}{B_4}\right)_{1234},$$

$$x_2 = x_4 \left(\frac{B_2}{B_4}\right)_{1234} + \left(\frac{B_4}{B_5}\right)_{1345} \left(\frac{B_2}{B_4}\right)_{1234},$$

$$x_3 = x_4 \left(\frac{B_3}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{1245} \left(\frac{B_3}{B_4}\right)_{1234},$$

$$x_4 = \frac{\left(\frac{B_4}{B_5}\right)_{2345}\left(\frac{B_1}{B_4}\right)_{1234} + \left(\frac{B_4}{B_5}\right)_{1245}\left(\frac{B_3}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{1345}\left(\frac{B_2}{B_4}\right)_{1234} - 1}{1 + \left(\frac{B_1}{B_4}\right)_{1234} + \left(\frac{B_2}{B_4}\right)_{1234} + \left(\frac{B_3}{B_4}\right)_{1234}}.$$

Proof of Corollary 3.3. From the assumption of the corollary we get:

$$\sum_{12345} B := (B_4)_{12345}\left(\frac{B_1}{B_4} + \frac{B_2}{B_4} + \frac{B_3}{B_4} + 1 + \frac{B_5}{B_4}\right)_{12345}$$
$$= (B_4)_{1234}\left(\frac{B_1}{B_4} + \frac{B_2}{B_4} + \frac{B_3}{B_4} + 1\right)_{1235}$$

By replacing to the above relation (44), (45) and (46), we derive:

$$(B_4)_{12345} \tag{69}$$
$$= \frac{\left(\frac{B_4}{B_5}\right)_{2345}\left(\frac{B_1}{B_4}\right)_{1234} + \left(\frac{B_4}{B_5}\right)_{1245}\left(\frac{B_3}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{1345}\left(\frac{B_2}{B_4}\right)_{1234} - 1}{1 + \left(\frac{B_1}{B_4}\right)_{1234} + \left(\frac{B_2}{B_4}\right)_{1234} + \left(\frac{B_3}{B_4}\right)_{1234}}(B_5)_{12345} + (B_4)_{1234}$$

or

$$(B_4)_{12345} = x_4(B_5)_{12345} + (B_4)_{1234}.$$

By replacing (69) in (44), (45) and (46), respectively, we get three relations:

$$(B_1)_{12345} = \left(x_4\left(\frac{B_1}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{2345}\left(\frac{B_1}{B_4}\right)_{1234}\right)(B_5)_{12345} + (B_1)_{1234}, \tag{70}$$

$$(B_2)_{12345} = \left(x_4\left(\frac{B_2}{B_4}\right)_{1234} + \left(\frac{B_4}{B_5}\right)_{1345}\left(\frac{B_2}{B_4}\right)_{1234}\right)(B_5)_{12345} + (B_2)_{1234}, \tag{71}$$

$$(B_3)_{12345} = \left(x_4\left(\frac{B_3}{B_4}\right)_{1234} - \left(\frac{B_4}{B_5}\right)_{1245}\left(\frac{B_3}{B_4}\right)_{1234}\right)(B_5)_{12345} + (B_3)_{1234}. \tag{72}$$

□

Example 3.4. Given the weighted Fermat-Torricelli point A_0 at time $t = 0$ with the vertices lie on five prescribed rays and suppose that we can select one vertex at each ray such that four vertices form the base of a pyramid with given exactly seven angles, $\alpha_{102} = 74.2549°$, $\alpha_{203} = 68.9375°$, $\alpha_{105} = 70.9964°$, $\alpha_{204} = 134.057°$, $\alpha_{103} = 110.736°$, $\alpha_{104} = 137.766°$, $\alpha_{205} = 111.097°$ (α_{304}, α_{405}, α_{305} are calculated by (62)) of the weighted Fermat-Torricelli problem for a given pyramid and the assumption that $\sum_{12345} B = \sum_{1234} B = \sum_{1345} B = \sum_{2345} B = 4$, we calculate the weights for the tetrahedra $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_1A_3A_4A_5$ and $A_1A_2A_4A_5$ according to (68):

$$\text{Tetrahedron} : (A_1A_2A_3A_4) : \quad (B_1)_{1234} = 1.28735, \quad (B_2)_{1234} = 0.40473,$$
$$(B_3)_{1234} = 0.806684, \quad (B_4)_{1234} = 1.50127,$$
$$\sum_{1234} B = 4,$$
$$\text{Tetrahedron} : (A_2A_3A_4A_5) : \quad (B_2)_{2345} = 1.3459, \quad (B_3)_{2345} = 0.441947,$$
$$(B_4)_{2345} = 1.16511, \quad (B_5)_{2345} = 1.04704,$$
$$\sum_{2345} B = 4,$$
$$\text{Tetrahedron} : (A_1A_3A_4A_5) : \quad (B_1)_{1345} = 1.39673, \quad (B_3)_{1345} = 1.01689,$$
$$(B_4)_{1345} = 1.24564, \quad (B_5)_{1345} = 0.340745,$$
$$\sum_{1345} B = 4,$$

$$\text{Tetrahedron} : (A_1A_2A_4A_5): \quad (B_1)_{1245} = 0.531531, \quad (B_2)_{1245} = 1.18143,$$
$$(B_4)_{1245} = 1.49794, \quad (B_5)_{1245} = 0.7891,$$
$$\sum_{1245} B = 4.$$

The equations of the initial system with respect to the weights $(B_i)_{12345}$ are:

$$(B_1)_{12345} = 0.857506(B_4)_{12345} - 0.954195(B_5)_{12345},$$
$$(B_2)_{12345} = 0.269574(B_4)_{12345} + 0.985461(B_5)_{12345},$$
$$(B_3)_{12345} = 0.537336(B_4)_{12345} - 1.02001(B_5)_{12345}.$$

From Proposition 3.2 and Corollary 3.3 the following results are derived:

$$(B_4)_{12345} - (B_4)_{1234} = -0.00422457(B_5)_{12345}, \tag{73}$$
$$(B_1)_{12345} - (B_1)_{1234} = -0.957818(B_5)_{12345}, \tag{74}$$
$$(B_2)_{12345} - (B_2)_{1234} = 0.984322(B_5)_{12345}, \tag{75}$$
$$(B_3)_{12345} - (B_3)_{1234} = -1.02228(B_5)_{12345}, \tag{76}$$

or

$$(B_1)_{12345} = 1.28735 - 0.957818(B_5)_{12345},$$
$$(B_2)_{12345} = 0.404703 + 0.984322(B_5)_{12345},$$
$$(B_3)_{12345} = 0.806685 - 1.02228(B_5)_{12345},$$
$$(B_4)_{12345} = 1.50127 - 0.00422457(B_5)_{12345},$$

and $\sum_{12345} B = 4$. The range of $(B_5)_{12345}$, $(B_4)_{12345}$, $(B_1)_{12345}$, $(B_2)_{12345}$, $(B_3)_{12345}$ is:

$$0 \leqslant (B_5)_{12345} \leqslant 0.789104,$$
$$1.28735 \geqslant (B_1)_{12345} \geqslant 0.78104,$$
$$0.404703 \leqslant (B_2)_{12345} \leqslant 1.18144,$$
$$0.806685 \geqslant (B_3)_{12345} \geqslant 0,$$
$$1.50127 \geqslant (B_4)_{12345} \geqslant 1.49794.$$

For instance, for $(B_5)_{12345} = 0.3$, we get:

$$(B_1)_{12345} = 1, \quad (B_2)_{12345} = 0.7, \quad (B_3)_{12345} = 0.5, \quad (B_4)_{12345} = 1.5,$$

for $(B_5)_{12345} = 0.4$, we get:

$$(B_1)_{12345} = 0.904218, \quad (B_2)_{12345} = 0.798431,$$
$$(B_3)_{12345} = 0.397773, \quad (B_4)_{12345} = 1.49958,$$

and for $(B_5)_{12345} = 0.7$, we obtain:

$$(B_1)_{12345} = 0.616873, \quad (B_2)_{12345} = 1.09373,$$
$$(B_3)_{12345} = 0.0910888, \quad (B_4)_{12345} = 1.49831.$$

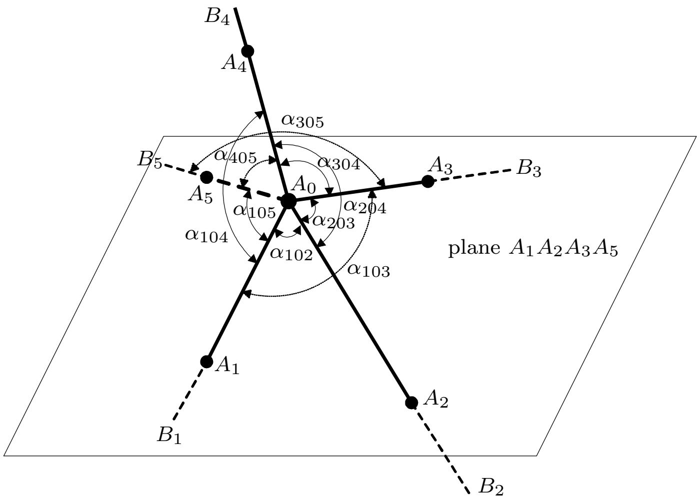

Figure 3.1.

Remark 3.5. By taking into consideration Figure 3.1, the weights B_1, B_3, B_4 decrease and the weights B_2, B_5 increase (see equations (73), (74), (75) and (76)). This result indicates the plasticity property of the evolution of pyramids.

Remark 3.6. For values of B_1, B_2, B_3, B_4, which depend on B_5 according to Corollary 3.3 and for any value of the vertex A_i which lies in the line A_0A_i such that the inequalities of the weighted floating case are satisfied (see Appendix A), the weighted Fermat-Torricelli point A_0 remains invariant.

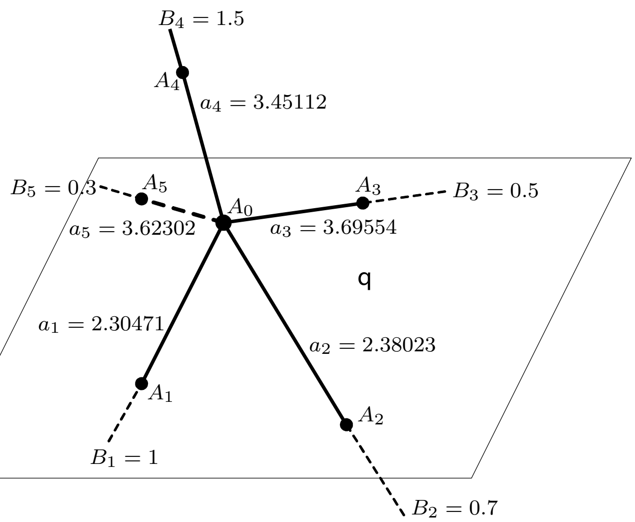

Figure 3.2.

Example 3.7. Let $A_1A_2A_3A_4A_5$ be the given pyramid as Example 3.4 with $a_1 = 2.30471$, $a_2 = 2.38023$, $a_3 = 3.69554$, $a_4 = 3.45112$, $a_5 = 3.62302$, $\alpha_{102} = 74.2549°$, $\alpha_{203} = 68.9375°$, $\alpha_{304} = 109.305°$, $\alpha_{405} = 111.004°$, $\alpha_{105} = 70.9964°$, $\alpha_{204} = 134.057°$, $\alpha_{103} = 110.736°$, $\alpha_{104} = 137.766°$, $\alpha_{205} = 111.097°$ and weights taken from the

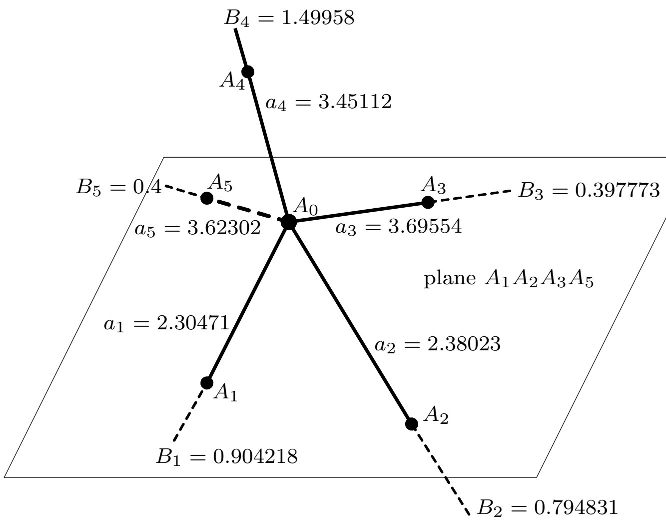

Figure 3.3.

plasticity equations of Example 3.4 for $(B_5)_{12345} = 0.3$:

$$(B_1)_{12345} = 1, \quad (B_2)_{12345} = 0.7,$$

$$(B_3)_{12345} = 0.5, \quad (B_4)_{12345} = 1.5.$$

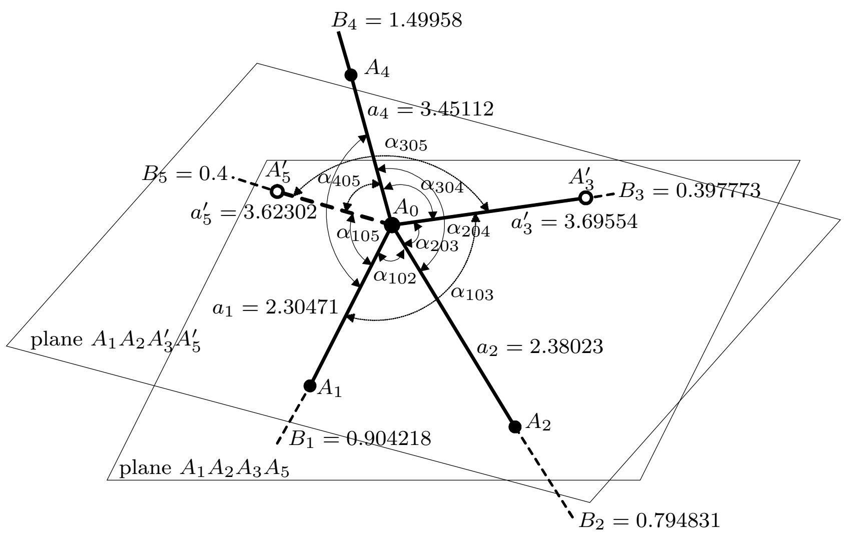

Figure 3.4.

The weighted Fermat-Torricelli point of the pyramid $A_1A_2A_3A_4A_5$ is A_0 (see Figure 3.2). The pyramid $A_1A_2A_3A_4A_5$ of Figure 3.2 has the same angles α_{i0j} and line segments a_i, $i, j = 1, 2, 3, 4, 5, i \neq j$ like in Figure 3.3 with weights

$$(B_1)_{12345} = 0.904218, \quad (B_2)_{12345} = 0.798431,$$

$$(B_3)_{12345} = 0.397773, \quad (B_4)_{12345} = 1.49958, \quad (B_5)_{12345} = 0.4$$

taken from the plasticity equations of Example 3.4. By comparing the Figure 3.2 with Figure 3.3, the weighted Fermat-Torricelli point remains the same. Let $A_1A_2A_3'A_4A_5'$ be a pyramid such that $A_1A_2A_3'A_5'$ be the base of the pyramid and A_i' is the vertex that exist at the line that connects the point A_0 of $A_1A_2A_3'A_4A_5'$ with A_i for $i = 3, 5$, such that $a_3' = 4.43464$, $a_5' = 4.34763$ with the angles α_{i0j} with the other line segments and weights B_i, for $i = 1, 2, 3, 4, 5$, to be the same as in Figure 3.4. The weighted Fermat-Torricelli point A_0 of Figure 3.3 and Figure 3.4 remains also the same. We call the plane $A_1A_2A_3'A_4A_5'$ defined by the base of the pyramid $A_1A_2A_3'A_4A_5'$ evolutionary plane. Let $A_1A_2A_3'A_4A_5'$ be a pyramid with weights taken from the plasticity equations of Example 3.4 (see Figure 3.5)

$$(B_1)_{12345} = 0.616873, \quad (B_2)_{12345} = 1.09373,$$

$$(B_3)_{12345} = 0.0910888, \quad (B_4)_{12345} = 1.49831, \quad (B_5)_{12345} = 0.7.$$

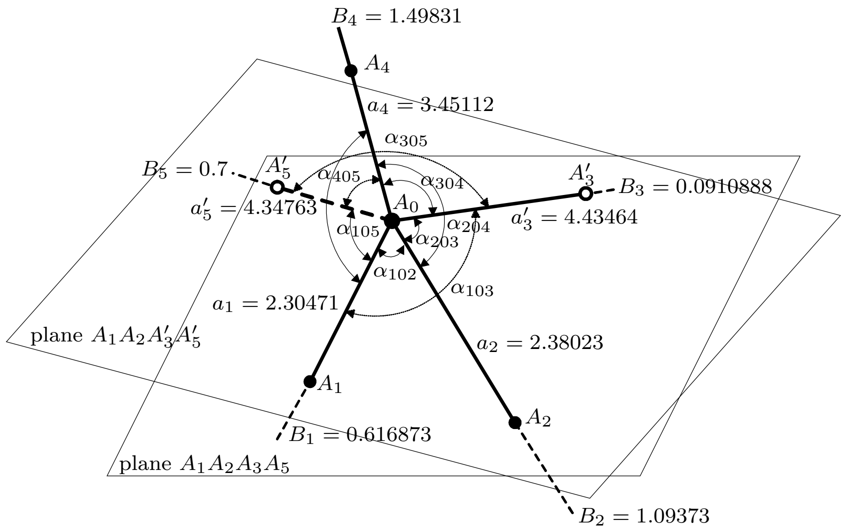

Figure 3.5.

The weighted Fermat-Torricelli point A_0 of Figure 3.4 and Figure 3.5 remains also invariant.

4. The generalized plasticity of pyramids in the three dimensional Euclidean Space

Proposition 4.1. *Let $A_1A_2A_3A_4A_5$ be a pyramid with the base $A_1A_2A_3A_5$ in $\mathbb{R}^3$ and with non-negative weights B_i that correspond to each vertex A_i, respectively, which satisfy the weighted inequalities of the floating case (Appendix A.I) and A_0 is the corresponding generalized Fermat-Torricelli point. Assume that every non-negative weight B_i is split into n_i non-negative weights B_{ik}:*

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

for $i = 1, 2, 3, 4, 5$. *The weight* $B_{i,k}$ *corresponds to every vertex* $A_{i,k}$ *which belongs to the line segment* A_0A_i, *for every* $k \neq n_i$ *and the weight* B_{i,n_i} *corresponds to the vertex* $A_i = A_{i,n_i}$. *Then the generalized Fermat-Torricelli point of* $\{A_{i,k}\}$ *coincides with the generalized Fermat-Torricelli point of* $\{A_1A_2...A_n\}$.

Proof of Proposition 4.1. We will prove that the minimum of $g(X)$ is attained at $X = A_0$.

$$g(X) = \sum_{i=1}^{5}\sum_{k=1}^{n_i} B_{i,k} \left\| A_{i,k} - X \right\|, \tag{77}$$

where $\|\|$ is the Euclidean norm in $\mathbb{R}^3$. The gradient of $g(X)$ gives:

$$\operatorname{grad}(g(X)) = \sum_{i=1}^{5}\sum_{k=1}^{n_i} B_{i,k}\vec{u}(X, A_{i,k}), \tag{78}$$

$X \in \mathbb{R}^3/\{A_{i,k}\}$, for $i = 1, 2, ..., n$, $k = 1, 2, ..., n_i$. We make use of the following result (see [1], page 238):

(1) If $X \in \mathbb{R}^3/\{A_{i,k}\}$, then X is the minimum point of g(X) if and only if the sum of the $\sum_{i=1}^{5} n_i$ from X to $\{A_{i,k}\}$ is zero. By replacing $X = A_0$ in (78) we have:

$$\operatorname{grad}(g(A_0)) = \sum_{i=1}^{5}\sum_{k=1}^{n_i} B_{i,k}\vec{u}(A_0, A_{i,k}) = \sum_{i=1}^{5} B_i\vec{u}(A_0, A_i) = \vec{0}.$$

This result follows from the parallel translation of the unit vectors $\vec{u}(A_0, A_{i,k})$ along the ray A_0A_i to A_i, the uniqueness property of the generalized Fermat-Torricelli point A_0 of $\{A_1A_2A_3A_4A_5\}$.

The uniqueness property of the generalized Fermat-Torricelli point A_0 of $\{A_{i,k}\}$, is deduced by the strict convexity of the Euclidean norm in $\mathbb{R}^3$. □

Example 4.2. Evolution of the weighted Fermat-Torricelli point due to the discretization of the weights along the five prescribed rays in the three-dimensional Euclidean Space.
Let $A_1A_2A_3'A_4A_5'$ be the same pyramid with the base $A_1A_2A_3'A_5'$ and A_0 is the weighted Fermat-Torricelli point with the weights $(B_i)_{12345}$ for $i = 1, 2, 3, 4, 5$ given from the plasticity equations from Example 3.4 for $(B_5)_{12345} = 0.7$ (see Figure 3.5). Let $A_{i,j}$ be points that lie on the prescribed ray A_0A_i for $i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, i \neq j$ and for $i = j$ $A_{i,i} = A_i$, with corresponding weights $B_{i,j}$ (see Figure 4.1):

$$B_{1,1} = 0.1, \quad B_{1,2} = 0.1, \quad B_{1,3} = 0.416873, \quad \sum_{j=1}^{3} B_{1,j} = 0.6168673,$$

$$B_{2,1} = 0.1, \quad B_{2,2} = 0.1, \quad B_{2,3} = 0.89373, \quad \sum_{j=1}^{3} B_{2,j} = 1.09373,$$

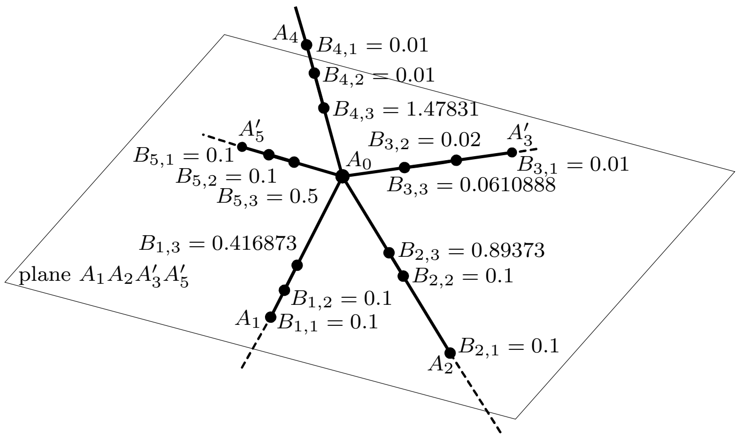

Figure 4.1.

$$B_{3,1} = 0.01, \quad B_{3,2} = 0.02, \quad B_{3,3} = 0.0610888, \quad \sum_{j=1}^{3} B_{3,j} = 0.0910888,$$

$$B_{4,1} = 0.01, \quad B_{4,2} = 0.01, \quad B_{4,3} = 1.47831, \quad \sum_{j=1}^{3} B_{4,j} = 1.49831,$$

$$B_{5,1} = 0.1, \quad B_{5,2} = 0.1, \quad B_{5,3} = 0.5, \quad \sum_{j=1}^{3} B_{5,j} = 0.7,$$

$$\sum_{i=1}^{5}\sum_{j=1}^{3} B_{i,j} = 4,$$

and

$$\sum_{j=1}^{3} B_{i,j} = (B_i)_{12345},$$

for $i = 1, 2, 3, 4, 5$. By using the Weiszfeld algorithm we calculate the weighted Fermat-Torricelli point A_0 of the pyramid $A_1A_2A'_3A_4A'_5$ with corresponding weights taken from Example 3.12 (see Figure 3.5). By using the Weiszfeld algorithm we calculate the weighted Fermat-Torricelli point A'_0 of $A_{i,j}$ with corresponding weights $B_{i,j}$ (see Figure 4.1). We obtain that: $A_0 = A'_0$.

We conclude with the following evolutionary scheme:

(1) Invariance of A_0 with respect to the variable discretization of the "weights" B_{ik} located on the corresponding i-th ray in space (variable lengths A_0A_{ik}) and quantum, under the condition:

$$\sum_{k=1}^{n_i} B_{ik} = B_i.$$

(2) Invariance of A_0 with respect to the variable B_i which fulfill the four equations of Proposition 3.2 (plasticity) for $i = 1, 2, 3, 4, 5$ of 5 prescribed rays that meet at A_0 (at least seven angles must be given):

$$\left(\frac{\sum_{k=1}^{n_3} B_{3k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} = \left(\frac{B_3}{B_4}\right)_{1234} \left(1 - \left(\frac{\sum_{k=1}^{n_5} B_{5k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{1245}\right), \tag{79}$$

$$\left(\frac{\sum_{k=1}^{n_1} B_{1k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} = \left(\frac{B_1}{B_4}\right)_{1234} \left(1 - \left(\frac{\sum_{k=1}^{n_5} B_{5k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{2345}\right), \tag{80}$$

$$\left(\frac{\sum_{k=1}^{n_2} B_{2k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} = \left(\frac{B_2}{B_4}\right)_{1234} \left(1 + \left(\frac{\sum_{k=1}^{n_5} B_{5k}}{\sum_{k=1}^{n_4} B_{4k}}\right)_{12345} \left(\frac{B_4}{B_5}\right)_{1345}\right), \tag{81}$$

and

$$\sum_{i=1}^{5} B_i = B_0.$$

We call this variation of B_{ik} and B_i in space and quantum "generalized plasticity".

By using "Steiner" trees as a consequence of Fermat-Torricelli points this plasticity will be reduced.

A. Appendix

We need the following results given in [1], Theorem 18.37, page 250, (see also [2]):

(I) The weighted Fermat-Torricelli point A_0 of the pyramid $A_1A_2A_3A_4A_5$ exists and is unique.

(i) If

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| > B_i, \quad i \neq j.$$

for $i, j = 1, 2, 3, 4, 5$, then the weighted Fermat-Torricelli point is an interior point of the pyramid $A_1A_2A_3A_4$ (Floating Case).

(ii) If there is some i with

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| \leq B_i, \quad i \neq j.$$

for $i, j = 1, 2, 3, 4, 5$, then the weighted Fermat-Torricelli point is the vertex A_i (Absorbed Case).

(II) Suppose that there is a closed polyhedron $A_1A_2 \ldots A_n$ in $\mathbb{R}^3$ and each vertex A_i has a non-negative weight B_i for $i = 1, 2, \ldots, n$. Assume that the floating case of the generalized weighted Fermat-Torricelli point A_0 point is valid:
for each $A_i \in \{A_1, \ldots, A_n\}$

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| > B_i, \quad i \neq j.$$

If A_0 is connected with every vertex A_i for $i = 1, 2, \ldots, n$ and a point A'_i is selected with a non-negative weight B_i of the line that is defined by the line segment A_0A_i and the n-convex polyhedron $A'_1A'_2...A'_n$ is constructed such that:

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A'_i, A'_j) \right\| > B_i, \quad i \neq j.$$

Then the generalized weighted Fermat-Torricelli point A'_0 is identical with A_0 (invariance property).

Proof of (II). The existence and uniqueness of the generalized weighted Fermat-Torricelli point given n non-collinear points $A_1, ...A_n \in \mathbb{R}^d$ has been established (see [1], Theorem 18.37, page 250). Furthermore, if for each point $A_i \in \{A_1, \ldots, A_n\}$

$$\left\| \sum_{j=1}^{n} B_j \vec{u}(A_i, A_j) \right\| > B_i, \; i \neq j$$

holds, then

(a) the weighted minimum point A_0 does not belong to $A_i \in \{A_1, \ldots, A_n\}$
(b)

$$\sum_{i=1}^{n} B_i \vec{u}(A_0, A_i) = \vec{0}, \quad i \neq j$$

(weighted floating case).

We consider the particular case for $d = 3$, regarding the n-convex polyhedron $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$. Let $A_0(x_0, y_0, z_0)$ be the coordinates of the weighted Fermat Torricelli point (critical).

The minimum conditions are:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} B_i \frac{(x - x_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} = 0,$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} B_i \frac{(y - y_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} = 0,$$

$$\frac{\partial f}{\partial z} = \sum_{i=1}^{n} B_i \frac{(z - z_i)}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} = 0.$$

We use the following transformation in spherical coordinates:

$$x - x_i = R_i \cos(\theta_i) \cos(\varphi_i),$$

$$y - y_i = R_i \cos(\theta_i) \sin(\varphi_i),$$

$$z - z_i = R_i \sin(\theta_i).$$

The minimum conditions of the objective function f(x,y,z) takes the form:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} B_i \cos(\theta_i)\cos(\varphi_i) = 0,$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} B_i \cos(\theta_i)\sin(\varphi_i) = 0,$$

$$\frac{\partial f}{\partial z} = \sum_{i=1}^{n} B_i \sin(\theta_i) = 0.$$

□

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