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# Validity of the Union of Uniform Closed Balls Conjecture

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A proof is provided for the union of uniform closed balls conjecture introduced in [7]; see also [8] and [9]. We also provide a generalization of this conjecture to the variable radius case.

*Keywords:* Interior and exterior sphere conditions, union of uniform closed balls property, proximal analysis, nonsmooth analysis

## 1. Introduction

Let $S \subset \mathbb{R}^n$ be a nonempty closed set. For $r > 0$, we can find in the control theoretic literature two definitions of the *interior $r$-sphere condition*. The first one (see [1, 2, 3]), used here, is that for each $x \in \operatorname{bdry} S$ (the boundary of $S$) there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S, \tag{1}$$

where $\bar{B}(z; \rho)$ denotes the closed ball of radius $\rho$ centered at $z$. The second definition (see [5, 6, 12]) says that for all $x \in S$ there exists $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

This means that $S$ is the union of closed $r$-balls. Equivalently, there exists $S_0 \subset S$ such that $S_0 + \bar{B}(0; r) = S$. Clearly, if $S$ is the union of closed $r$-balls then it satisfies the interior $r$-sphere condition. The following example (see [7, Example 4.1]) shows that the reverse implication is not necessarily true, and therefore the two definitions are *not equivalent*.

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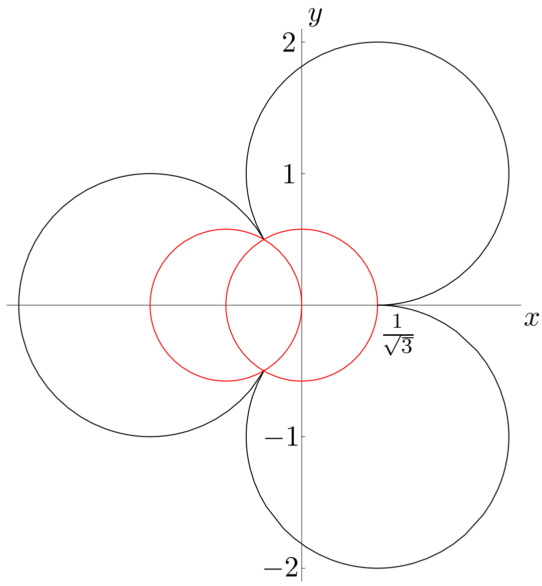


Figure 1.1: Example 1.1

**Example 1.1.** Let $S$ be the closed region inside the three outer circles of Figure 1.1. Clearly this set satisfies the interior 1-sphere condition (in the first sense) since the three circles are of radius 1. But the origin cannot be covered by a 1-ball contained in $S$; in fact, for this configuration, the maximal radius for a family of covering balls is $\frac{1}{\sqrt{3}}$. Therefore the interior sphere condition does not hold for $S$ in the second sense.

While $S$ of the previous example is not the union of closed 1-balls, but it certainly *is* the union of closed $r$-balls for $r \leq \frac{1}{\sqrt{3}}$. This lead Nour, Stern and Takche [7] to frame the following conjecture:

**Conjecture 1.2.** *Suppose that $S \subset \mathbb{R}^n$ is a nonempty closed set satisfying the interior $r$-sphere condition. Then there exists $r'$ such that $S$ is the union of closed $r'$-balls.*

In [9], Nour, Stern and Takche generalized Example 1.1 to $\mathbb{R}^n$, see [9, Example 11]. Specifically, they provided, for any $r > 0$, a set $S$ in $\mathbb{R}^n$ which satisfies the interior $r$-sphere condition but which $S$ fails to be the union of closed balls with radius $r' > \frac{nr}{2\sqrt{n^2-1}}$. Therefore they introduced the following new version of Conjecture 1.2, called the *union of uniform closed balls conjecture*:

**Conjecture 1.3.** *Suppose that $S \subset \mathbb{R}^n$ is a nonempty closed set satisfying the interior $r$-sphere condition. Then there exists $r' \leq \frac{nr}{2\sqrt{n^2-1}}$ such that $S$ is the union of closed $r'$-balls.*

We can find in [7] a proof of the union of uniform closed balls conjecture, but under the assumption that $S$ is *wedged* with compact boundary, see [7, Corollary 4.2]. Recall that a set $S$ is said to be wedged (or *epi-Lipschitz*) if near any boundary point, $S$ can be viewed, after application of an orthogonal transformation, as the epigraph of a Lipschitz continuous function. The proof employed a result which asserts that under the wedgedness and compactness hypotheses, *proximal smoothness* of $(\operatorname{int} S)^c$ (the complement of the interior of $S$) and the interior sphere condition of $S$ coincide; see [7, Corollary 3.12].

The goal of the present article is to provide a proof (which we believe to be the first) of Conjecture 1.3. Specifically, we will demonstrate that $r'$ can be taken to be $\frac{r}{2}$, which is less than $\frac{nr}{2\sqrt{n^2-1}}$ for all $n \geq 2$. Our proof is a direct one which only uses some simple results from nonsmooth and proximal analysis. A generalization of this conjecture to the case in which the radius of the balls can be taken to be a continuous function is also given at the end of the article.

In the next section we present, after giving some definitions and establishing notation, the details of the proof of the union of uniform closed balls conjecture. Section 3 is devoted to the generalization of this conjecture to the continuous radius case.

## 2. Proof of the conjecture

First we will provide some definitions and notation from nonsmooth analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [4]; see also [11].

We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $B$ and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\bar{B}(x;\rho) := x + \rho\bar{B}$. For a set $A \subset \mathbb{R}^n$, $A^c$, $\operatorname{int} A$, $\operatorname{bdry} A$ and $\operatorname{cl} A$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of $A$, respectively. We also denote by $A'$ the complement of the interior of $A$, that is, $A' := (\operatorname{int} A)^c$. The distance from a point $x$ to a set $A$ is denoted by $d_A(x)$. We also denote by $\operatorname{proj}_A(x)$ the set of closest points in $A$ to $x$, that is, the set of points $a \in A$ which satisfy $d_A(x) = \|a - x\|$.

Let $A$ be a nonempty closed subset of $\mathbb{R}^n$. For $x \in A$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to $A$ at $x$* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, a - x\rangle \leq \sigma\|a - x\|^2 \quad \forall a \in A. \tag{2}$$

The relation (2) is commonly referred to as the *proximal normal inequality.* No nonzero $\zeta$ satisfying (2) exists if $x \in \operatorname{int} A$, but this may also occur for $x \in \operatorname{bdry} A$ (as is the case when $A$ is the epigraph of the function $f(z) = -|z|$ and $x = (0,0)$). For such points, the only proximal normal is $\zeta = 0$. In view of (2), the set of all proximal normals to $A$ at $x$ is a convex cone, and we denote it by $N_A^P(x)$. Now let $x \in \operatorname{bdry} A$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap A = \emptyset. \tag{3}$$

Then $\zeta$ is a proximal normal to $A$ at $x$ and we say that $\zeta$ is *realized by an $r$-sphere.* Note that $\zeta$ is then also realized by an $r'$-sphere for any $0 < r' < r$. One can show that $\zeta$ being realized by an $r$-sphere is equivalent to the proximal normal inequality holding for the normalization of $\zeta$, with $\sigma = \frac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x\right\rangle \leq \frac{1}{2r}\|a - x\|^2 \quad \forall a \in A. \tag{4}$$

Therefore using the fact that (1) is equivalent to $B(y_x;r)\cap S'=\emptyset$, we deduce that a closed set $A$ satisfies the interior $r$-sphere condition if and only if for all $x\in \operatorname{bdry} A$ there exists $0\neq\zeta\in N^P_{A'}(x)$ such that $\zeta$ is realized by an $r$-sphere.

The following is the proof of the union of uniform closed balls conjecture.

**Proof of Conjecture 1.3.** Let $S\subset\mathbb{R}^n$ be a nonempty closed set satisfying the interior $r$-sphere condition. We will prove that $S$ is the union of $\frac{r}{2}$-balls; that is, for all $x\in S$ there exists $y_x\in S$ such that $x\in\bar{B}(y_x;\frac{r}{2})\subset S$.

Let $x\in S$. If $x\in\operatorname{bdry} S$, then since $S$ satisfies the interior sphere condition we obtain the existence of $z_x\in S$ such $x\in\bar{B}(z_x;r)\subset S$. Taking $y_x:=x+\frac{r}{2}\frac{z_x-x}{\|z_x-x\|}$ we, have

$$x\in\bar{B}\left(y_x;\frac{r}{2}\right)\subset\bar{B}(z_x;r)\subset S.$$

Now we assume that $x\notin\operatorname{bdry} S$, that is, $x\in\operatorname{int} S$. We consider $s\in\operatorname{proj}_{\operatorname{bdry} S}(x)$ and we denote by $r_0:=\|x-s\|\neq 0$. If $r_0\geq\frac{r}{2}$, then for $y_x:=x$ we have

$$x\in\bar{B}\left(y_x;\frac{r}{2}\right)\subset\bar{B}(x;r_0)\subset S.$$

So we assume that $0<r_0<\frac{r}{2}$. For every $\epsilon\in]0,\frac{r_0}{2}[$ we denote by $z_\epsilon$ a point in $\bar{B}(s;\epsilon)\cap S^c$ (which exists since $s$ is a boundary point) and

$$x_\epsilon:=z_\epsilon+\operatorname{proj}_\zeta(x-z_\epsilon)=z_\epsilon+\langle x-z_\epsilon,\zeta\rangle\zeta=z_\epsilon+t_\epsilon\zeta,$$

where $\zeta:=\frac{x-s}{\|x-s\|}$ and $t_\epsilon:=\langle x-z_\epsilon,\zeta\rangle$; see Figure 2.1. We claim that $x_\epsilon-x$ and $\zeta$ are orthogonal and that $x_\epsilon\in\bar{B}(x;r_0)$. Indeed,

$$\langle x_\epsilon-x,\zeta\rangle=\langle z_\epsilon+t_\epsilon\zeta-x,\zeta\rangle=\langle t_\epsilon\zeta,\zeta\rangle-\langle x-z_\epsilon,\zeta\rangle=t_\epsilon-t_\epsilon=0.$$

On the other hand,

$$\begin{aligned}\|z_\epsilon-s\|^2 &= \|(x_\epsilon-t_\epsilon\zeta)-(x-r_0\zeta)\|^2\\ &= \|(x_\epsilon-x)+(r_0-t_\epsilon)\,\zeta\|^2\\ &= \|x_\epsilon-x\|^2+(r_0-t_\epsilon)^2+2\,(r_0-t_\epsilon)\,\langle x_\epsilon-x,\zeta\rangle\\ &= \|x_\epsilon-x\|^2+(r_0-t_\epsilon)^2.\end{aligned}$$

This gives, using the fact that $\|z_\epsilon-s\|\leq\epsilon$, that

$$\|x_\epsilon-x\|^2\leq\epsilon^2-(r_0-t_\epsilon)^2. \tag{5}$$

Therefore $\|x_\epsilon-x\|\leq\epsilon\leq r_0$, and so $x_\epsilon\in\bar{B}(x;r_0)$. We can also deduce from (5) that $\epsilon^2-(r_0-t_\epsilon)^2\geq 0$, and then

$$\epsilon\leq t_\epsilon\leq r_0+\epsilon. \tag{6}$$

Since $z_\epsilon\notin S$ and $x_\epsilon\in S$, the segment joining these two points will intersect the boundary of $S$ at a point $s_\epsilon=z_\epsilon+t'_\epsilon\zeta=x_\epsilon+(t'_\epsilon-t_\epsilon)\zeta$, where $0<t'_\epsilon\leq t_\epsilon$.

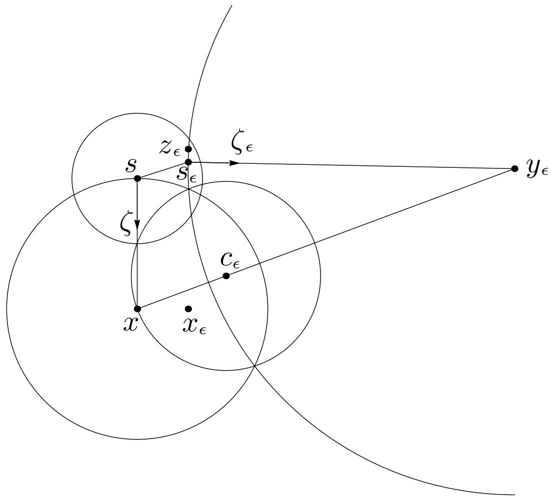


Figure 2.1: Proof figure

**Lemma 2.1.** $\|s_\epsilon - z_\epsilon\| = t'_\epsilon \leq \epsilon$.

To prove this, we first note that

$$\|s_\epsilon - x\|^2 = \|(x_\epsilon - x) - (t_\epsilon - t'_\epsilon)\zeta\|^2 = \|x_\epsilon - x\|^2 + (t_\epsilon - t'_\epsilon)^2.$$

But $s_\epsilon \notin B(x; r_0)$, and therefore

$$\|x_\epsilon - x\|^2 \geq r_0^2 - (t_\epsilon - t'_\epsilon)^2. \tag{7}$$

Now by (5) and (7) we obtain

$$2r_0(r_0 - t_\epsilon) \leq \epsilon^2 - t_\epsilon^2 + (t_\epsilon - t'_\epsilon)^2. \tag{8}$$

On the other hand, the inequality (6) gives that $t_\epsilon(r_0 - \epsilon) \leq r_0^2 - \epsilon^2$, and then

$$\epsilon(\epsilon - t_\epsilon) \leq r_0(r_0 - t_\epsilon).$$

If we combine this inequality with (8), then we obtain get

$$\begin{aligned} 2\epsilon(\epsilon - t_\epsilon) \leq \epsilon^2 - t_\epsilon^2 + (t_\epsilon - t'_\epsilon)^2 \quad &\Longrightarrow \quad (t_\epsilon - \epsilon)^2 \leq (t_\epsilon - t'_\epsilon)^2 \\ &\Longrightarrow \quad 0 \leq t_\epsilon - \epsilon \leq t_\epsilon - t'_\epsilon \\ &\Longrightarrow \quad t'_\epsilon \leq \epsilon, \end{aligned}$$

which completes the proof of Lemma 2.1.

We continue the proof of the conjecture and consider $\zeta_\epsilon$, a unit proximal normal vector to $S'$ at $s_\epsilon$ realized by an $r$-sphere (which exists since $S$ satisfies the interior $r$-sphere condition). Clearly the center of this $r$-sphere is $y_\epsilon := s_\epsilon + r\zeta_\epsilon$. The proximal normal inequality gives that

$$\langle \zeta_\epsilon, z - s_\epsilon \rangle \leq \frac{1}{2r}\|z - s_\epsilon\|^2$$

for all $z \in S'$. Since $z_\epsilon \notin S$ (and then $z_\epsilon \in \operatorname{int} S'$) we obtain

$$\langle \zeta_\epsilon, \zeta \rangle = -\langle \zeta_\epsilon, -\zeta \rangle = -\left\langle \zeta_\epsilon, \frac{z_\epsilon - s_\epsilon}{\|z_\epsilon - s_\epsilon\|} \right\rangle > \frac{-1}{2r}\|z_\epsilon - s_\epsilon\|. \tag{9}$$

If $\|y_\epsilon - x\| \le r$ then for $y_x := x + \frac{r}{2}\frac{y_\epsilon - x}{\|y_\epsilon - x\|}$ we obtain that

$$x \in \bar{B}\left(y_x; \frac{r}{2}\right) \subset \bar{B}(y_\epsilon; r) \subset S.$$

So we assume that $\|y_\epsilon - x\| > r$.

**Lemma 2.2.** *For $r_\epsilon := \frac{r_0^2\|y_\epsilon - x\|}{\|y_\epsilon - x\|^2 + r_0^2 - r^2}$ and $c_\epsilon = x + r_\epsilon \frac{y_\epsilon - x}{\|y_\epsilon - x\|}$, we have*

$$\bar{B}(c_\epsilon; r_\epsilon) \subset \bar{B}(x; r_0) \cup \bar{B}(y_\epsilon; r) \subset S.$$

First we note that one can prove that $r_\epsilon$ is the radius if the largest closed ball containing $s$ and contained in $\bar{B}(x; r_0) \cup \bar{B}(y_\epsilon; r)$, see Figure 2.1. For the proof of the lemma, let $w \in \bar{B}(c_\epsilon; r_\epsilon)$ and assume that $w \notin \bar{B}(x; r_0)$. We need to prove that $w \in \bar{B}(y_\epsilon; r)$. Since $w \in \bar{B}(c_\epsilon; r_\epsilon)$, we have that

$$\|w - x\|^2 \le 2r_\epsilon \left\langle w - x, \frac{y_\epsilon - x}{\|y_\epsilon - x\|} \right\rangle.$$

Then using the preceding inequality and the facts that $\|w - x\| \ge r_0$ and $\|y_\epsilon - x\|^2 > r^2$, we get that

$$\begin{aligned}
\|w - y_\epsilon\|^2 &= \|(w - x) - (y_\epsilon - x)\|^2 \\
&= \|w - x\|^2 + \|y_\epsilon - x\|^2 - 2\langle w - x, y_\epsilon - x \rangle \\
&\le \|w - x\|^2 + \|y_\epsilon - x\|^2 - \frac{\|y_\epsilon - x\|}{r_\epsilon}\|w - x\|^2 \\
&\le \|y_\epsilon - x\|^2 - \left(\frac{\|y_\epsilon - x\|}{r_\epsilon} - 1\right)\|w - x\|^2 \\
&\le \|y_\epsilon - x\|^2 - \left(\frac{\|y_\epsilon - x\|^2 - r^2}{r_0^2}\right)\|w - x\|^2 \\
&\le \|y_\epsilon - x\|^2 - \left(\frac{\|y_\epsilon - x\|^2 - r^2}{r_0^2}\right) r_0^2 \\
&= r^2.
\end{aligned}$$

This completes the proof of Lemma 2.2.

**Lemma 2.3.** $\|y_\epsilon - x\| \le \sqrt{r^2 + r_0^2} + 4\epsilon.$

To prove this, first note

$$\begin{aligned}
\|y_\epsilon - x\|^2 &= \|(y_\epsilon - s_\epsilon) + (s_\epsilon - s) + (s - x)\|^2 \\
&= \|r\zeta_\epsilon + (s_\epsilon - s) - r_0\zeta\|^2 \\
&= r^2 + \|s_\epsilon - s\|^2 + r_0^2 + 2r\langle \zeta_\epsilon, s_\epsilon - s \rangle - 2rr_0\langle \zeta_\epsilon, \zeta \rangle - 2r_0\langle s_\epsilon - s, \zeta \rangle \\
&\le r^2 + \|s_\epsilon - s\|^2 + r_0^2 + 2r_0\|s_\epsilon - s\| + 2r\|s_\epsilon - s\| - 2rr_0\langle \zeta_\epsilon, \zeta \rangle.
\end{aligned}$$

On the other hand, using (9) and Lemma 2.1 we have that

$$-2rr_0\langle \zeta_\epsilon, \zeta\rangle < r_0\|z_\epsilon - s_\epsilon\| \le \epsilon r_0,$$

and

$$\|s_\epsilon - s\| \le \|s_\epsilon - z_\epsilon\| + \|z_\epsilon - s\| \le \epsilon + \epsilon = 2\epsilon.$$

Therefore

$$\begin{aligned}\|y_\epsilon - x\|^2 &\le r^2 + 4\epsilon^2 + r_0^2 + 4r_0\epsilon + 4r\epsilon + r_0\epsilon\\ &= r^2 + r_0^2 + 4\epsilon^2 + \epsilon(5r_0 + 4r)\\ &\le r^2 + r_0^2 + 4\epsilon^2 + \frac{13}{2}\epsilon r \quad \left(\text{since } r_0 < \frac{r}{2}\right)\\ &\le r^2 + r_0^2 + 16\epsilon^2 + 8\epsilon\sqrt{r^2 + r_0^2}\\ &= \left(\sqrt{r^2 + r_0^2} + 4\epsilon\right)^2.\end{aligned}$$

This completes the proof of Lemma 2.3.

**Lemma 2.4.** *There exists $\bar{\epsilon} > 0$ such that $r_{\bar{\epsilon}} \ge \frac{r}{2}$.*

Note that $r_\epsilon \ge \frac{r}{2}$ is equivalent (by replacing $r_\epsilon$ by its definition) to

$$r\|y_\epsilon - x\|^2 - 2r_0^2\|y_\epsilon - x\| + r(r_0^2 - r^2) \le 0.$$

Hence it is sufficient to prove the existence of an $\bar{\epsilon} > 0$ such that $\|y_{\bar{\epsilon}} - x\|$ is between the two roots $\frac{r_0^2 \pm \sqrt{\Delta'}}{r}$ where $\Delta' = (r^2 - r_0^2)^2 + r^2 r_0^2 > 0$. But

$$\frac{r_0^2 - \sqrt{\Delta'}}{r} < 0 \le \|y_\epsilon - x\|,$$

so it is sufficient to prove the existence of an $\bar{\epsilon} > 0$ such that

$$\|y_{\bar{\epsilon}} - x\| \le \frac{r_0^2 + \sqrt{\Delta'}}{r}.$$

We can readily show that $\frac{r_0^2 + \sqrt{\Delta'}}{r} - \sqrt{r^2 + r_0^2} > 0$. In fact, this inequality is equivalent to

$$\sqrt{(r^2 - r_0^2)^2 + r^2 r_0^2} > r^2 - r_0^2.$$

Now let $\bar{\epsilon} := \min\{\frac{r_0}{2}, \frac{1}{4}(\frac{r_0^2 + \sqrt{\Delta'}}{r} - \sqrt{r^2 + r_0^2}\,)\} > 0$. By Lemma 2.3 we have

$$\|y_{\bar{\epsilon}} - x\| \le \sqrt{r^2 + r_0^2} + 4\bar{\epsilon} \le \frac{r_0^2 + \sqrt{\Delta'}}{r},$$

and the proof of Lemma 2.4 is completed.

For $y_x := x + \frac{r}{2}\frac{y_{\bar{\epsilon}} - x}{\|y_{\bar{\epsilon}} - x\|}$ we have by Lemma 2.2 that

$$x \in \bar{B}\left(y_x; \frac{r}{2}\right) \subset \bar{B}(c_{\bar{\epsilon}}; r_{\bar{\epsilon}}) \subset \bar{B}(x; r_0) \cup \bar{B}(y_{\bar{\epsilon}}; r) \subset S.$$

This terminates the proof of Conjecture 1.3. □

We have not yet found a counterexample in which $S$ fails to be the union of closed balls of radius $\frac{nr}{2\sqrt{n^2-1}}$. Moreover, we believe that the above proof cannot be adopted to prove that $S$ is the union of $\frac{nr}{2\sqrt{n^2-1}}$-balls since the dimension $n$ is not involved in our arguments. So the following is a stronger version of the union of uniform closed balls conjecture which will be a topic of future research.

**Conjecture 2.5.** *Suppose that $S \subset \mathbb{R}^n$ is a nonempty closed set satisfying the interior $r$-sphere condition. Then $S$ is the union of closed $\frac{nr}{2\sqrt{n^2-1}}$-balls.*

## 3. Generalization

We begin this section by defining the *$\theta$-interior sphere condition*, which is a generalization of the interior $r$-sphere condition used in the previous section. A closed set $S$ is said to satisfy the $\theta$-interior sphere condition if there exists a continuous function $\theta : \operatorname{bdry} S \longrightarrow [0, +\infty[$ such that for all $x \in \operatorname{bdry} S$ one can find a point $y_x \in S$ satisfying:

- $x \in \bar{B}(y_x; \frac{1}{2\theta(x)}) \subset S$, if $\theta(x) > 0$,
- $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\theta(x) = 0$.

By the $\theta_0$-interior sphere condition we mean the $\theta$-interior sphere condition with $\theta \equiv \theta_0$, a constant. Then the $\theta_0$-interior sphere condition (with $\theta_0 > 0$) is equivalent to the existence, for each $x \in \operatorname{bdry} S$, of $y_x \in S$ satisfying:

$$x \in \bar{B}\left(y_x; \frac{1}{2\theta_0}\right) \subset S.$$

Therefore, the $\theta_0$-interior sphere condition coincides with the interior $\frac{1}{2\theta_0}$-sphere condition. Using the proximal normal inequality, as in the constant radius case, we can easily see that the $\theta$-interior sphere condition is equivalent to the existence of a continuous function $\theta : \operatorname{bdry} S \longrightarrow [0, +\infty[$ such that for all $x \in \operatorname{bdry} S$ one can find a vector $0 \neq \zeta \in N^P_{S'}(x)$ such that

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \theta(x) \, \|s - x\|^2 \quad \forall s \in S'.$$

In other words, the $\theta$-interior sphere condition is equivalent to the existence of a continuous function $\theta : \operatorname{bdry} S \longrightarrow [0, +\infty[$ such that for all $x \in \operatorname{bdry} S$ one can find a vector $0 \neq \zeta \in N^P_{S'}(x)$ which is realized by a $\frac{1}{2\theta(x)}$-sphere if $\theta(x) \neq 0$ and by an $r$-sphere for all $r > 0$ if $\theta(x) = 0$. For more information about the $\theta$-interior sphere condition, we invite the reader to see [10] where the *$\theta$-exterior sphere condition* (the $\theta$-interior sphere condition satisfied by $S'$) and its relation to *$\varphi$-convexity* were studied.

We proceed to define the *$\psi$-union of closed balls* property, which will generalize the union of uniform closed balls property. A closed set $S$ is said to be the $\psi$-union of closed balls if there exists a function $\psi : S \longrightarrow [0, +\infty[$ such that:

(i) $\psi$ is upper semicontinous on $S$ and continuous on $\operatorname{bdry} S$.
(ii) For all $x \in S$ there exists $y_x \in S$ such that:

- $x \in \bar{B}(y_x; \frac{1}{2\psi(x)}) \subset S$, if $\psi(x) > 0$,
- $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\psi(x) = 0$.

Clearly the $\psi_0$-union of closed balls property (with $\psi_0 > 0$) is equivalent to the union of closed $\frac{1}{2\psi_0}$-balls property. The following is the generalization of the union of uniform closed balls conjecture.

**Theorem 3.1.** *Let $S \subset \mathbb{R}^n$ be a nonempty closed set which satisfies the $\theta$-interior sphere condition. Then $S$ is the $\psi$-union of closed balls where $\psi(\cdot)$ is defined by*

$$\psi(x) = 2 \max\{\theta(s) : s \in \operatorname{proj}_{\operatorname{bdry} S}(x)\} \quad \forall x \in S.$$

**Proof.** Let $S \subset \mathbb{R}^n$ be a nonempty closed set satisfying the $\theta$-interior sphere condition. Then there exists a continuous function $\theta : \operatorname{bdry} S \longrightarrow [0, +\infty[$ such that for all $x \in \operatorname{bdry} S$ one can find a point $y_x \in S$ satisfying (ii) above. Note that since the set $\operatorname{proj}_{\operatorname{bdry} S}(x)$ is compact for each $x \in S$ and by the continuity of $\theta(\cdot)$ on $\operatorname{bdry} S$, we can define the function $\psi(\cdot)$ on $S$ by

$$\psi(x) = 2 \max\{\theta(s) : s \in \operatorname{proj}_{\operatorname{bdry} S}(x)\} \quad \forall x \in S.$$

Since $\psi(\cdot) = 2\theta(\cdot)$ on $\operatorname{bdry} S$, we have that $\psi(\cdot)$ is continuous on $\operatorname{bdry} S$.

**Claim.** *The function $\psi(\cdot)$ is upper semicontinuous on $S$.*

To prove this, let $x_i$ be a sequence in $S$ such that $x_i$ converges to $x_0 \in S$. It is sufficient to prove that there is a subsequence $x_{i_k}$ of $x_i$, such that

$$\lim_{k \longrightarrow \infty} \psi(x_{i_k}) \leq \psi(x_0).$$

Let $s_i$ be a sequence in $\operatorname{bdry} S$ such that

$$s_i \in \operatorname{proj}_S(x_i) \quad \text{and} \quad \psi(x_i) = 2\,\theta(s_i).$$

Clearly the sequence $s_i$ is bounded. Then it admits a subsequence $s_{i_k}$ which converges to $s_0 \in \operatorname{bdry} S$. Using the closedness of the $\operatorname{proj}_{\operatorname{bdry} S}(\cdot)$ map we get that $s_0 \in \operatorname{proj}_S(x_0)$. Therefore

$$\psi(x_0) \geq 2\,\theta(y_0) = \lim_{k \longrightarrow \infty} 2\ ,\theta(s_{i_k}) = \lim_{k \longrightarrow \infty} \psi(x_{i_k}),$$

and this verifies the claim.

We continue with the proof of the theorem and we consider $x \in S$. If $x \in \operatorname{bdry} S$, then there exists $y_x \in S$ such that

- $x \in \bar{B}(y_x; \frac{1}{2\theta(x)}) \subset S$, if $\theta(x) > 0$,
- $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\theta(x) = 0$.

Since $\psi(x) = 2\theta(x)$ we get that

- $x \in \bar{B}(y_x; \frac{1}{2\psi(x)}) \subset \bar{B}(y_x; \frac{1}{2\theta(x)}) \subset S$, if $\psi(x) > 0$,
- $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\psi(x) = 0$.

Now we assume that $x \in \operatorname{int} S$.

*Case 1:* $\psi(x) \neq 0$. We consider $s \in \operatorname{proj}_{\operatorname{bdry} S}(x)$ such that $\psi(x) = 2\theta(s) \neq 0$. By the continuity of $\theta(\cdot)$ on $\operatorname{bdry} S$ we obtain, for $i \in \mathbb{N}^*$, the existence of $0 < \delta_i < \frac{1}{i}$ such that

$$0 < \theta(s') \leq \theta(s) + \frac{1}{i} = \frac{1}{2}\psi(x) + \frac{1}{i} \quad \forall s' \in B(s; \delta_i) \cap \operatorname{bdry} S.$$

Then using the fact that for all $s' \in B(s;\delta_i) \cap \operatorname{bdry} S$, $S'$ has a proximal normal at $s'$ realized by a $\frac{1}{2\theta(s')}$-sphere, we find that for all $s' \in B(s;\delta_i) \cap \operatorname{bdry} S$, $S'$ has a proximal normal at $s'$ realized by an $\frac{i}{i\psi(x)+2}$-sphere. Now using the same ideas as in the proof of Conjecture 1.3, we can prove the existence of $y_i \in S$ such that

$$x \in \bar{B}\left(y_i\,; \frac{i}{2i\psi(x)+4}\right) \subset S.$$

Since $\|y_i - x\| \leq \frac{i}{2i\psi(x)+4} \leq \frac{1}{2\psi(x)}$, we can assume that the sequence $(y_i)_i$ converges to a point $y_x \in S$. We claim that

$$x \in \bar{B}\left(y_x\,; \frac{1}{2\psi(x)}\right) \subset S.$$

Indeed, let $z \in B(y_x; \frac{1}{2\psi(x)})$ and assume that $z \notin \bar{B}(y_i; \frac{i}{2i\psi(x)+4})$ for all $i \in \mathbb{N}^*$. We shall derive a contradiction. Since $z \notin \bar{B}(y_i; \frac{i}{2i\psi(x)+4})$ for all $i \in \mathbb{N}^*$ we have that

$$\|z - y_i\| > \frac{i}{2i\psi(x)+4} \quad \forall i \in \mathbb{N}^*.$$

Taking $i \longrightarrow +\infty$, we get that

$$\|z - y_x\| \geq \frac{1}{2\psi(x)}$$

which contradicts the fact that $z \in B(y_x; \frac{1}{2\psi(x)})$. Then there exists $i_0 \in \mathbb{N}^*$ such that $z \in \bar{B}(y_{i_0}; \frac{i_0}{2i_0\psi(x)+4}) \subset S$. Hence $B(y_x; \frac{1}{2\psi(x)}) \subset S$ and this gives (since $S$ is closed) that $\bar{B}(y_x; \frac{1}{2\psi(x)}) \subset S$. On the other hand, the fact that

$$\|x - y_i\| \leq \frac{i}{2i\psi(x)+4}$$

for all $i \in \mathbb{N}^*$ yields (after taking $i \longrightarrow +\infty$) that $\|x - y_x\| \leq \frac{1}{2\psi(x)}$. Therefore

$$x \in \bar{B}\left(y_x\,; \frac{1}{2\psi(x)}\right) \subset S.$$

*Case 2:* $\psi(x) = 0$. We consider $s \in \operatorname{proj}_{\operatorname{bdry} S}(x)$. Clearly we have that $\theta(s) = 0$. By the continuity of $\theta(\cdot)$ on $\operatorname{bdry} S$ we get for $i \in \mathbb{N}^*$ the existence of $0 < \delta_i < \frac{1}{i}$ such that

$$0 \leq \theta(s') \leq \frac{1}{i} \quad \forall s' \in B(s;\delta_i) \cap \operatorname{bdry} S.$$

Then using the fact that for all $s' \in B(s;\delta_i) \cap \operatorname{bdry} S$, $S'$ has a proximal normal at $s'$ realized by a $\frac{1}{2\theta(s')}$-sphere if $\theta(s') \neq 0$ and $r$-sphere for all $r > 0$ if $\theta(s') = 0$, we get that for all $s' \in B(s;\delta_i) \cap \operatorname{bdry} S$, $S'$ has a proximal normal at $s'$ realized by an $\frac{i}{2}$-sphere. Now using the same ideas as in the proof of Conjecture 1.3 and the fact that $r_0 := \|x - s\|$ is less than $\frac{i}{4}$ (for $i$ sufficiently large), we can prove the existence of $y_i \in S$ such that

$$x \in \bar{B}\left(y_i; \frac{i}{4}\right) \subset S \quad \text{and} \quad \|x - y_i\| = \frac{i}{4}.$$

We denote by $\zeta_i$ the unit vector $\frac{y_i - x}{\|y_i - x\|}$. Since the sequence $(\zeta_i)_i$ is bounded, we can assume that it converges to a unit vector $\zeta_0 \in \mathbb{R}^n$. For $y_x := x + \zeta_0$, we claim that $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$. Indeed, let $z \in B(x + t(y_x - x); t)$ and assume that for all $i \in \mathbb{N}^*$, $z \notin \bar{B}(y_i; \frac{i}{4})$. We shall derive a contradiction. There exists $\epsilon > 0$ such that

$$\|z - x - t(y_x - x)\| = \|z - x - t\zeta_0\| = t - \epsilon \quad \text{and} \quad \|z - y_i\| > \frac{i}{4} \quad \forall i \in \mathbb{N}^*.$$

Hence

$$\begin{aligned}
\frac{i}{4} < \|z - y_i\| &= \left\| z - x - \frac{i}{4}\zeta_i \right\| \\
&= \left\| z - x - t\zeta_0 + t\zeta_0 - t\zeta_i + t\zeta_i - \frac{i}{4}\zeta_i \right\| \\
&\leq t - \epsilon + t\|\zeta_0 - \zeta_i\| + \left| t - \frac{i}{4} \right| \\
&= t - \epsilon + t\|\zeta_0 - \zeta_i\| + \frac{i}{4} - t \quad \text{(for } i \text{ sufficiently large)} \\
&= \frac{i}{4} - \epsilon + t\|\zeta_0 - \zeta_i\|.
\end{aligned}$$

Then $\epsilon < t\|\zeta_0 - \zeta_i\|$ which gives, since $\zeta_i \longrightarrow \zeta_0$, the desired contradiction. Hence there exists $i_0 \in \mathbb{N}^*$ such that $z \in \bar{B}(y_{i_0}; \frac{i_0}{4}) \subset S$. This gives that $B(x + t(y_x - x); t) \subset S$ and then $\bar{B}(x + t(y_x - x); t) \subset S$. Therefore

$$x \in \bar{B}(x + t(y_x - x); t) \subset S$$

for all $t > 0$, and this completes the proof of the theorem. □

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