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Generalized Monotone Operators, Generalized Convex Functions and Closed Countable Sets

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In this paper we deal with operators which are monotone in several generalized sense. We show that if such property holds locally on the complement of a certain type of closed set, then the same property holds globally on the whole domain under some mild conditions. Our results extend similar statements already established for the classical Minty-Browder monotonicity. As applications we obtain some global generalized convexity results based on local generalized convexity property and some extra analytical requirements.

Keywords: Generalized monotone map, generalized convex function, locally generalized monotone operator, closed countable set

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1. Introduction

In the present paper we are going to analyze some conditions which ensure that a local generalized monotonicity of an operator such as pseudo and quasimonotonicity, provide the global generalized monotonicity for that operator.

Recall that an operator A defined on a subset D of a real Banach space X, taking values in its dual X^*, is called Minty-Browder monotone, if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in D$, (see for instance [3, 4, 5, 25, 26]).

The behavior of a locally Minty-Browder monotone operator on the complement of a finite set C was studied first in [22]. The results obtained there were extended to the case when the set C is closed and countable in [21]. Also in [21], the authors gave an example of a continuous locally Minty-Browder monotone operator, defined on a connected but not convex subset of $\mathbb{R}^2$, which is not even globally quasimonotone. This shows that the convexity of the domain is essential when extending the local monotonicity to the global monotonicity. Extending the results of [21], we will show that if the domain of the operator is open and convex, most of the local generalized monotonicity concepts can be extended to their global counterparts. However there is an exception: we will give an example of a continuous, locally quasimonotone

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real valued function of one real variable, defined on the whole space $\mathbb{R}$, which is not globally quasimonotone on $\mathbb{R}$.

Let X be a real Banach space, X^* its dual, $D \subseteq X$ a subset of X, and $A : D \longrightarrow X^*$ an operator. We denote by $\operatorname{int} Y$ the interior of the set $Y \subseteq X$, and by (x, y) the open line segment in X with the endpoints x and y, i.e. $(x, y) = \{z \in X : z = x + t(y - x),\, t \in (0, 1)\}$. The closed segment $[x, y]$ with the endpoints $x, y \in X$ is defined as usual, i.e. $[x, y] = \{z \in X : z = x + t(y - x),\, t \in [0, 1]\}$.

The paper deals with four types of generalized monotonicity and generalized convexity concepts, namely quasimonotonicity and quasiconvexity, strict quasimonotonicity and strict quasiconvexity, pseudomonotonicity and pseudoconvexity, respectively strict pseudomonotonicity and strict pseudoconvexity.

We recall that the operator A is called pseudomonotone (see [9, 11, 13, 17, 19]), if for all $x, y \in D$, $\langle Ax, y - x\rangle \geq 0$ implies $\langle Ay, y - x\rangle \geq 0$, or equivalently, for all $x, y \in D$, $\langle Ax, y - x\rangle > 0$ implies $\langle Ay, y - x\rangle > 0$.

A is called strictly pseudomonotone (see [13, 18, 19]), if for all $x, y \in D$, $x \neq y$, $\langle Ax, y - x\rangle \geq 0$ implies $\langle Ay, y - x\rangle > 0$.

The operator A is called quasimonotone (see [9, 11, 13, 15, 18, 19]), if for all $x, y \in D$, $\langle Ax, y - x\rangle > 0$ implies $\langle Ay, y - x\rangle \geq 0$.

Let D be convex. A is called strictly quasimonotone (see [9, 13, 14]), if A is quasimonotone, and for all $x, y \in D$, $x \neq y$ there exists $z \in (x, y)$ such that $\langle Az, y - x\rangle \neq 0$.

Remark 1.1. Obviously the definition of quasimonotonicity is equivalent to the condition:

$$\min\{\langle Ax, y - x\rangle, \langle Ay, x - y\rangle\} \leq 0, \quad \text{for all } x, y \in D.$$

It can be easily observed that for a strictly quasimonotone operator A, we have $\operatorname{int}\{t : t \in (0, 1), \langle A(x + t(y - x)), y - x\rangle = 0\} = \emptyset$, for all $x, y \in D$, $x \neq y$, even more, in the one-dimensional case, a quasimonotone function defined on the interval I, is strictly quasimonotone if $\operatorname{int} f^{-1}(0) = \emptyset$.

If the operator A is strictly pseudomonotone then A is pseudomonotone, and the pseudomonotonicity of A implies the quasimonotonicity of A. If D is convex, then the following implication also holds: A is strictly pseudomonotone implies A is strictly quasimonotone. Obviously the strict quasimonotonicity of A implies the quasimonotonicity of A, (see for instance [13, 18, 19]).

The next example provides a quasimonotone operator that is not pseudomonotone.

Example 1.2. Let us consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} -x - 1, & \text{if } x < -1 \\ 0, & \text{if } x \in [-1, 1] \\ x - 1, & \text{if } x > 1. \end{cases}$$

Since f is nonnegative is obvious that it is quasimonotone. But f is not pseudomonotone, since for $x = 0$ and $y = -2$ we have $f(x)(y - x) = 0$ and $f(y)(y - x) = -2 < 0$.

Obviously f is not strictly quasimonotone either, since $\operatorname{int} f^{-1}(0) = (-1, 1)$.

In [19], the authors gave a geometrical characterization to one dimensional quasimonotone, pseudomonotone, and strictly pseudomonotone maps $f : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$.

(i) f is quasimonotone on I if and only if, for any $x \in I$, we have $f(x) > 0$ implies $f(y) \geq 0$, for all $y \in I$, $y > x$.
(ii) f is pseudomonotone on I if and only if , for any $x \in I$, we have $f(x) > 0$ implies $f(y) > 0$, for all $y \in I$, $y > x$, and $f(x) < 0$ implies $f(y) < 0$, for all $y \in I$, $y < x$.
(iii) f is strictly pseudomonotone on I if and only if , for any $x \in I$, we have $f(x) > 0$ implies $f(y) > 0$, for all $y \in I$, $y > x$, and $f(x) < 0$ implies $f(y) < 0$, for all $y \in I$, $y < x$, and $f(x) = 0$ has at most one real root.

We complete this characterization with strict quasimonotonicity, i.e.:

(iv) f is strictly quasimonotone on the real interval I, if and only if, for any $x \in I$, we have $f(x) > 0$ implies $f(y) \geq 0$, for all $y \in I$, $y > x$, and the interior of the set $f^{-1}(0)$ is empty.

The paper is organized as follows. In Section 2 is proved that the local generalized monotonicity of a real valued function of one real variable defined on the open interval $J \subseteq \mathbb{R}$ is equivalent to its global counterpart, excepting the case of local quasimonotonicity. Also here an example of a locally quasimonotone continuous real valued map defined on the whole space $\mathbb{R}$ is given, which is not globally quasimonotone. As a main result of the section we prove, by means of Cantor-Bendixson theory on derived sets and ordinals (see [7] and [24]), that a continuous real-valued function of one real variable which has a local generalized monotonicity property on the complement of a closed countable set, has that property globally. This fact is used in Section 3 to prove similar results for an operator defined on a convex open subset of a Banach space. In Section 4 our results are applied to some theorems involving generalized convex functions. Also here an example of a locally quasiconvex continuously differentiable function, defined on the whole space $\mathbb{R}$ is given, which is not globally quasiconvex.

2. Local generalized monotonicity of the real valued functions of one real variable

In this section we prove that most of the local generalized monotonicity of real-valued functions of one real variable, on the complement of a closed countable set, provide their global counterpart. However the case of quasimonotonicity is an exception, for which a counterexample is given.

Using the definitions from the previous section we are able to define the notion of local generalized monotonicity of one dimensional maps. Let $I \subseteq \mathbb{R}$ be open and let $f : I \longrightarrow \mathbb{R}$ be a function. One says that:

(i) f is locally quasimonotone, if for all $t \in I$ there exists an open interval $J_t \subseteq I$, with $t \in J_t$, such that the restriction $f|_{J_t}$ is quasimonotone.
(ii) f is locally strictly quasimonotone, if for all $t \in I$ there exists an open interval $J_t \subseteq I$, with $t \in J_t$, such that the restriction $f|_{J_t}$ is strictly quasimonotone.

(iii) f is locally pseudmonotone, if for all $t \in I$ there exists an open interval $J_t \subseteq I$, with $t \in J_t$, such that the restriction $f|_{J_t}$ is pseudomonotone.
(iv) f is locally strictly pseudmonotone, if for all $t \in I$ there exists an open interval $J_t \subseteq I$, with $t \in J_t$, such that the restriction $f|_{J_t}$ is strictly pseudomonotone.

Remark 2.1. Obviously, if $f : I \longrightarrow \mathbb{R}$ is a locally strictly quasimonotone function defined on the open interval I, we have $\operatorname{int} f^{-1}(0) = \emptyset$.

The next theorem shows, that for a real valued function of one real variable, defined on an open interval, the local strict quasimonotonicity implies the global strict quasimonotonicity.

Theorem 2.2. *Let $J \subseteq \mathbb{R}$ be an open interval and let $f : J \longrightarrow \mathbb{R}$ be a function. If f is locally strictly quasimonotone on J, then f is globally strictly quasimonotone on J.*

Proof. For $t \in J$ let us denote by $J_t \subseteq J$ an open interval, with $t \in J_t$, such that f is strictly quasimonotone on J_t, that is $\min\{f(x)(y-x), f(y)(x-y)\} \leq 0$, for all $x, y \in J_t$, and $\operatorname{int}\{x \in J_t : f(x) = 0\} = \emptyset$. Without loss of generality one can assume that the interval J_t is centered at 1, that is, $J_t = (t-p, t+p)$ for some $p > 0$. We show that $\min\{f(a)(b-a), f(b)(a-b)\} \leq 0$, for every $a, b \in J$, $a < b$. In this respect we extract from the open covering $\{J_t\}_{t\in[a,b]}$ of the compact interval $[a, b]$ a finite subcover, say $J_{t_1}, \ldots, J_{t_k}$, $t_1, \ldots, t_k \in [a, b]$, minimal in the sense that none of the intervals can be omitted, and assume that $t_1 < \cdots < t_k$ as well as $J_{t_i} \cap J_{t_{i+1}} \neq \emptyset$ for all $i \in \{1, \ldots, k-1\}$. We next consider $a < s_1 < \cdots < s_{k-1} < b$ such that $s_i \in J_{t_i} \cap J_{t_{i+1}}$ and $f(s_i) \neq 0$ for all $i \in \{1, \ldots, k-1\}$. Obviously $\min\{f(a)(s_1-a), f(s_1)(a-s_1)\} \leq 0$, $\min\{f(s_1)(s_2-s_1), f(s_2)(s_1-s_2)\} \leq 0, \ldots$, $\min\{f(s_{k-1})(b-s_{k-1}), f(b)(s_{k-1}-b)\} \leq 0$. If $f(a) \leq 0$, then $f(a)(b-a) \leq 0$, which leads to $\min\{f(a)(b-a), f(b)(a-b)\} \leq 0$. Otherwise, we have $f(a)(s_1-a) > 0$, and combined with $\min\{f(a)(s_1-a), f(s_1)(a-s_1)\} \leq 0$ implies $f(s_1)(a-s_1) \leq 0$. Since $f(s_1) \neq 0$ we obtain $f(s_1) > 0$. Therefore $f(s_1)(s_2-s_1) > 0$ and combined with $\min\{f(s_1)(s_2-s_1), f(s_2)(s_1-s_2)\} \leq 0$ implies $f(s_2)(s_1-s_2) \leq 0$. Since $f(s_2) \neq 0$ we obtain $f(s_2) > 0$. Continuing the procedure, we obtain finally that $f(s_{k-1})(b-s_{k-1}) > 0$ implies $f(b)(s_{k-1}-b) \leq 0$ which leads to $f(b) \geq 0$. Therefore $f(b)(a-b) \leq 0$, consequently $\min\{f(a)(b-a), f(b)(a-b)\} \leq 0$. Combining with Remark 2.1 we obtain the conclusion. □

We conclude similar results for pseudomonotonicity. The next theorem shows, that for a real valued function of one real variable, defined on an open interval, the local pseudomonotonicity implies the global pseudomonotonicity.

Theorem 2.3. *Let $J \subseteq \mathbb{R}$ be an open interval and let $f : J \longrightarrow \mathbb{R}$ be a function. If f is locally pseudomonotone on J, then f is globally pseudomonotone on J.*

Proof. For $t \in J$ let us denote by $J_t \subseteq J$ an open interval, with $t \in J_t$, such that f is pseudomonotone on J_t. Let $a, b \in J$. Without loss of generality one may suppose that $a < b$ and that the interval J is centered at t. We have to show, that $f(a)(b-a) > 0$ implies $f(b)(b-a) > 0$, which is equivalent to the fact that $f(a) > 0$ implies $f(b) > 0$. In this respect we extract from the open covering $\{J_t\}_{t\in[a,b]}$ of the compact interval $[a, b]$ a finite subcover, say $J_{t_1}, \ldots, J_{t_k}$, $t_1, \ldots, t_k \in [a, b]$, minimal in the sense that none of the intervals can be omitted, and assume that $t_1 < \cdots < t_k$ as well as

$J_{t_i} \cap J_{t_{i+1}} \neq \emptyset$ for all $i \in \{1, \ldots, k-1\}$. We next consider $a < s_1 < \cdots < s_{k-1} < b$ such that $s_i \in J_{t_i} \cap J_{t_{i+1}}$, $i \in \{1, \ldots, k-1\}$. Assume that $f(a) > 0$. Obviously $f(a) > 0$ implies $f(s_1) > 0$, since $a, s_1 \in J_{t_1}$, and $a < s_1$, $f(s_1) > 0$ implies $f(s_2) > 0$ since $s_1, s_2 \in J_{t_2}$, and $s_1 < s_2$, and continuing the procedure, we obtain finally that $f(s_{k-1}) > 0$ implies $f(b) > 0$. It follows that $f(a) > 0$ implies $f(b) > 0$ and the proof is completely done. □

One may easily deduce that the same conclusion holds for strict pseudomonotonicity. The next theorem shows, that for a real valued function of one real variable, defined on an open interval, the local strict pseudomonotonicity implies the global strict pseudomonotonicity.

Theorem 2.4. *Let $J \subseteq \mathbb{R}$ be an open interval and let $f : J \longrightarrow \mathbb{R}$ be a function. If f is locally strictly pseudomonotone on J, then f is globally strictly pseudomonotone on J.*

Proof. Since f is locally strictly pseudomonotone, f is locally pseudomonotone and the equation $f(x) = 0$ has at most one root in the open interval $J_t \subseteq J$, where the local strict pseudomonotonicity holds. According to Theorem 2.3 f is globally pseudomonotone. We must prove that the equation $f(x) = 0$ has at most one root in J. Suppose that there are two roots x_1 and x_2, $x_1 < x_2$ and there are no other roots of f in the interval (x_1, x_2). This can be assumed, sice supposing the contrary, i.e. for every $x, y \in J$, $x \neq y$ satisfying $f(x) = f(y) = 0$ there exists $z \in (x, y)$ such that $f(z) = 0$, we obtain that, there exists $z_1 \in (x, z)$ such that $f(z_1) = 0$, there exists $z_2 \in (x, z_1)$ such that $f(z_2) = 0$, and continuing the procedure we obtain a sequence z_n, $n > 1$ converging to x, satisfying $f(z_n) = 0$, $n > 1$. Therefore every neighborhood of x contains an element $x' \neq x$ such that $f(x') = 0$, which leads to contradiction with the local strict pseudomonotonicity of f. For $t \in (x_1, x_2)$ we have $f(t)(x_1 - t) > 0$ or $f(t)(x_2 - t) > 0$, and by the pseudomonotonicity of f we obtain $f(x_1)(x_1 - t) > 0$ or $f(x_2)(x_2 - t) > 0$, which contradicts the fact that $f(x_1) = f(x_2) = 0$. □

Remark 2.5. Obviously if the function $f : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is locally quasimonotone, (respectively locally strictly quasimonotone, locally pseudomonotone, locally strictly pseudomonotone) on every $J_i \subseteq J$, $i \in \mathfrak{I}$, where J_i is open, for all $i \in \mathfrak{I}$, then f is locally quasimonotone, (respectively locally strictly quasimonotone, locally pseudomonotone, locally strictly pseudomonotone) on $\bigcup_{i \in \mathfrak{I}} J_i$.

However local quasimonotonicity does not imply global quasimonotonicity even if the function f is continuous, as the next example shows. Let us mention that the local $h-$monotonicity, defined and studied in the work *On preimages of a class of generalized monotone operators* by Kassay and Pintea (see [20]), does not imply its global counterpart either.

Example 2.6. Let us consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} -x - 1, & \text{if } x < -1 \\ 0, & \text{if } x \in [-1, 1] \\ -x + 1, & \text{if } x > 1. \end{cases}$$

It is easy to check that f is locally quasimonotone on $\mathbb{R}$. On the other hand for $x=-2$ and $y=2$ we have $\min\{f(x)(y-x), f(y)(x-y)\}=4$ which shows that f is not globally quasimonotone, (see Fig. 2.1).

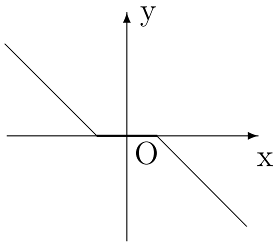

Figure 2.1

In order to continue our analysis we need the following definition of the Cantor-Bendixson derivative: let Y be a subset of a topological space X and denote by Y' the set of accumulation points of Y. For the ordinal number α, the α-th Cantor-Bendixson derivative of Y is defined as follows: $Y^{(0)}=Y$, if α is a successor ordinal then $Y^{(\alpha)}=\left(Y^{(\alpha-1)}\right)'$, if α is a limit ordinal then $Y^{(\alpha)}=\bigcap_{\beta<\alpha} Y^{(\beta)}$.

A separable, completely metrizable topological space X, is said to be a polish space. The Cantor-Bendixson theorem affirms that for a polish space X, there exists a countable ordinal α_0 such that $X^{(\alpha)}=X^{(\alpha_0)}$ for all $\alpha \geq \alpha_0$. ($X^{(\alpha_0)}$ is the perfect kernel).

The smallest ordinal α such that $X^{(\alpha+1)}=X^{(\alpha)}$ is called the Cantor-Bendixson rank of X. Consequently a polish space has countable α_0 Cantor-Bendixson rank. For details see [2], [7] or [23]. A set Y having zero Cantor-Bendixson rank is said to be perfect, that is $Y'=Y$. If Y is a closed countable set and α is its Cantor-Bendixson rank, then obviously $Y^{(\alpha)}=\emptyset$ since perfect sets are uncountable and $Y^{(\alpha)}$ is a subset of Y.

Next we present the principle of transfinite induction. Suppose that $P(\alpha)$ is a property defined for every ordinal α. The principle of transfinite induction states that whenever $P(\beta)$ holds for all $\beta<\alpha$ implies $P(\alpha)$ holds, then $P(\alpha)$ holds for every ordinal α.

The next result shows that the local generalized monotonicity of a function can be extended in some circumstances.

Lemma 2.7. *Let $J \subseteq \mathbb{R}$ be an open interval and $f: J \longrightarrow \mathbb{R}$ be a continuous function. If $Y \subseteq J$ is a closed set such that f is locally quasimonotone, (respectively locally strictly quasimonotone, locally pseudomonotone, locally strictly pseudomonotone) on $J \setminus Y$ and $f(x) \neq 0$ for all $x \in Y$, then f is locally quasimonotone, (respectively locally strictly quasimonotone, locally pseudomonotone, locally strictly pseudomonotone) on $J \setminus Y^{(\alpha)}$ for every ordinal α.*

Proof. We prove the statement for local quasimonotonicity, the other cases can be proved in a similar way. We use the principle of transfinite induction. In this respect we denote by $P(\alpha)$ the statement "f is locally quasimonotone on $J \setminus Y^{(\alpha)}$" and observe that $P(0)$ is precisely the hypothesis of the theorem. We first treat the case of a successor ordinal $\alpha+1$, that is $Y^{(\alpha+1)} = \left(Y^{(\alpha)}\right)'$ and observe that $Y^{(\alpha+1)} = Y^{(\alpha)} \setminus I(Y^{(\alpha)})$, where $I(Y^{(\alpha)})$ is the set of isolated points of $Y^{(\alpha)}$. Consequently $J \setminus Y^{(\alpha+1)} = (J \setminus Y^{(\alpha)}) \cup I(Y^{(\alpha)})$. Since f is locally quasimonotone on $J \setminus Y^{(\alpha)}$, we only need to show that f is locally quasimonotone at the points of $I(Y^{(\alpha)})$. For $t \in I(Y^{(\alpha)})$ there exists an open interval, say $J_t \subseteq J$, such that $J_t \cap Y^{(\alpha)} = \{t\}$. Since Y is closed it is obvious that $t \in Y$, therefore $f(t) \neq 0$, and by the continuity of f we conclude that there exists an open neighborhood U of t, such that $f(x) \neq 0$ for all $x \in U$. Therefore, if $f(x) > 0$ for some $x \in U \cap J_t$, we obtain that $f(y) > 0$ for all $y \in U \cap J_t$, $y > x$, which shows that f is quasimonotone on $U \cap J_t$. Hence f is locally quasimonotone at t.

Assume next that α is a limit ordinal and that $P(\beta)$ holds for all $\beta < \alpha$. Since $Y^{(\alpha)} = \bigcap_{\beta<\alpha} Y^{(\beta)}$, it follows that $J \setminus Y^{(\alpha)} = J \setminus \bigcap_{\beta<\alpha} Y^{(\beta)} = \bigcup_{\beta<\alpha}(J \setminus Y^{(\beta)})$. Since $P(\beta)$ holds for all $\beta < \alpha$, it follows that f is locally quasimonotone on $J \setminus Y^{(\beta)}$ for all $\beta < \alpha$. Consequently, according to Remark 2.5, f is locally quasimonotone on $\bigcup_{\beta<\alpha}(J \setminus Y^{(\beta)}) = J \setminus Y^{(\alpha)}$. □

Remark 2.8. The assumption $f(x) \neq 0$ for all $x \in Y$, within Lemma 2.7 is essential. Indeed, let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = -x + 1$, and let $Y = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$.

It can be easily checked that f is locally strictly pseudomonotone on $\mathbb{R} \setminus Y$, as well that Y is closed countable and $Y' = \{0\}$. But f is not locally quasimonotone on $\mathbb{R} \setminus \{0\}$. Indeed, let J be an open interval containing 1, and $x, y \in J$, such that $x \neq 0$ and $x < 1 < y$. Then $f(x) > 0$ and $f(y) < 0$, therefore $\min\{f(x)(y-x), f(y)(x-y)\} > 0$.

The next result provides a sufficient condition for global strict quasimonotonicity, (respectively global pseudomonotonicity, global strict pseudomonotonicity).

Theorem 2.9. *Let $J \subseteq \mathbb{R}$ be an open interval and $f : J \longrightarrow \mathbb{R}$ be a continuous function. If $Y \subseteq J$ is a closed countable set such that f is locally strictly quasimonotone, (respectively locally pseudomonotone, locally strictly pseudomonotone) on $J \setminus Y$, and $f(x) \neq 0$ for all $x \in Y$, then f is strictly quasimonotone, (respectively pseudomonotone, strictly pseudomonotone) on J.*

Proof. Denote by α the Cantor-Bendixson rank of Y. Since $Y \subseteq J$ is closed and countable, it follows that $Y^{(\alpha)} = \emptyset$. According to Lemma 2.7 the function f is locally strictly quasimonotone, (respectively locally pseudomonotone, locally strictly pseudomonotone) on $J \setminus Y^{(\alpha)} = J$. The statement follows from Theorem 2.2, (respectively from Theorem 2.3 and from Theorem 2.4). □

3. Generalized monotone operators on the complementary of a closed countable set

In this section we extend the results from Section 2 for generalized monotone operators defined on an open and convex subset of a real Banach space. In [21] the authors proved, that for an operator defined on the open and convex subset D of the real Banach space X, the local Minty-Browder monotonicity of the operator on the complement of a closed countable set implies its global Minty-Browder monotonicity. However, for generalized monotone maps this implication is no more true in the absence of further conditions.

Next we give several definitions for local generalized monotonicity of operators on a Banach space.

Definition 3.1. Let X be a real Banach space, X^* its dual, $D \subseteq X$ an open subset of X, and $A : D \longrightarrow X^*$ an operator. One says that:

(i) A is locally quasimonotone, if for all $z \in D$ there exists an open neighborhood $U_z \subseteq D$ of z, such that the restriction $A|_{U_z}$ is quasimonotone, i.e. for all $x, y \in U_z$,
$$\langle Ax, y - x \rangle > 0 \Longrightarrow \langle Ay, y - x \rangle \geq 0,$$

(ii) A is locally strictly quasimonotone, if for all $z \in D$ there exists an open and convex neighborhood $U_z \subseteq D$ of z, such that the restriction $A|_{U_z}$ is strictly quasimonotone, i.e. for all $x, y \in U_z$,
$$\langle Ax, y - x \rangle > 0 \Longrightarrow \langle Ay, y - x \rangle \geq 0,$$
and for all $x, y \in U_z$, $x \neq y$ there exists $z \in (x, y)$, such that $\langle Az, y - x \rangle \neq 0$,

(iii) A is locally pseudomonotone, if for all $z \in D$ there exists an open neighborhood $U_z \subseteq D$ of z, such that the restriction $A|_{U_z}$ is pseudomonotone, i.e. for all $x, y \in U_z$,
$$\langle Ax, y - x \rangle \geq 0 \Longrightarrow \langle Ay, y - x \rangle \geq 0,$$

(iv) A is locally strictly pseudomonotone, if for all $z \in D$ there exists an open neighborhood $U_z \subseteq D$ of z, such that the restriction $A|_{U_z}$ is strictly pseudomonotone, i.e. for all $x, y \in U_z$, $x \neq y$,
$$\langle Ax, y - x \rangle \geq 0 \Longrightarrow \langle Ay, y - x \rangle > 0.$$

In what follows, X denotes a real Banach space, and let $C \subseteq D \subseteq X$ with D open and convex, and C closed relative to D, with empty interior, such that the intersection $[x, y] \cap C$ is countable, possibly empty, for all $x, y \in D \setminus C$.

Remark 3.2. Examples of subsets $C \subset D \subseteq \mathbb{R}^n$ which satisfy the above mentioned requirements, consist in finite families of spheres $S(p, r) := \{x \in \mathbb{R}^n : ||x - p|| = r\}$ in D, since spheres do not contain segments (see [21]). However there are sets C containing segments still satisfying these requirements as the figure below shows.

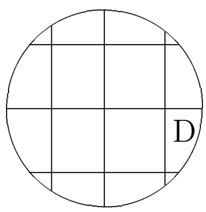

Figure 3.1

Here D is an open disk from $\mathbb{R}^2$ (see Figure 3.1), and C is the union of a finite number of open line segments having their endpoints on the boundary of D.

Definition 3.3. Let $A : X \longrightarrow X^*$ be an operator. We say that A is hemicontinuous at $x \in X$, if for all $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $t_n \longrightarrow 0$, $n \longrightarrow \infty$ and $y \in X$, we have $A(x + t_n y) \rightharpoonup^* Ax$, $n \longrightarrow \infty$, where " $\rightharpoonup^*$ " denotes the convergence with respect to the *weak** topology of X^*.

In order to continue our analysis we need the following lemmas.

Lemma 3.4. *For all $u \in D$ there exist $x, y \in D \setminus C$ such that $u \in (x, y)$.*

Proof. Let $u \in D$. Since $C \subseteq D$ has empty interior, we have $V \cap (D \setminus C) \neq \emptyset$, for every neighborhood V of u, in particular for all $r > 0$, $B(u, r) \cap (D \setminus C) \neq \emptyset$. Let $r > 0$ such that $B(u, r) \subseteq D$, and $x_0 \in B(u, r) \cap (D \setminus C)$. Since $D \setminus C$ is open, we obtain, that there exists $r' > 0$, such that $B(x_0, r') \subseteq B(u, r) \cap (D \setminus C)$. For $y_0 = 2u - x_0$ we have $B(y_0, r') \subseteq B(u, r)$, and let $y \in B(y_0, r') \cap (D \setminus C)$. Then, for $x = 2u - y \in B(x_0, r')$ we have $u \in (x, y)$. $\square$

The next lemma plays an essential role in the proof of Theorem 3.6 bellow.

Lemma 3.5. *For all $u \in D$ there exist $(t_n) \subseteq \mathbb{R}$, $t_n \longrightarrow 0$, $n \longrightarrow \infty$, and $z \in X$, such that $u + t_n z \in D \setminus C$ for all $n \geq 1$.*

Proof. Let $u \in D$. According to Lemma 3.4 there exist $x, y \in D \setminus C$ such that $u \in (x, y)$, that is there exist $t_0 \in (0, 1)$, such that $u = x + t_0(y - x)$. It can be easily verified that all $x' \in (x, y)$ may be written in the form $x' = u + t(x - u)$, where $t \in \left(1 - \frac{1}{t_0}, 1\right)$. Since $[x, y] \cap C$ is countable, possibly empty, is obvious that the set $Y = \left\{t \in \left(1 - \frac{1}{t_0}, 1\right) : u + t(x - u) \in C\right\}$ is countable, possibly empty. Let $t_n \in \left(1 - \frac{1}{t_0}, 1\right) \setminus Y$ for all $n \geq 1$, such that $t_n \longrightarrow 0$, $n \longrightarrow \infty$, and $z = x - u$. Then $u + t_n z \in D \setminus C$ for all $n \geq 1$. $\square$

The next result provides, in a Banach space context, a sufficient condition for global strict quasimonotonicity.

Theorem 3.6. *If $A : D \longrightarrow X^*$ is a hemicontinuous operator with the property that $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ and whose restriction $A|_{D\setminus C}$ is locally strictly quasimonotone, then A is strictly quasimonotone on D.*

Proof. For $u, v \in D \setminus C$, $u \neq v$ consider an open interval J containing $[0, 1]$, such that $u + t(v - u) \in D$ for all $t \in J$ and $u + t(v - u) \in D \setminus C$, for all $t \in J$, $t < 0$ and for all $t \in J$, $t > 1$, and the map

$$\phi : J \longrightarrow \mathbb{R}, \quad \phi(t) := \langle A(u + t(v - u)), v - u\rangle.$$

For $s, t \in J$ we have

$$\begin{aligned} & \min\{\phi(s)(t - s), \phi(t)(s - t)\} \\ = \; & \min\{\langle A(u + s(v - u)), (t - s)(v - u)\rangle, \langle A(u + t(v - u)), (s - t)(v - u)\rangle\} \\ = \; & \min\{\langle Au_s, (t - s)(v - u)\rangle, \langle Au_t, (s - t)(v - u)\rangle\} \\ = \; & \min\{\langle Au_s, u_t - u_s\rangle, \langle Au_t, u_s - u_t\rangle\}, \end{aligned}$$

where we denoted $u_s = u + s(v - u)$, $u_t = u + t(v - u)$.

Obviously the set $Y = \{t \in J : u + t(v - u) \in C\}$ is a closed countable subset of J, and by using the hypothesis, i.e. for all $z \in D \setminus C$ there exists an open and convex neighborhood U_z of z, such that $\min\{\langle Ax, y - x\rangle, \langle Ay, x - y\rangle\} \leq 0$ and there exists $w \in (x, y)$ such that $\langle Aw, y - x\rangle \neq 0$, for all $x, y \in U_z$, $x \neq y$ we obtain, that for all $z' \in (u, v) \setminus C$, $z' = u + t'(v - u)$ there exists an open and convex neighborhood of z', say $U_{z'}$, such that $\min\{\langle Au_s, u_t - u_s\rangle, \langle Au_t, u_s - u_t\rangle\} \leq 0$ and there exists $w \in (u_s, u_t)$ such that $\langle Aw, u_s - u_t\rangle \neq 0$ for all $u_s, u_t \in U_{z'}$, $u_s \neq u_t$. Let us denote $J_{t'} = \{t \in (0, 1) : u + t(v - u) \in U_{z'}\}$. Then $\min\{\phi(s)(t - s), \phi(t)(s - t)\} \leq 0$ for all $t, s \in J_{t'}$ and $\operatorname{int}\{t \in J_{t'} : \phi(t) = 0\} = \emptyset$. In other words, for each $t' \in J \setminus Y$, there exists an open interval $J_{t'}$, containing t', such that $\min\{\phi(s)(t - s), \phi(t)(s - t)\} \leq 0$, for all $t, s \in J_{t'}$ and $\operatorname{int}\{t \in J_{t'} : \phi(t) = 0\} = \emptyset$. These latter relations show that ϕ is locally strictly quasimonotone on $J \setminus Y$, and by using Theorem 2.9 we obtain that the function ϕ is strictly quasimonotone on J, that is $\min\{\langle Au_s, u_t - u_s\rangle, \langle Au_t, u_s - u_t\rangle\} \leq 0$, for all $s, t \in J$, which particularly shows that

$$\min\{\langle Au, v - u\rangle, \langle Av, u - v\rangle\} \leq 0.$$

In the general case of arbitrary $u, v \in D$, according to Lemma 3.5 consider the sequences $u_n = u + t_n z$, $n \geq 1$ and $v_n = v + s_n w$, $n \geq 1$ such that $(u_n), (v_n) \subset D \setminus C$, where $z, w \in X$, t_n, $s_n \in \mathbb{R}$, $n \geq 1$, with t_n, $s_n \longrightarrow 0, n \longrightarrow \infty$. According to the first part of the proof, $\min\{\langle Au_n, v_n - u_n\rangle, \langle Av_n, u_n - v_n\rangle\} \leq 0$ for all $n \geq 1$, which shows by using the hemicontinuity of the operator A, that

$$\min\{\langle Au, v - u\rangle, \langle Av, u - v\rangle\} \leq 0.$$

For completing the proof, we have to prove that there exists $w \in (u, v)$ such that $\langle Aw, v - u\rangle \neq 0$.

If $(u, v) \cap (D \setminus C) \neq \emptyset$, let $z \in (u, v) \cap (D \setminus C)$. Then z has an open neighborhood U, where A is strictly quasimonotone, consequently for every $z_1, z_2 \in U \cap (u, v)$, $z_1 \neq z_2$ there exists $w \in (z_1, z_2)$ such that $\langle Aw, z_2 - z_1\rangle \neq 0$. But, since $z_1 = u + t_1(v - u)$, $z_2 = u + t_2(v - u)$ for some $t_1, t_2 \in (0, 1)$, $t_1 \neq t_2$, it follows that $\langle Aw, v - u\rangle = \frac{1}{(t_2 - t_1)}\langle Aw, z_2 - z_1\rangle \neq 0$.

If $(u, v) \cap (D \setminus C) = \emptyset$, then $(u, v) \subseteq C$, therefore according to assumption of the theorem i.e. $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ we obtain the conclusion. □

In what follows we conclude similar results for locally pseudomonotone operators. However, in the proof of Theorem 3.7 bellow, we cannot use directly the hemicontinuity of the operator A, as we did in the proof of Theorem 3.6.

Theorem 3.7. *If $A : D \longrightarrow X^*$ is a hemicontinuous operator with the property that $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ and whose restriction $A|_{D \setminus C}$ is locally pseudomonotone, then A is pseudomonotone on D.*

Proof. We divide the proof into two cases.

Case I. For $u, v \in D \setminus C$, $u \neq v$ consider the open interval J and the map ϕ as in the proof of Theorem 3.6.

Denote $u_s = u + s(v - u)$, $u_t = u + t(v - u)$. We have, for all $z' \in (u, v) \setminus C$, $z' = u + t'(v - u)$ that there exists an open neighborhood of z', say $U_{z'}$ such that $\langle Au_t, u_s - u_t \rangle > 0$ implies $\langle Au_s, u_s - u_t \rangle > 0$ for $u_s, u_t \in U_{z'}$. Obviously, the set $Y = \{t \in J : u + t(v - u) \in C\}$ is a closed countable subset of J. Let us denote $J_{t'} = \{t \in (0, 1) : u + t(v - u) \in U_{z'}\}$, then $\phi(t)(s - t) > 0$ implies $\phi(s)(s - t) > 0$, for $s, t \in J_{t'}$. In other words, for each $t' \in J \setminus Y$, there exists an open interval, $J_{t'}$, containing t' such that $\phi(t)(s - t) > 0$ implies $\phi(s)(s - t) > 0$, for $s, t \in J_{t'}$, and by using Theorem 2.9 we obtain that the function ϕ is pseudomonotone on J, that is $\langle Au_t, u_s - u_t \rangle > 0$ implies $\langle Au_s, u_s - u_t \rangle > 0$ for all $s, t \in J$, which particulary shows that

$$\langle Au, v - u \rangle > 0 \Longrightarrow \langle Av, v - u \rangle > 0.$$

Case II. For $u \in C$ and $v \in D$, suppose that $\langle Au, v - u \rangle \geq 0$ and $\langle Av, v - u \rangle < 0$. Then, since $\langle Au, v - u \rangle \neq 0$, it is enough to assume that $\langle Au, v - u \rangle > 0$ and $\langle Av, v - u \rangle < 0$.

According to Lemma 3.4, there exist $x, y, z, w \in D \setminus C$ such that $u \in (x, y)$ and $v \in (z, w)$, that is, there exist $t_0, t_1 \in (0, 1)$ such that $u = x + t_0(y - x)$ and $v = z + t_1(w - z)$. It can be easily verified, that all $x' \in (x, y)$ may be written in the form $x' = u + t(x - u)$, where $t \in \left(1 - \frac{1}{t_0}, 1\right)$, as well that all $z' \in (z, w)$ may be written in the form $z' = v + t(z - v)$, where $t \in \left(1 - \frac{1}{t_1}, 1\right)$.

Let us define the functions,

$$f : \left(1 - \frac{1}{t_0}, 1\right) \longrightarrow \mathbb{R}, \quad f(t) = \langle A(u + t(x - u)), (v + t(z - v)) - (u + t(x - u)) \rangle$$

and

$$g : \left(1 - \frac{1}{t_1}, 1\right) \longrightarrow \mathbb{R}, \quad g(t) = \langle A(v + t(z - v)), (v + t(z - v)) - (u + t(x - u)) \rangle.$$

Using the hemicontinuity of A we obtain that for all $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $t_n \longrightarrow 0$, $n \longrightarrow \infty$ we have $f(t_n) \longrightarrow f(0)$, $n \longrightarrow \infty$ and $g(t_n) \longrightarrow g(0)$, $n \longrightarrow \infty$, which show that the functions f and g are continuous at $t = 0$.

On the other hand, we have $f(0) = \langle Au, v - u \rangle > 0$ and $g(0) = \langle Av, v - u \rangle < 0$, and by the continuity of these functions at 0 we conclude that there exists $\epsilon > 0$, such that $f(t) > 0$, $g(t) < 0$ for all $t \in (-\epsilon, \epsilon) \subseteq \left(1 - \frac{1}{t_0}, 1\right) \cap \left(1 - \frac{1}{t_1}, 1\right)$.

Since $[x, y] \cap C$ respectively $[z, w] \cap C$ are countable, possibly empty, and $\{u + t(x - u) : t \in (-\epsilon, \epsilon)\} \subseteq [x, y]$, respectively $\{v + t(z - v) : t \in (-\epsilon, \epsilon)\} \subseteq [z, w]$, we obtain that the sets $\{u + t(x - u) : t \in (-\epsilon, \epsilon)\} \cap C$ respectively $\{v + t(z - v) : t \in (-\epsilon, \epsilon)\} \cap C$ are countable, possibly empty.

Next we will show that there exists some $l \in (-\epsilon, \epsilon)$ such that $u + l(x - u)$, $v + l(z - v) \in D \setminus C$.
Indeed, supposing the contrary, that is $u + l(x - u) \in D \setminus C$ implies $v + l(z - v) \in C$, since the set $\{u + l(x - u) \in D \setminus C : l \in (-\epsilon, \epsilon)\}$ is uncountable we obtain that the

set $\{v + l(z - v) \in C : l \in (-\epsilon, \epsilon)\}$ is uncountable, which contradicts the fact that $\{v + t(z - v) : t \in (-\epsilon, \epsilon)\} \cap C$ is countable, possibly empty.

Let $l \in (-\epsilon, \epsilon)$ such that $u + l(x - u)$, $v + l(z - v) \in D \setminus C$. Then

$$\langle A(u + l(x - u)), (v + l(z - v)) - (u + l(x - u))\rangle = f(l) > 0,$$

and according to *Case I.* we obtain that

$$\langle A(v + l(z - v)), (v + l(z - v)) - (u + l(x - u))\rangle = g(l) > 0,$$

which contradicts the fact that $g(t) < 0$ for all $t \in (-\epsilon, \epsilon)$. □

The next result provides, in a Banach space context, a sufficient condition for global strict pseudomonotonicity.

Theorem 3.8. *If $A : D \longrightarrow X^*$ is a hemicontinuous operator with the property that $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ and whose restriction $A|_{D\setminus C}$ is locally strictly pseudomonotone, then A is strictly pseudomonotone on D.*

Proof. Let $u, v \in D \setminus C$, $u \neq v$. In this case the proof is similar to the proof of *Case I.* in Theorem 3.7 and therefore we omit it.

If $u \in C$ then by the assumption of the theorem $\langle Au, v - u\rangle \neq 0$ and according to Theorem 3.7 A is globally pseudomonotone, i.e. $\langle Au, v-u\rangle > 0$ implies $\langle Av, v-u\rangle > 0$, which combined gives

$$\langle Au, v - u\rangle \geq 0 \Longrightarrow \langle Av, v - u\rangle > 0.$$

If $u \in D \setminus C$, $v \in C$ then by the assumption of the theorem $\langle Av, v - u\rangle \neq 0$ and according to Theorem 3.7 A is globally pseudomonotone, i.e. $\langle Au, v - u\rangle \geq 0$ implies $\langle Av, v - u\rangle \geq 0$, which combined gives

$$\langle Au, v - u\rangle \geq 0 \Longrightarrow \langle Av, v - u\rangle > 0,$$

and this completes the proof. □

Remark 3.9. The assumption of convexity on D in the above theorems is essential. In [21] an example of locally Minty-Browder monotone operator is given, (consequently is locally quasimonotone, and locally pseudomonotone as well) defined on a connected but non convex subset of $\mathbb{R}^2$, which is not even quasimonotone globally.

We wonder if the assumption $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ in the above theorems is essential or can be replaced by $\langle Az, y - x\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D \setminus C$, $x \neq y$?

4. Applications to generalized convex functions

In this section we apply the results from Section 3 to prove the generalized convexity of some locally generalized convex functions under certain classical hypothesis, which are imposed on the complement of the same type of sets as before.

In what follows H denotes a real Hilbert space, and let C and D the same sets as in Section 3. We shall need the following definitions and results related to generalized convex functions.

A real valued function f defined on the open convex subset D of H, is called quasiconvex (see [9, 11, 16, 27]), respectively strictly quasiconvex (see [9, 27]), if for all $x, y \in D$ and $t \in [0,1]$, we have

$$f(y) \leq f(x) \Longrightarrow f(tx + (1-t)y) \leq f(x),$$

respectively for all $x, y \in D$, $x \neq y$ and $t \in (0,1)$, we have

$$f(y) \leq f(x) \Longrightarrow f(tx + (1-t)y) < f(x),$$

or equivalently for all $x, y \in D$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

respectively for all $x, y \in D$, $x \neq y$ and $t \in (0,1)$, we have

$$f(tx + (1-t)y) < \max\{f(x), f(y)\}.$$

Remark 4.1. A differentiable quasiconvex function f can be characterized by its differential (see [13]), i.e. f is quasiconvex on the open convex subset D of H, if and only if, for every pair of points $x, y \in D$ we have

$$f(y) \leq f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle \leq 0,$$

where ∇f denotes the gradient operator.

However, in general, strictly quasiconvex functions cannot be characterized by their differential in a similar way, but if a differentiable function f, defined on the open convex subset D of H has the property that $\nabla f(x) \neq 0$ for all $x \in D$, then f is strictly quasiconvex on D (see [6]), if and only if, for every pair of points $x, y \in D$, $x \neq y$ we have

$$f(y) \leq f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle < 0.$$

The following statement holds (see [8, 10, 13, 18]):

Proposition 4.2. *A differentiable function f defined on the open convex subset D of H is quasiconvex, (respectively strictly quasiconvex) on D, if and only if, ∇f is quasimonotone, (respectively strictly quasimonotone) on D.*

A real valued function f defined on the open convex subset D of H, is called pseudoconvex (see [16]), if for all $x, y \in D$ and $t \in (0,1)$, whenever $f(tx + (1-t)y) \geq f(x)$ it holds that $f(tx + (1-t)y) \leq f(y)$.

In the differentiable case we have another characterization of pseudoconvexity, respectively strict pseudoconvexity.

A real valued differentiable function f defined on the open convex subset D of H, is called pseudoconvex (see [1, 8, 9, 10, 12]), respectively strictly pseudoconvex (see [8, 11, 9, 10, 12]) on D, if for every pair of distinct points $x, y \in D$ we have

$$\langle \nabla f(x), y - x\rangle \geq 0 \Longrightarrow f(y) \geq f(x),$$

respectively

$$\langle \nabla f(x), y - x\rangle \geq 0, \quad x \neq y \Longrightarrow f(y) > f(x).$$

The next result is well-known, see for instance [6, 13, 18].

Proposition 4.3. *Let f be differentiable on the open convex subset D of H. Then f is pseudoconvex, (respectively strictly pseudoconvex) on D, if and only if, ∇f is pseudomonotone, (respectively strictly pseudomonotone) on D.*

Next we give the definitions of some locally generalized convex functions.

Definition 4.4. We say that a function $f : D \longrightarrow \mathbb{R}$ is locally quasiconvex, (respectively locally strictly quasiconvex, locally pseudoconvex, locally strictly pseudoconvex) on D, if for all $z \in D$ there is an open and convex neighborhood $U_z \subseteq D$ of z, where f is quasiconvex, (respectively strictly quasiconvex, pseudoconvex, strictly pseudoconvex).

In what follows we provide, in a Hilbert space context, a sufficient condition for strict quasiconvexity of a locally strictly quasiconvex function.

Theorem 4.5. *Let $f : D \longrightarrow \mathbb{R}$ be a continuously differentiable, locally strictly quasiconvex function on $D \setminus C$. If ∇f has the property, that $\langle \nabla f(z), x - y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ then f is globally strictly quasiconvex on D.*

Proof. Since f locally strictly quasiconvex on $D \setminus C$, according to Proposition 4.2, ∇f is locally strictly quasimonotone on $D \setminus C$. Using the fact that $\langle \nabla f(z), x - y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ according to Theorem 3.6, we obtain that ∇f is strictly quasimonotone on D, which shows that f is strictly quasiconvex on D. □

In what follows we conclude a similar result for locally pseudoconvex functions.

Theorem 4.6. *Let $f : D \longrightarrow \mathbb{R}$ be a continuously differentiable, locally pseudoconvex function on $D \setminus C$. If ∇f has the property, that $\langle \nabla f(z), x - y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ then f is globally pseudoconvex on D.*

Proof. Since f is locally pseudoconvex on $D \setminus C$, according to Proposition 4.3, ∇f is locally pseudomonotone on $D \setminus C$. Using the fact that $\langle \nabla f(z), x - y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ according to Theorem 3.7, we obtain that ∇f is pseudomonotone on D, which shows that f is pseudoconvex on D. □

The next result provides, in a Hilbert space context, a sufficient condition for global strict pseudoconvexity.

Theorem 4.7. *Let $f : D \longrightarrow \mathbb{R}$ be a continuously differentiable, locally strictly pseudoconvex function on $D \setminus C$. If ∇f has the property, that $\langle \nabla f(z), x - y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ then f is globally strictly pseudoconvex on D.*

Proof. Since f is locally strictly pseudoconvex on $D \setminus C$, according to Proposition 4.3, ∇f is locally strictly pseudomonotone on $D \setminus C$. Using the fact that $\langle \nabla f(z), x-y\rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ according to Theorem 3.8 we conclude, that ∇f is strictly pseudomonotone on D, which shows that f is strictly pseudoconvex on D. □

Remark 4.8. Actually it is enough to assume that ∇f is hemicontinuous. However, as we have seen in Example 2.6, the local quasimonotonicity does not imply the global quasimonotonicity. Next we will give an example of a continuously differentiable locally quasiconvex function, which is not globally quasiconvex.

Example 4.9. Let us consider the function

$$F : \mathbb{R} \longrightarrow \mathbb{R},\ F(x) = \begin{cases} -\frac{x^2}{2} - x, & \text{if } x < -1 \\ \frac{1}{2}, & \text{if } x \in [-1, 1] \\ -\frac{x^2}{2} + x, & \text{if } x > 1. \end{cases}$$

It can be easily checked that F is an antiderivative of f given in Example 2.6, consequently F is continuously differentiable.

We know that any monotone (increasing/decreasing) function from $\mathbb{R}$ to $\mathbb{R}$ is quasiconvex. Since F is locally monotone we obtain that F is locally quasiconvex.

On the other hand, for $x = -2$ and $y = 2$ we have: $F\left(\frac{1}{2} \cdot (-2) + \left(1 - \frac{1}{2}\right) \cdot (2)\right) = F(0) = \frac{1}{2} > \max\{F(-2), F(2)\} = 0$, which shows that F is not globally quasiconvex, (see Fig. 4.1).

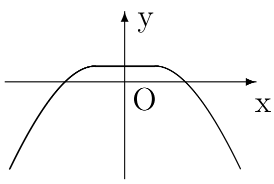

Figure 4.1

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