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Pettis Integrability of Multifunctions with Values in Arbitrary Banach Spaces

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There is a rich literature describing integrability of multifunctions that take weakly compact convex subsets of a separable Banach space as their values. Most of the papers concern the Bochner type integration, but there is also quite a number of papers dealing with the Pettis integral. On the other hand almost nothing is known in case of non-separable Banach spaces. Only recently the papers [5] and [6] have been published, where the authors proved the existence of scalarly measurable selections of scalarly measurable multifunctions with weakly compact values. The aim of this paper is to fill in partially that gap by presenting a number of theorems that characterize Pettis integrable multifunctions with (weakly) compact non-separable sets as their values. Having applied the above results, I obtained a few convergence theorems, that generalize results known in case of Pettis integrable functions and in case of separably valued multifunctions.

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Introduction.

Looking through the existing abandon literature concerning integrability of multifunctions which values are weakly compact subsets of a separable Banach space, one can immediately notice that the most exploited property of the integrals is the fact that they are of the Aumann type, that is their definitions or further properties depend on the existence of measurable selections. Predominantly the integrals of the Bochner type have been investigated and I dare to say that this topic is almost exhausted, at least when values are weakly compact convex sets. The situation is a little bit more complicated in case of the Pettis integral. I recommend the paper of El Amri and Hess [10] for more details and corresponding literature.

As there were no appropriate selection theorems for scalarly measurable multifunctions with values in non-separable Banach spaces, there was also no Pettis integration theory for such multifunctions. However, a few papers published in the last decade suggest a definition of the Pettis integrability independent of selections. One can find it, for instance, in papers of Ziat [27] and El Amri - Hess [10]. I follow that way. Most

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of the main results are proved without invoking the existence of any selections. As a consequence, the methods of proofs applied here are, in several points, completely different from those used in case of separable Banach spaces. The technique applied is closer to the theory of Pettis integration of functions with values in non-separable Banach spaces. But even if one would like to adapt the methods from the separable case, it cannot be done immediately. First of all, the selections are in general not strongly measurable and this excludes general approximation approach via simple functions. The second reason is that the weak topology restricted to weakly compact sets may be now not metrizable and this immediately eliminates some methods of proofs that used to be applied in case of separable Banach spaces. From time to time I will use the beautiful result of Cascales, Kadetz and Rodriguez ([5] and [6]) who proved that each scalarly measurable multifunction with weakly compact (not necessarily convex) values has a scalarly measurable selection. But in general I will try to avoid selections.

Here are the most essential results of the paper.

1. Two complete characterizations of scalarly integrable multifunctions with convex weakly compact values that are Pettis integrable in the family of convex weakly compact sets (Theorems 2.5 and 4.6). The proof of Theorem 2.5, when restricted to functions, gives a new proof of the corresponding result of Talagrand [25] for Pettis integrable functions. The characterizations are new also in case of separable Banach spaces.

2. There is a well known result of Diestel (cf. [8]) and Dimitrov [9] that if a separable Banach space X does not contain any isomorphic copy of c_0, then each X-valued scalarly integrable function is Pettis integrable. It is also known that the result fails for non-separable spaces. I present a non-separable version of the above result for multifunctions with convex weakly compact values (Theorem 2.13). The result is new also for functions (see Theorem 2.14).

3. If Γ is a multifunction with weakly compact convex values that is Pettis integrable in the collection of weakly compact convex sets, then the sublinear operator $T_\Gamma : X^* \to L_1(\mu)$, defined by $T_\Gamma(x^*) = \mathrm{s}(x^*, \Gamma)$, (see the next section) is always weakly compact. If such a Γ is Pettis integrable in the family of compact convex sets, then the operator may fail to be compact.

I prove that each multifunction Γ with weakly compact convex values and Pettis integrable in the collection of compact convex sets has a representation $\Gamma = G + g$, where g is a Pettis integrable function and T_G is compact. In particular, all Pettis integrable selections of G have norm relatively compact ranges of their integrals (Theorems 3.3 and 3.6). This is a significant simplification of the theory.

4. I apply the above results to obtain a few convergence theorems and one Fatou type theorem for multifunctions that generalize known facts about functions and separable valued multifunctions.

5. I introduce a property of a Banach space called the subsequential weak* lifting property (Definition 3.8) that should be – in my opinion – interesting for functional analysts, but I was not able to find it anywhere.

1. Basic facts.

This section contains definitions, notation and several facts that are already mostly known but not necessarily published.

Throughout (Ω, Σ, μ) is a complete probability space, Σ_μ^+ is the collection of all sets of positive measure and X is a Banach space with its dual X^*. The closed unit ball of X is denoted by $B(X)$. $\mathrm{cwk}(X)$ denotes the family of all nonempty convex weakly compact subsets of X and $\mathrm{ck}(X)$ is the collection of all nonempty convex and compact subsets of X. $\mathrm{cb}(X)$ is the collection of all nonempty closed bounded and convex subsets of X and $\mathrm{c}(X)$ denotes the collection of all nonempty closed convex subsets of X. For every $C \in \mathrm{c}(X)$ the *support function of* C is denoted by $\mathrm{s}(\cdot, C)$ and defined on X^* by $\mathrm{s}(x^*, C) = \sup\{\langle x^*, x\rangle \colon\ x \in C\}$, for each $x^* \in X^*$.

$\tau(X^*, X)$ denotes the topology of uniform convergence on elements of $\mathrm{cwk}(X)$ and $\tau_c(X^*, X)$ is the topology of uniform convergence on convex compact subsets of X. It is known that $\mathrm{s}(\cdot, C)$ is $\tau_c(X^*, X)$-continuous if and only if it is weak*-continuous on $B(X^*)$. The weak*-topology of X^* will be denoted by $\sigma(X^*, X)$. If Y is a subspace of X, then $\tau(X^*, Y)$ denotes the topology of uniform convergence on weakly compact subsets of Y, $\sigma(X^*, Y)$ denotes the topology of pointwise convergence on Y and $Y^\perp$ is the annihilator of Y in X^*.

X^* is weak*-angelic if for each bounded set $B \subset X^*$ the weak*-closure of B is equal to the set of weak*-limits of sequences from B.

A set $V \subset X$ is called limited if $x_n^* \to 0$ uniformly on V, for every sequence $x_n^* \to 0$ in the weak* topology of X^*. It is well known that limited sets are conditionally weakly compact (i.e. sequences have weakly Cauchy subsequences) [3], and if X is weakly compactly generated, then limited subsets of X are norm relatively compact.

A map $\Gamma \colon \Omega \to \mathrm{c}(X)$ is called a *multifunction*. The multifunction Γ is non-negative, if for each $x^* \in X^*$, we have $\mathrm{s}(x^*, \Gamma) \geq 0$, a.e. A multifunction $\widetilde{\Gamma} \colon \Omega \to \mathrm{c}(X)$ is dominated by Γ if $\widetilde{\Gamma}(\omega) \subseteq \Gamma(\omega)$, for every $\omega \in \Omega$. A multifunction $\widetilde{\Gamma} \colon \Omega \to \mathrm{c}(X)$ is called to be an *extremal face* of $\Gamma \colon \Omega \to \mathrm{c}(X)$, if there is a functional $x_0^* \in X^*$ such that $\widetilde{\Gamma}(\omega) = \{x \in \Gamma(\omega) : x_0^*(x) = \mathrm{s}(x_0^*, \Gamma(\omega))\}$, for every $\omega \in \Omega$. We associate with each Γ the set

$$\mathcal{Z}_\Gamma := \{\mathrm{s}(x^*, \Gamma) : \|x^*\| \leq 1\},$$

where we consider functions, not equivalence classes of a.e. equal functions.

A function $f \colon \Omega \to X$ is called a *selection of* Γ if $f(\omega) \in \Gamma(\omega)$, for every $\omega \in \Omega$. $f \colon \Omega \to X$ is called a *quasi selection of* Γ if $x^* f(\omega) \in x^* \Gamma(\omega)$ a.e., for each $x^* \in X^*$ separately. One can easily check that a multifunction Γ is non-negative if and only if the zero function is its quasi selection. If $A \subset X$ is nonempty, then we write $|A| := \sup\{\|x\| \colon x \in A\}$.

A map $M \colon \Sigma \to \mathrm{c}(X)$ is called a *weak multimeasure* if $\mathrm{s}(x^*, M(\cdot))$ is a measure, for every $x^* \in X^*$. If M is a point map, then we talk about measure. If $M \colon \Sigma \to \mathrm{c}(X)$ is countably additive in the Hausdorff metric, then it is called an h-multimeasure. It is known that if $M \colon \Sigma \to \mathrm{cwk}(X)$, then M is a weak multimeasure if and only if it is an h-multimeasure. A weak multimeasure $M \colon \Sigma \to \mathrm{c}(X)$ is said to be *μ-continuous* (or *absolutely continuous with respect to* μ) if $\mu(E) = 0$ yields $M(E) = \{0\}$, for every $E \in \Sigma$.

A vector measure $m \colon \Sigma \to X$ such that $m(A) \in M(A)$, for every $A \in \Sigma$, is called a *selection* of M. $\mathcal{S}(M)$ will denote the set of all countably additive selections of M.

A family $W \subset L_1(\mu)$ is uniformly integrable if W is bounded in $L_1(\mu)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\sup_{f\in W} \int_A |f| \, d\mu < \varepsilon$.

Definition 1.1. A multifunction Γ is said to be *scalarly measurable* (or *weakly measurable*) if for every $x^* \in X^*$, the map $\mathrm{s}(x^*, \Gamma(\cdot))$ is measurable. A multifunction $\Gamma \colon \Omega \to \mathrm{c}(X)$ is *scalarly integrable* if $\mathrm{s}(x^*, \Gamma)$ is integrable for every $x^* \in X^*$. $\Gamma \colon \Omega \to \mathrm{c}(X)$ is scalarly bounded if there is a constant $M \geq 0$ such that for every $x^* \in X^*$

$$|\,\mathrm{s}(x^*, \Gamma)| \leq M\|x^*\| \quad \text{a.e.}$$

A function $f \colon \Omega \to \mathbb{R}$ is quasi-integrable (cf. [22]) if the integral $\int_\Omega f \, d\mu$ exists. A multifunction $\Gamma \colon \Omega \to \mathrm{c}(X)$ is *scalarly quasi-integrable* (see [10]) if $\mathrm{s}(x^*, \Gamma)$ is quasi-integrable for every $x^* \in X^*$. A scalarly quasi-integrable multifunction $\Gamma \colon \Omega \to \mathrm{c}(X)$ is *Pettis integrable* in $\mathrm{c}(X)$ [$\mathrm{cb}(X)$, $\mathrm{ck}(X)$, $\mathrm{cwk}(X)$] if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in \mathrm{c}(X)$ [$\mathrm{cb}(X)$, $\mathrm{ck}(X)$, $\mathrm{cwk}(X)$, respectively] such that

$$\mathrm{s}(x^*, M_\Gamma(A)) = \int_A \mathrm{s}(x^*, \Gamma) \, d\mu \quad \text{for every } x^* \in X^*. \tag{1}$$

We set $M_\Gamma(A) := (P) \int_A \Gamma \, d\mu$ and call $M_\Gamma(A)$ the *Pettis integral of* Γ over A. It follows from (1) that M_Γ is a weak multimeasure that is μ-continuous. □

As observed in [10], if $\Gamma \colon \Omega \to \mathrm{c}(X)$ is Pettis integrable in $\mathrm{c}(X)$, then

$$\int_\Omega \mathrm{s}(x^*, \Gamma)^- \, d\mu < \infty, \quad \text{for every } x^* \in X^*. \tag{2}$$

(In the formula above $\mathrm{s}(x^*, \Gamma)^-$ is the negative part of $\mathrm{s}(x^*, \Gamma)$).

Similarly, if Γ is Pettis integrable in $\mathrm{cb}(X)$, then Γ is scalarly integrable. And in fact, in this paper, I will mainly concentrate on multifunctions which are integrable in the family of weakly compact sets. Therefore, the assumption of scalar integrability will be often applied.

If Γ is an X-valued function, then we have a Pettis integrable function. Identifying scalarly equivalent functions we obtain a linear space $\mathbb{P}(\mu, X)$ of X-valued Pettis integrable functions. It is well known that $\mathbb{P}(\mu, X)$ can be normed by $\|f\|_P := \sup_{\|x^*\|\leq 1} \int_\Omega |x^* f| \, d\mu$ and that this norm is equivalent to $|||f|||_P := \sup\{\|M_f(E)\| \colon E \in \Sigma\}$. Moreover, $\mathbb{P}(\mu, X)$ is non-complete if X is infinite dimensional and μ is not purely atomic.

Proposition 1.2 (Compare with [20, Proposition 3.1]). *Let* $\Gamma \colon \Omega \to \mathrm{cb}(X)$ *be scalarly measurable. Then* Γ *can be represented in the form*

$$\Gamma(\omega) = \sum_n \Gamma(\omega)\chi_{E_n} \tag{3}$$

where the sets $E_n \in \Sigma$ are pairwise disjoint, $\mu\left(\bigcup_n E_n\right) = \mu(\Omega)$ and each $\Gamma\colon E_n \to$ $\mathrm{cb}(X)$ is scalarly bounded.

Proof. Since $|\,\mathrm{s}(x^*, \Gamma(\omega))| \le |\Gamma(\omega)|\|x^*\|$ for every ω and x^*, we get the existence of a measurable function $\psi_\Gamma\colon \Omega \to [0, +\infty)$ such that:

(α) for each $x^* \in X^*$ $|\,\mathrm{s}(x^*, \Gamma(\omega))| \le \psi_\Gamma(\omega)\|x^*\|$ for almost every ω;
(β) $\psi_\Gamma(\omega) \le |\Gamma(\omega)|$ for every ω;
(γ) If φ is another measurable function satisfying the conditions (α) and (β) (with ψ_Γ replaced by φ), then $\psi_\Gamma \le \varphi$ a.e.

Now, if $E_n := \{\omega\colon n-1 \le \psi_\Gamma(\omega) < n\}$ and $E \in \Sigma$, then we get the representation (3). □

I am going to prove a characterization of the Pettis integrability of multifunctions analogous to the operator characterization of the Pettis integrability: $f\colon \Omega \to X$ is Pettis integrable if and only if the operator $T_f\colon X^* \to L_1(\mu)$ given by $T_f(x^*) = x^*f$ is weak*-weakly continuous. In case of scalarly integrable $\Gamma\colon \Omega \to \mathrm{c}(X)$ we introduce the operator $T_\Gamma\colon X^* \to L_1(\mu)$, defined by $T_\Gamma(x^*) := \mathrm{s}(x^*, \Gamma)$.

We say that T_Γ is compact (weakly compact) if the set $T_\Gamma(B(X^*))$ is norm relatively compact (weakly relatively compact) in $L_1(\mu)$.

The next proposition is a well known result that is usually formulated without any relation to Pettis integration (see [7], Theorem II.16).

Proposition 1.3. *Let X be an arbitrary Banach space and $\Gamma\colon \Omega \to \mathrm{c}(X)$ be scalarly quasi-integrable. Then Γ is Pettis-integrable in $\mathrm{c}(X)$ if and only if the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Gamma)\, d\mu$ is weak* lower semicontinuous for every $E \in \Sigma$.*

If Γ is scalarly integrable, then the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Gamma)\, d\mu$ is weak lower semicontinuous for every $E \in \Sigma$ if and only if Γ is Pettis-integrable in $\mathrm{cb}(X)$. The operator T_Γ is then bounded and norm-weakly continuous. Moreover, the set $M_\Gamma(\Sigma) := \bigcup_{E\in\Sigma} M_\Gamma(E)$ is bounded.*

Proof. We will prove only the last assertion. Assume that Γ is Pettis integrable in $\mathrm{cb}(X)$. If $x^* \in X^*$, then

$$\begin{aligned} \int_\Omega |\,\mathrm{s}(x^*, \Gamma(\omega))|\, d\mu(\omega) &\le 2 \sup_{E\in\Sigma} \left| \int_E \mathrm{s}(x^*, \Gamma(\omega))\, d\mu(\omega) \right| \\ &= 2 \sup_{E\in\Sigma} |\,\mathrm{s}(x^*, M_\Gamma(E))| < \infty, \end{aligned} \tag{4}$$

where the last inequality follows from the fact that $\mathrm{s}(x^*, \cdot)$ is, for every x^*, a real measure.

It follows now from (4) and the Banach–Steinhaus Theorem that $M_\Gamma(\Sigma)$ is bounded and so (4) yields the boundedness of T_Γ:

$$\begin{aligned} \|T_\Gamma\| = \sup_{\|x^*\|\le 1} \|T_\Gamma(x^*)\|_{L_1} &= \sup_{\|x^*\|\le 1} \int_\Omega |\,\mathrm{s}(x^*, \Gamma(\omega))|\, d\mu(\omega) \\ &\le \sup\{\|x\|\colon x \in M_\Gamma(\Sigma)\} < \infty. \end{aligned}$$

We have for each $E \in \Sigma$ and $x^* \in X^*$

$$\mathrm{s}(x^*, M_\Gamma(E)) = \int_E \mathrm{s}(x^*, \Gamma)\, d\mu = \langle T_\Gamma(x^*), \chi_E \rangle, \tag{5}$$

where $M_\Gamma(E) \in \mathrm{cb}(X)$ and the left hand side term of (5) is norm continuous. So the same holds true for the right hand side term. It follows that T_Γ is norm-weak continuous. □

The following result is the basic operator characterization of Pettis integrability. It is a direct generalization of the classical operator characterization of Pettis integrable functions (cf. [21, Theorem 4.1]).

Theorem 1.4. *Let X be an arbitrary Banach space and $\Gamma\colon \Omega \to \mathrm{c}(X)$ be scalarly integrable. Then:*

Γ is Pettis-integrable in $\mathrm{cwk}(X)$ *if and only if the operator T_Γ is $\tau(X^*, X)$-weakly continuous.*

If Γ is non-negative, then T_Γ is $\tau(X^*, X)$-norm continuous.

Γ is Pettis-integrable in $\mathrm{ck}(X)$ *if and only if the operator T_Γ is $\tau_c(X^*, X)$-weakly continuous on $B(X^*)$.*

If Γ is non-negative, then T_Γ is $\tau_c(X^*, X)$-norm continuous on $B(X^*)$.

Proof. Assume the $\tau(X^*, X)$-weak continuity of T_Γ on X^* and define for an arbitrary $E \in \Sigma$ of positive measure a function $\varphi_E\colon X^* \to \mathbb{R}$ by the formula

$$\varphi_E(x^*) := \int_E \mathrm{s}(x^*, \Gamma(\omega))\, d\mu = \langle T_\Gamma(x^*), \chi_E \rangle.$$

According to the continuity assumption, the right hand side of the equality is $\tau(X^*, X)$-continuous on X^*. Hence, the same holds true for the left one. This means however that φ_E is sublinear and $\tau(X^*, X)$-continuous functional on X^*. We are going to prove that φ_E is also weak* lower semicontinuous. It is enough to show that for each $\alpha \in \mathbb{R}$ the set $\{x^* \in X^* \colon \varphi_E(x^*) \le \alpha\}$ is weak*-closed, but this is immediate, since this set is convex and $\tau(X^*, X)$-closed, due to the $\tau(X^*, X)$-continuity of φ_E.

Thus, there is a closed convex set C_E such that $\varphi_E(x^*) = \mathrm{s}(x^*, C_E)$, for every x^*. Since φ_E is $\tau(X^*, X)$-continuous, the set C_E is weakly compact.

If T_Γ is weak*-weak continuous on $B(X^*)$, then the proof is even simpler because the required weak* lower semicontinuity is immediate.

Assume now that $\Gamma\colon \Omega \to X$ is Pettis-integrable in $\mathrm{cwk}(X)$. For each $E \in \Sigma$ there is $M_\Gamma(E) \in \mathrm{cwk}(X)$ such that for every $x^* \in X^*$ we have

$$\mathrm{s}(x^*, M_\Gamma(E)) = \int_E \mathrm{s}(x^*, \Gamma(t))\, d\mu(\omega) = \langle T_\Gamma(x^*), \chi_E \rangle, \tag{6}$$

and T_Γ is bounded (by Proposition 1.3).

According to [13, Theorem 8.5.10] the range $M_\Gamma(\Sigma) = \bigcup_{E\in\Sigma} M_\Gamma(E)$ of the multimeasure M_Γ is a relatively weakly compact subset of X. We get from (4) the inequalities

$$\begin{aligned} \|T_\Gamma(x^*)\|_{L_1} &= \int_\Omega |\,\mathrm{s}(x^*,\Gamma(\omega))|\,d\mu(\omega) \\ &\le 2\sup_{E\in\Sigma} |\,\mathrm{s}(x^*, M_\Gamma(E))| \le 2|\,\mathrm{s}(x^*, \overline{\mathrm{conv}} M_\Gamma(\Sigma))| \end{aligned} \tag{7}$$

Since $\overline{\mathrm{conv}} M_\Gamma(E)$ is weakly compact, the support function $\mathrm{s}(\cdot, M_\Gamma(E))$ is $\tau(X^*,X)$-continuous. Take now an arbitrary net $\{x^*_\alpha\} \subset X^*$ that is $\tau(X^*,X)$-convergent to zero and $g \in L_\infty(\mu)$. Then

$$|\langle T_\Gamma(x^*_\alpha), g\rangle| \le \|T_\Gamma(x^*_\alpha)\|_{L_1} \cdot \|g\|_{L_\infty} < 2|\,\mathrm{s}(x^*_\alpha, \overline{\mathrm{conv}} M_\Gamma(\Sigma))|\,\|g\|_{L_\infty}.$$

It follows that $\langle T_\Gamma(\cdot), g\rangle$ is $\tau(X^*,X)$-continuous at zero, what proves the $\tau(X^*,X)$-weak continuity of T_Γ.

If Γ is non-negative and $\{x^*_\alpha\} \subset X^*$ is $\tau(X^*,X)$-convergent to zero, then

$$0 \le \lim_\alpha \int_\Omega \mathrm{s}(x^*_\alpha, \Gamma)\,d\mu = \lim_\alpha \mathrm{s}(x^*_\alpha, M_\Gamma(\Omega)) = 0.$$

But this is just the convergence in $L_1(\mu)$.

Assume now that $\Gamma : \Omega \to X$ is Pettis-integrable in $\mathrm{ck}(X)$. We have the relation (6) with $M_\Gamma(E) \in \mathrm{ck}(X)$.

We are going to prove now the weak*-weak continuity of T_Γ on $B(X^*)$. The proof has to be a little bit different from that for weakly compact sets because the set $M_\Gamma(\Sigma)$ is not always relatively compact. Since $M_\Gamma(E)$ is compact, the support function $\mathrm{s}(\cdot, M_\Gamma(E))$ is weak* continuous on $B(X^*)$ and so it follows that $\langle T_\Gamma(\cdot), \chi_E\rangle$ is a weak*-continuous sublinear functional on $B(X^*)$. Hence, for each real-valued simple function h the functional $\langle T_\Gamma(\cdot), h\rangle$ is weak*-continuous on $B(X^*)$. Take now an arbitrary $g \in L_\infty(\mu)$, and let $\{x^*_\alpha\} \subset B(X^*)$ be an arbitrary net which is weak*-convergent to zero. Let us fix $\varepsilon > 0$. Then there is a simple function $h_\varepsilon : \Omega \to \mathbb{R}$ with $\|g - h_\varepsilon\|_{L_\infty} < \varepsilon$. By the weak*-continuity of $\langle T_\Gamma(\cdot), h_\varepsilon\rangle$ there is α_0 such that for every $\alpha > \alpha_0$ we have $|\langle T_\Gamma(x^*_\alpha), h_\varepsilon\rangle| < \varepsilon$. Consequently, if $\alpha > \alpha_0$, then

$$\begin{aligned} |\langle T_\Gamma(x^*_\alpha), g\rangle| &\le |\langle T_\Gamma(x^*_\alpha), g\rangle - \langle T_\Gamma(x^*_\alpha), h_\varepsilon\rangle| + |\langle T_\Gamma(x^*_\alpha), h_\varepsilon\rangle| \\ &\le \|T_\Gamma(x^*_\alpha)\|_{L_1} \cdot \|g - h_\varepsilon\|_{L_\infty} + \varepsilon < \|T_\Gamma\|\varepsilon + \varepsilon\,. \end{aligned}$$

Hence $\langle T_\Gamma(\cdot), g\rangle$ is weak*-continuous on $B(X^*)$, what proves the $\tau_c(X^*,X)$-weak continuity of T_Γ.

If Γ is non-negative, then the proof is similar to the previous one. □

Another version of the above result has been recently and independently obtained by Cascales, Kadets and Rodriguez [5]. Instead of the operator T_Γ they considered for every $E \in \Sigma$ the sublinear functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Gamma)\,d\mu$.

The next result, proved independently in [5], is essential for our further investigations.

Corollary 1.5. *Let $\varGamma\colon \varOmega \to \mathrm{c}(X)$ and $\widetilde{\varGamma}\colon \varOmega \to \mathrm{c}(X)$ be multifunctions such that for every $x^* \in X^*$ the inequality $\mathrm{s}(x^*, \widetilde{\varGamma}(\omega)) \le \mathrm{s}(x^*, \varGamma(\omega))$ holds true for almost every $\omega \in \varOmega$ (the exceptional sets depend on x^*). Assume that $\varGamma$ is Pettis integrable in* $\mathrm{cwk}(X)$ *(or in* $\mathrm{ck}(X)$*). Then, if $\widetilde{\varGamma}$ is scalarly measurable, then it is also Pettis integrable in* $\mathrm{cwk}(X)$ *(resp. in* $\mathrm{ck}(X)$*). In particular, every scalarly measurable selection of $\varGamma$ is then Pettis integrable.*

Proof. Let us consider the case when $\varGamma$ is Pettis integrable in $\mathrm{cwk}(X)$. We have for every $x^* \in X^*$ the inequalities

$$-\,\mathrm{s}(-x^*, \varGamma(\omega)) \le -\,\mathrm{s}(-x^*, \widetilde{\varGamma}(\omega)) \le \mathrm{s}(x^*, \widetilde{\varGamma}(\omega)) \le \mathrm{s}(x^*, \varGamma(\omega)) \quad \mu - a.e.$$

Hence, if $E \in \Sigma$, then

$$\begin{aligned} -\,\mathrm{s}(-x^*, M_\varGamma(E)) = &-\int_E \mathrm{s}(-x^*, \varGamma)\, d\mu \le -\int_E \mathrm{s}(-x^*, \widetilde{\varGamma})\, d\mu \\ &\le \int_E \mathrm{s}(x^*, \widetilde{\varGamma})\, d\mu \le \int_E \mathrm{s}(x^*, \varGamma)\, d\mu = \mathrm{s}(x^*, M_\varGamma(E)). \end{aligned}$$

The boundary functions are $\tau(X^*, X)$-continuous, by Theorem 1.4, and so this yields the $\tau(X^*, X) - \sigma(L_1(\mu), L_\infty(\mu))$-continuity of $T_{\widetilde{\varGamma}}$. Consequently, according to Corollary 1.4, the multifunction $\widetilde{\varGamma}$ is Pettis integrable in $\mathrm{cwk}(X)$. □

It is well known that $\tau_c(X^*, X)$-continuity is equivalent to the $\sigma(X^*, X)$-continuity on $B(X^*)$. It turns out that $\tau(X^*, X)$-continuity of $T_\varGamma$ on $B(X^*)$ provides a sufficient condition for Pettis integrability in $\mathrm{cb}(X)$.

Proposition 1.6. *Let X be an arbitrary Banach space and $\varGamma\colon \varOmega \to \mathrm{c}(X)$ be scalarly integrable. If the operator $T_\varGamma$ is $\tau(X^*, X)$-weakly continuous on $B(X^*)$, then $\varGamma$ is Pettis-integrable in* $\mathrm{cb}(X)$*.*

Proof. Assume the $\tau(X^*, X)$-weak continuity of $T_\varGamma$ on $B(X^*)$ and define for an arbitrary $E \in \Sigma$ of positive measure a function $\varphi_E\colon X^* \to \mathbb{R}$ by the formula

$$\varphi_E(x^*) := \int_E \mathrm{s}(x^*, \varGamma(\omega))\, d\mu = \langle T_\varGamma(x^*), \chi_E \rangle.$$

According to the continuity assumption, the right hand side of the equality is $\tau(X^*, X)$-continuous on $B(X^*)$. But $T_\varGamma$ is positively homogeneous and so $T_\varGamma$ is $\tau(X^*, X)$-continuous on all balls $B(X^*, r)$ centered at zero and of positive radius r. Hence, the same holds true for φ_E. This means, however, that φ_E is sublinear on X^* and $\tau(X^*, X)$-continuous functional on all balls $B(X^*, r)$. We are going to prove that φ_E is also weak* lower semicontinuous. It is enough to show that for each $\alpha \in \mathbb{R}$ the set $\{x^* \in X^*\colon \varphi_E(x^*) \le \alpha\}$ is weak*-closed. Since it is also convex, it follows from the Krein-Šmulian theorem that it is weak*-closed if and only if its intersections with all closed balls $B(X^*, r)$ centered at zero are weak*-closed. But by the continuity of φ_E the sets $\{x^* \in B(X^*, r)\colon \varphi_E(x^*) \le \alpha\}$ are $\tau(X^*, X)$-closed. Since they are convex, they are also weak*-closed.

Thus, there is a closed convex set $C_E \subset X$ such that $\varphi_E(x^*) = \mathrm{s}(x^*, C_E)$, for every x^*. Since Γ is scalarly integrable C_E is bounded and so Γ is Pettis integrable in $\mathrm{cb}(X)$. □

The reverse implication in Proposition 1.6 fails (see Example 1.11).

Proposition 1.7. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be Pettis integrable in $\mathrm{cb}(X)$. If each extremal face of Γ is Pettis integrable in $\mathrm{cb}(X)$, then Γ is Pettis integrable in $\mathrm{cwk}(X)$.*

Proof. Let $(P)\int_\Omega \Gamma\, d\mu = M(\Omega) \in \mathrm{cb}(X)$. We are going to prove that $M(\Omega) \in \mathrm{cwk}(X)$. To do it, let $x_0^* \in X^*$ be arbitrary. We shall prove that x_0^* attains its supremum on $M(\Omega)$. It will follow from the result of James [15] that $M(\Omega)$ is weakly compact. Let $G\colon \Omega \to \mathrm{cwk}(X)$ be the extremal face of Γ defined by

$$G(\omega) := \{x \in \Gamma(\omega)\colon x_0^*(x) = \mathrm{s}(x_0^*, \Gamma(\omega))\}.$$

Due to [26, Lemma 3] the multifunction G is scalarly measurable. Moreover, by the assumption, G is also Pettis integrable in $\mathrm{cb}(X)$. Thus, we obtain the existence of a set $P_\Omega \in \mathrm{cb}(X)$ such that

$$\mathrm{s}(x^*, P_\Omega) = \int_\Omega \mathrm{s}(x^*, G)\, d\mu \quad \text{for every } x^* \in X^*.$$

Moreover, since $\mathrm{s}(x^*, P_\Omega) \le \mathrm{s}(x^*, M(\Omega))$, for every $x^* \in X^*$, we have $P_\Omega \subseteq M(\Omega)$. Directly from the definition of G we have

$$\mathrm{s}(-x_0^*, G(\omega)) = \sup_{x\in G(\omega)} [-x_0^*(x)] = -\inf_{x\in G(\omega)} x_0^*(x) = -\,\mathrm{s}(x_0^*, \Gamma(\omega)).$$

As a result, we obtain the following series of equalities:

$$\begin{aligned}\mathrm{s}(x_0^*, P_\Omega) &= \int_\Omega \mathrm{s}(x_0^*, G)\, d\mu = \int_\Omega \mathrm{s}(x_0^*, \Gamma)\, d\mu \\ &= -\int_\Omega \mathrm{s}(-x_0^*, G)\, d\mu = -\,\mathrm{s}(-x_0^*, P_\Omega) = \inf_{x\in P_\Omega} x_0^*(x).\end{aligned}$$

It follows that x_0^* is constant on the set P_Ω and so x_0^* attains its supremum on P_Ω. But

$$\mathrm{s}(x_0^*, P_\Omega) = \int_\Omega \mathrm{s}(x_0^*, G)\, d\mu = \int_\Omega \mathrm{s}(x_0^*, \Gamma)\, d\mu = \mathrm{s}(x_0^*, M(\Omega)),$$

and so x_0^* attains its supremum on the set $M(\Omega)$. In the above proof the set Ω may be replaced by any $E \in \Sigma$.

Thus, we have proven that Γ is Pettis integrable in $\mathrm{cwk}(X)$. □

One may ask whether the assumption of the Pettis integrability of Γ in $\mathrm{cb}(X)$, in Proposition 1.7, would be sufficient. A negative answer is given by Example 1.12. But with the help of Proposition 1.7 we achieve the following remarkable fact supplementing Theorem 1.4 and Proposition 1.6:

Theorem 1.8. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction. Then Γ is Pettis integrable in* $\mathrm{cwk}(X)$ *if and only if T_Γ is $\tau(X^*,X)$-weakly continuous on $B(X^*)$.*

Proof. One of the implications is obvious so let us assume that T_Γ is $\tau(X^*,X)$-weakly continuous on $B(X^*)$. By Proposition 1.6 Γ is Pettis integrable in $\mathrm{cb}(X)$. If $\Delta : \Omega \to \mathrm{cwk}(X)$ is scalarly measurable and dominated by Γ, then T_Δ is also $\tau(X^*,X)$-weakly continuous on $B(X^*)$. Consequently, Δ is also Pettis integrable in $\mathrm{cb}(X)$. Proposition 1.7 yields the Pettis integrability of Γ in $\mathrm{cwk}(X)$. □

The next theorem is a consequence of the just proved theorem. It is a generalization of [25, Theorem 6-1-2] formulated in the language of stable sets.

Definition 1.9 (Fremlin, Talagrand). Let $\mathcal{H}$ be a collection of real valued functions defined on Ω. $\mathcal{H}$ is said to be stable if for each $A \in \Sigma_\mu^+$ and arbitrary reals $\alpha < \beta$ there exist $k, l \in \mathbb{N}$ satisfying the inequality

$$\mu_{k+l}^*\left(\bigcup_{f\in\mathcal{H}} \{f<\alpha\}^k \times \{f>\beta\}^l \cap A^{k+l}\right) < \mu(A)^{k+l}\,,$$

where μ_{k+l} is the direct product of $k+l$ copies of μ. □

If $\mathcal{H}$ is stable and pointwise bounded, then its pointwise closure is also stable (see [25]).

Theorem 1.10. *Let $\Gamma : \Omega \to \mathrm{cwk}(X)$ be scalarly integrable. If $\mathcal{Z}_\Gamma$ is stable and uniformly integrable, then Γ is Pettis integrable in* $\mathrm{cwk}(X)$.

Proof. Let $\langle x_\alpha^*\rangle_{\alpha\in A}$ be a net of points from $B(X^*)$ converging in the Mackey topology $\tau(X^*,X)$ to $x_0^* \in B(X^*)$. Since each set $\Gamma(\omega)$ is weakly compact, we have the pointwise convergence $\lim_\alpha \mathrm{s}(x_\alpha^*,\Gamma(\omega)) = \mathrm{s}(x_0^*,\Gamma(\omega))$. But as the set $\mathcal{Z}_\Gamma$ is stable, it follows from [25, Theorem 9-5-2] that $\lim_\alpha \mathrm{s}(x_\alpha^*,\Gamma) = \mathrm{s}(x_0^*,\Gamma)$ in measure. But $\mathcal{Z}_\Gamma$ is uniformly integrable and so we may apply the Vitali convergence theorem for nets (see [22, Proposition II.5.4+II.5.6]). As a result we obtain the convergence

$$\lim_\alpha \int_\Omega |\,\mathrm{s}(x_\alpha^*,\Gamma) - \mathrm{s}(x_0^*,\Gamma)|\,d\mu = 0.$$

But the above convergence means that T_Γ is $\tau(X^*,X)$-norm continuous on $B(X^*)$. We may apply Theorem 1.8 to get the Pettis integrability of Γ in $\mathrm{cwk}(X)$. □

Example 1.11. (The separable version of this example can be found in [10].) Let X be an arbitrary (but preferably non-separable) Banach space. Let $f\colon \Omega \to X$ be a scalarly integrable function and let $r\colon \Omega \to (0,\infty)$ be an integrable function. Define $\Gamma\colon \Omega \to \mathrm{cb}(X)$ by $\Gamma(\omega) := B(f(\omega), r(\omega))$, where $B(x,r)$ is the closed ball with its center in x and of radius r. One can easily check that $\mathrm{s}(x^*,\Gamma(\omega)) = x^*f(\omega)+r(\omega)\|x^*\|$ and so Γ is scalarly integrable.

If we assume that f is Pettis integrable, then $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cb}(X)$ and

$$(P)\qquad \int_E \mathit{\Gamma}\,d\mu = B\left(\int_E f\,d\mu, \int_E r\,d\mu\right) \quad \text{for every } E\in\mathit{\Sigma}.$$

One can easily see that the family $\{\mathrm{s}(x^*,\mathit{\Gamma})\colon \|x^*\|\le 1\}$ of support functions is uniformly integrable (because the set $\{x^*f\colon \|x^*\|\le 1\}$ is uniformly integrable) and so $T_{\mathit{\Gamma}}$ is weakly compact.

If $X=l_1$ and $e_n^*=\langle\delta_{n,k}\rangle_{k=1}^\infty$ is the standard weak* base of l_∞, then $e_n^*\to 0$ in $\sigma(l_\infty,l_1)$ and $\mathrm{s}(e_n^*,\mathit{\Gamma}(\omega))\to r(\omega)>0$. Thus, $T_{\mathit{\Gamma}}$ is not weak*-weakly continuous on $B(l_\infty)$. But as l_1 has the Shur property, we have $\tau(l_\infty,l_1)=\sigma(l_\infty,l_1)$ on $B(l_\infty)$. Consequently, $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cb}(l_1)$ but $T_{\mathit{\Gamma}}$ is not $\tau(l_\infty,l_1)$-weakly continuous on $B(l_\infty)$. □

Example 1.12. Let $f\colon[0,1]\to c_0$ be a scalarly integrable function (with respect to the Lebesgue measure) that is not Pettis integrable (cf. [8]). In particular the operator $T_f:[0,1]\to L_1[0,1]$ is not weakly compact. Define a $\mathrm{ck}(c_0)$-valued multifunction by $\mathit{\Gamma}(t):=\mathrm{conv}\{0,f(t)\},\ t\in[0,1]$. One can easily see that $\mathit{\Gamma}$ is scalarly integrable. As the zero function is a selection of $\mathit{\Gamma}$, it follows from [10, Theorem 3.7] that $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cb}(c_0)$. On the other hand, the function f is a non-Pettis integrable selection of $\mathit{\Gamma}$ and so $\mathit{\Gamma}$ is not Pettis integrable in $\mathrm{cwk}(c_0)$ (see [10, Theorem 5.4] or apply Corollary 1.5.) □

Remark 1.13. The above examples show that in case of multifunctions integrable in $\mathrm{cb}(X)$ the operator $T_{\mathit{\Gamma}}$ may be weakly compact but may also fail to be weakly compact. We will see that its behaviour is more stable in case of integrability in $\mathrm{cwk}(X)$. Moreover, if $\mathit{\Gamma}$ is Pettis integrable only in $\mathrm{cb}(X)$, then the conclusion about Pettis integrability of $\widetilde{\mathit{\Gamma}}$ in Corollary 1.5 may fail. □

We finish this section with a Fatou type lemma, that seems to be much less popular than the classical one for non-negative functions. It will be applied in the proof of Proposition 6.3.

Proposition 1.14. *If $\langle f_n\rangle$ is a uniformly integrable sequence of real valued functions defined on $(\mathit{\Omega},\mathit{\Sigma},\mu)$, then $\limsup_n f_n$ and $\liminf_n f_n$ are quasi-integrable and*

$$\int_E \liminf_n f_n\,d\mu \le \liminf_n \int_E f_n\,d\mu \le \limsup_n \int_E f_n\,d\mu \le \int_E \limsup_n f_n\,d\mu,$$

for every $E\in\mathit{\Sigma}$.

2. Pettis integrability in $\mathrm{cwk}(X)$.

Definition 2.1. We say that a space $Y\subset X$ determines a multifunction $\mathit{\Gamma}\colon\mathit{\Omega}\to \mathrm{c}(X)$ if $\mathrm{s}(x^*,\mathit{\Gamma})=0$ μ-a.e. for each $x^*\in Y^\perp$ (the exceptional sets depend on x^*). □

One can easily see that a space $Y\subset X$ determines a multifunction $\mathit{\Gamma}\colon\mathit{\Omega}\to\mathrm{c}(X)$ if and only if $\mathrm{s}(x_1^*,\mathit{\Gamma})=\mathrm{s}(x_2^*,\mathit{\Gamma})$ μ-a.e. for each $(x_1^*,x_2^*)\in X^*\times X^*$ such that $x_1^*-x_2^*\in Y^\perp$ (the exceptional sets depend on (x_1^*,x_2^*)). Indeed, if Y determines $\mathit{\Gamma}$ and

$x_1^* - x_2^* \in Y^\perp$, then $|\,\mathrm{s}(x_1^*, \Gamma) - \mathrm{s}(x_2^*, \Gamma)| \le \max\{|\,\mathrm{s}(x_1^* - x_2^*, \Gamma)|, |\,\mathrm{s}(x_2^* - x_1^*, \Gamma)|\} = 0$ a.e.

It follows that if x_1^* and x_2^* are arbitrary extensions of $y^* \in Y^*$, then $\mathrm{s}(x_1^*, \Gamma) = \mathrm{s}(x_2^*, \Gamma)$ a.e.

It is well known that Pettis integrable functions are determined by WCG spaces. It is my aim now to show that a similar property is shared also by $\mathrm{cwk}(X)$-integrable multifunctions.

Proposition 2.2. *If $\Gamma\colon \Omega \to \mathrm{c}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$, then it is determined by a WCG subspace of X and T_Γ is weakly compact.*

Proof. According to [13, Theorem 8.5.10] the range $M_\Gamma(\Sigma) = \bigcup_{E\in\Sigma} M_\Gamma(E)$ of the weak multimeasure M_Γ is a relatively weakly compact subset of X. Let $Y \subset X$ be the closed linear space generated by $M_\Gamma(\Sigma)$. Then Y is a weakly compactly generated subspace of X. If x_1^*, x_2^* are such that $x_1^* - x_2^* \in Y^\perp$, then we have

$$\int_A \mathrm{s}(x_1^*, \Gamma)\, d\mu = \langle x_1^*, M_\Gamma(A)\rangle = \langle x_2^*, M_\Gamma(A)\rangle = \int_A \mathrm{s}(x_2^*, \Gamma)\, d\mu$$

for every $A \in \Sigma$. Consequently $\mathrm{s}(x_1^*, \Gamma) = \mathrm{s}(x_2^*, \Gamma)$ μ-a.e.

If Γ is integrable in $\mathrm{ck}(X)$, then the weak compactness of T_Γ follows immediately from Theorem 1.4. So assume that $\Gamma\colon \Omega \to \mathrm{c}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$. Since M_Γ is a weak multimeasure with values in $\mathrm{cwk}(X)$, it is countably additive in the Hausdorff metric of $\mathrm{cwk}(X)$ (cf. [13, Theorem 8.4.10]). In the terminology of support functions, this means that if $\langle A_n\rangle$ is a sequence of pairwise disjoint elements of Σ, then

$$\lim_m \sup_{\|x^*\|\le 1} \left| \sum_{i=1}^m \mathrm{s}(x^*, M_\Gamma(A_i)) - \mathrm{s}\left(x^*, M_\Gamma\left(\bigcup_{i=1}^\infty A_i\right)\right)\right| = 0.$$

Thus, the collection $\{\mathrm{s}(x^*, M_\Gamma)\colon \|x^*\| \le 1\}$ of scalar measures is bounded (see Proposition 1.3) and uniformly σ-additive. It follows from [8, Corollary I.2.5] that the measures of the family are uniformly absolutely continuous with respect to μ. Consequently, the set $\{\mathrm{s}(x^*, \Gamma)\colon \|x^*\| \le 1\}$ is a weakly relatively compact subset of $L_1(\mu)$, what yields the weak compactness of T_Γ. □

Remark 2.3. A Pettis integrable function $f\colon \Omega \to X$ can always be treated as a $\mathrm{ck}(X)$-valued multifunction. It is well known that, in general, it is not determined by a separable subspace of X. Thus, in case of $\mathrm{ck}(X)$-valued multifunctions, Proposition 2.2 cannot be strengthened.

Theorem 2.4. *Let $\Gamma\colon \Omega \to \mathrm{c}(X)$ be a scalarly integrable multifunction possessing the following two properties:*

(WC) $T_\Gamma\colon X^* \to L_1(\mu)$ is weakly compact;
(D^*) Γ is determined by a space $Y \subseteq X$ such that Y^* is weak-angelic.*

Then Γ is Pettis integrable in $\mathrm{cb}(X)$. If Γ is $\mathrm{cwk}(X)$-valued, then it is Pettis integrable in $\mathrm{cwk}(X)$.

Proof. Through this proof y_e^* will denote an arbitrary extension of a functional $y^* \in Y^*$ to a functional on X. Assume that for a fixed scalarly integrable $\Gamma\colon \Omega \to \mathrm{c}(X)$ the conditions (WC) and (D^*) are fulfilled and define a sublinear functional $\varphi\colon Y^* \to (-\infty, +\infty)$ by the formula

$$\varphi(y^*) := \int_\Omega \mathrm{s}(y_e^*, \Gamma)\, d\mu.$$

We want to show now that there exists $M(\Omega) \in \mathrm{cb}(Y)$ such that $\int_\Omega \mathrm{s}(x^*, \Gamma)\, d\mu = \mathrm{s}(x^*, M(\Omega))$ for all functionals x^*. We shall prove first that φ is w^*–lower semicontinuous, i.e. that for each real α the set

$$Q(\alpha) := \{y^* \in Y^*\colon \varphi(y^*) \le \alpha\}$$

is w^*–closed. According to the Krein-Šmulian theorem it suffices to show that $Q(\alpha) \cap B(Y^*)$ is w^*–closed. Due to angelicity of Y^*, it suffices to show that $Q(\alpha)$ is sequentially weak* closed. So let $y_n^* \in Q(\alpha)$, $n \in \mathbb{N}$, be a sequence $\sigma(Y^*, Y)$-converging to $\overline{y}^*$ and, let $(y_n^*)_e$ be, for each $n \in \mathbb{N}$, a norm preserving extension of y_n^* to X. Since T_Γ is weakly compact, we can extract a subsequence of $\langle T_\Gamma((y_n^*)_e\rangle$ that is $\sigma(L_1(\mu), L_\infty(\mu))$-convergent to a function $h \in L_1(\mu)$. Assume that the sequence itself is already convergent. Clearly, we have $\int_\Omega h\, d\mu \le \alpha$.

It follows from Mazur's theorem that there is an increasing sequence $\langle n_k\rangle$ of natural numbers and non-negative reals $\{a_{ki}\colon k \in \mathbb{N},\ n_k < i \le n_{k+1}\}$ such that $\sum_{i=n_k+1}^{n_{k+1}} a_{ki} = 1$ and the functions

$$w_k = \sum_{i=n_k+1}^{n_{k+1}} a_{ki}\, \mathrm{s}((y_i^*)_e, \Gamma)$$

are converging to h in $L_1(\mu)$ and a.e.

Then, let

$$z_k^* = \sum_{i=n_k+1}^{n_{k+1}} a_{ki} y_i^* \quad \text{and} \quad (z_k^*)_e := \sum_{i=n_k+1}^{n_{k+1}} a_{ki} (y_i^*)_e\,.$$

We have $z_k^* \to \overline{y}^*$ in the weak*-topology of Y^*.

Let us now fix $\varepsilon > 0$, $\delta > 0$ and a set $A_\varepsilon \in \Sigma_\mu^+$ such that:

(a) For each $A \in \Sigma$, if $\mu(A) < \delta$, then $\int_A |h|\, d\mu < \varepsilon$;
(b) For each $A \in \Sigma$, if $\mu(A) < \delta$, then $\int_A |\,\mathrm{s}(x^*, \Gamma)|\, d\mu < \varepsilon$, for every $x^* \in B(X^*)$ (possible, because T_Γ is weakly compact).
(c) $\mu(A_\varepsilon^c) < \delta$ and $w_k|A_\varepsilon \to h|A_\varepsilon$ uniformly.

Then, let $k_\varepsilon \in \mathbb{N}$ be such that

$$\forall\, k \ge k_\varepsilon\ \forall\, \omega \in A_\varepsilon |w_k(\omega) - h(\omega)| < \varepsilon.$$

Since $B(X^*)$ is weak*-compact we can find a net $\{v_p^* \in B(X^*)\colon p \in \mathbb{P}\}$ (where $\mathbb{P}$ is a directed set) being a subnet of the sequence $\langle (z_k^*)_e\rangle_{k \ge k_\varepsilon}$ and a functional $(\overline{y}^*)_e \in X^*$ such that

$$v_p^* \to (\overline{y}^*)_e \ \text{ in } \sigma(X^*, X).$$

Since $\Gamma(\omega) \in \mathrm{c}(X)$, we have

$$\mathrm{s}((\overline{y}^*)_e, \Gamma(\omega)) \leq \liminf_p \mathrm{s}(v_p^*, \Gamma(\omega)) \quad \text{for every } \omega \in \Omega.$$

Now notice that

$$\mathrm{s}((z_k^*)_e, \Gamma) = \mathrm{s}\left(\sum_{i=n_k+1}^{n_{k+1}} a_{ki}(y_i^*)_e, \Gamma\right) \leq \sum_{i=n_k+1}^{n_{k+1}} a_{ki}\, \mathrm{s}((y_i^*)_e, \Gamma) = w_k \quad \text{everywhere,}$$

and so

$$\mathrm{s}(v_p^*, \Gamma) \leq w_p \quad \text{everywhere.}$$

If $\omega \in A_\varepsilon$, then

$$w_p(\omega) \leq h(\omega) + \varepsilon \quad \text{for every } \omega \in A_\varepsilon \text{ and } p \in \mathbb{P}.$$

Consequently, if $p \in \mathbb{P}$, then

$$\mathrm{s}(v_p^*, \Gamma(\omega)) \leq h(\omega) + \varepsilon \quad \text{and so} \quad \mathrm{s}((\overline{y}^*)_e, \Gamma(\omega)) \leq h(\omega) + \varepsilon \quad \text{for every } \omega \in A_\varepsilon\,.$$

Thus,

$$\begin{aligned}\varphi(\overline{y}^*) = & \int_\Omega \mathrm{s}((\overline{y}^*)_e, \Gamma)\, d\mu \leq \int_{A_\varepsilon} (h+\varepsilon)\, d\mu + \int_{A_\varepsilon^c} \mathrm{s}((\overline{y}^*)_e, \Gamma)\, d\mu \\ \leq & \int_\Omega |h|\, d\mu + \int_{A_\varepsilon^c} |h|\, d\mu + \varepsilon\mu(A_\varepsilon) + \varepsilon \leq \int_\Omega |h|\, d\mu + 3\varepsilon \leq \alpha + 3\varepsilon.\end{aligned}$$

Since ε was arbitrary, we get $\overline{y}^* \in Q(\alpha)$, i.e. $Q(\alpha)$ is weak*-closed.

Consequently, the function φ is w^*–lower semicontinuous and so, according to [7, Theorem II.16] (or Proposition 1.3), there exists a non-empty bounded closed convex set $M(\Omega) \subset Y$ such that $\varphi(y^*) = \mathrm{s}(y^*, M(\Omega))$, for every $y^* \in Y^*$. Equivalently,

$$\mathrm{s}(x^*, M(\Omega)) = \int_\Omega \mathrm{s}(x^*, \Gamma)\, d\mu \quad \text{for every } x^* \in X^*.$$

As Ω may be replaced by an arbitrary $E \in \Sigma$ we have proven the integrability in $\mathrm{cb}(X)$.

Assume now that $\Gamma\colon \Omega \to \mathrm{cwk}(X)$. According to the first part of the proof, for each $E \in \Sigma$ there exists a set $M(E) \in \mathrm{cb}(X)$ such that

$$\mathrm{s}(x^*, M(E)) = \int_E \mathrm{s}(x^*, \Gamma)\, d\mu \quad \text{for every } x^* \in X^*.$$

We are going to prove that $M(E) \in \mathrm{cwk}(X)$. According to Proposition 1.7 it is enough to prove that every extremal face of Γ is Pettis integrable in $\mathrm{cb}(X)$. To do it, let $x_0^* \in X^*$ be arbitrary and let $G\colon \Omega \to \mathrm{cwk}(X)$ be defined by

$$G(\omega) := \{x \in \Gamma(\omega) \colon x_0^*(x) = \mathrm{s}(x_0^*, \Gamma(\omega))\}.$$

Due to [26, Lemma 3] the multifunction G is scalarly measurable. One can easily see also that T_G is weakly compact and G is determined by the same WCG space Y. Consequently, applying the first part of Theorem 2.4, we obtain the existence of a set $P_E \in \mathrm{cb}(X)$ such that

$$\mathrm{s}(x^*, P_E) = \int_E \mathrm{s}(x^*, G)\, d\mu \quad \text{for every } x^* \in X^*.$$

Thus, G is Pettis integrable in $\mathrm{cb}(X)$, what yields Pettis integrability of $\varGamma$ in $\mathrm{cwk}(X)$. □

The next result is a generalization of the classical characterization of Pettis integrable functions, usually formulated in terms of weakly compactly generated determining spaces.

Theorem 2.5. *A scalarly integrable multifunction $\varGamma\colon \varOmega \to \mathrm{cwk}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$ if and only if it satisfies the following conditions*

(WC) $T_\varGamma\colon X^* \to L_1(\mu)$ *is weakly compact;*
(D) $\varGamma$ *is determined by a WCG space $Y \subseteq X$.*

Proof. Assume that $\varGamma\colon \varOmega \to \mathrm{cb}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$. The conditions (WC) and (D) are then consequences of Proposition 2.2. The reverse implication is a particular case of Theorem 2.4. □

The following three results are immediate consequences of Theorems 2.4 and 2.5.

Theorem 2.6. *Let $\varGamma\colon \varOmega \to \mathrm{c}(X)$ be a scalarly measurable multifunction. Assume that there exists a multifunction $\varDelta\colon \varSigma \to \mathrm{cwk}(X)$ that is Pettis integrable in $\mathrm{cwk}(X)$ and for each $x^* \in X^*$*

$$|\,\mathrm{s}(x^*, \varGamma)| \le |\,\mathrm{s}(x^*, \varDelta)| \quad \text{a.e.}$$

Then $\varGamma$ is Pettis integrable in $\mathrm{cb}(X)$. If $\varGamma$ is $\mathrm{cwk}(X)$-valued, then it is Pettis integrable in $\mathrm{cwk}(X)$.

Theorem 2.7. *Let $\varGamma\colon \varOmega \to \mathrm{c}(X)$ be a scalarly measurable multifunction that is determined by a WCG subspace of X. If there exists a function $h \in L_1(\mu)$ such that for each $x^* \in B(X^*)$*

$$|\,\mathrm{s}(x^*, \varGamma)| \le h \quad \textit{a.e.},$$

then $\varGamma$ is Pettis integrable in $\mathrm{cb}(X)$. If $\varGamma$ is $\mathrm{cwk}(X)$-valued, then it is Pettis integrable in $\mathrm{cwk}(X)$.

Corollary 2.8. *If $\varGamma\colon \varOmega \to \mathrm{cwk}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$, then also $\varDelta\colon \varOmega \to \mathrm{cwk}(X)$ defined pointwise by $\varDelta(\omega) := \overline{\mathrm{conv}}(\varGamma(\omega) \cup -\varGamma(\omega))$ is Pettis integrable in $\mathrm{cwk}(X)$.*

Proof. It is obvious that $\varDelta$ is determined by the same WCG space as $\varGamma$ is. If $x^* \in X^*$ and $\omega \in \varOmega$, then

$$\mathrm{s}(x^*, \varDelta(\omega)) = \max\{\mathrm{s}(x^*, \varGamma(\omega)), \mathrm{s}(-x^*, \varGamma(\omega))\}$$

and so $\{\mathrm{s}(x^*, \varDelta)\colon \|x^*\| \le 1\}$ is relatively weakly compact in $L_1(\mu)$. □

Remark 2.9. The result formulated in the above corollary has been proven in [1] for ck(X)-valued multifunctions with values in a separable X. □

The next result is a very special case of the general situation described by Theorem 2.5. It will be applied in the proof of Theorem 5.1.

Corollary 2.10. *If a multifunction $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ is such that for each x^* the support function $\mathrm{s}(x^*, \Gamma)$ is constant a.e. (the constant depending on x^*) and Γ is determined by a WCG subspace of X, then Γ is Pettis integrable in* cwk(X). *In particular, if μ is a two-valued measure, then each scalarly measurable multifunction $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ that is determined by a WCG subspace of X is Pettis integrable in* cwk(X).

Proof. To prove the required assertion we are going to apply Proposition 1.2 and formula (3). We have then $\Gamma(\omega) = \sum_n \Gamma(\omega)\chi_{E_n}(\omega)$. Given x^* let $c_{x^*} := \mathrm{s}(x^*, \Gamma)$ a.e., for each $x^* \in X^*$. If E_m is the first set of positive measure, then for each x^* we must have $\mathrm{s}(x^*, \Gamma|E_m) = c_{x^*}$ a.e. Consequently Γ is scalarly bounded and T_Γ is weakly compact. □

The following result is a particular case of [6, Theorem 3.8] but its proof is much simpler.

Corollary 2.11. *If $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ is determined by a WCG subspace of X, then Γ has a scalarly measurable selection.*

Proof. Γ can be represented in the form

$$\Gamma(\omega) = \sum_n \Gamma(\omega)\chi_{E_n}(\omega) \tag{8}$$

where the sets $E_n \in \Sigma$ are pairwise disjoint, $\mu\left(\bigcup_n E_n\right) = \mu(\Omega)$ and each $\Gamma|E_n$ is scalarly bounded. We apply now Theorem 2.5 to each multifunction $\Gamma|E_n$ getting its Pettis integrability. Then we take a Pettis integrable selection for each such a multifunction (existing in virtue of [5]) and stick them together. □

The following consequence is a generalization of a known result of Ionescu-Tulcea concerning functions taking their values in a fixed weakly compact set.

Proposition 2.12. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly measurable multifunction such that $\Gamma(\omega) \subset W$, where $W \in \mathrm{cwk}(X)$ is fixed. Then each scalarly measurable selection of Γ is weakly equivalent to a strongly measurable selection of Γ and Γ is Pettis integrable in* cwk(X).

Proof. The Pettis integrability of Γ follows directly from Theorem 2.5, because Γ is scalarly bounded. According to Corollary 2.11 there is a weakly measurable selection $\gamma\colon \Omega \to X$ of Γ. But now for each x^* we have also

$$x^*\gamma(\omega) \in x^*\Gamma(\omega) \subset x^*W.$$

If ρ is a lifting on the measure space, then setting $\langle\widetilde{\gamma}(\omega), x^*\rangle = \rho(x^*\gamma)(\omega)$, for every ω and x^*, we have (for each x^*) $x^*\gamma = x^*\widetilde{\gamma}$ a.e. and the function $\widetilde{\gamma}\colon \Omega \to X^{**}$ is such that

$$\text{for each } x^* \in X^* \quad x^*\widetilde{\gamma}(\omega) \in x^*\Gamma(\omega) \subset x^*W \quad \mu - a.e.$$

But x^*W is a closed interval (or a point) and liftings respect weak inequalities, therefore we have $x^*\widetilde{\gamma}(\omega) \in x^*W$, for every $\omega \in \Omega$. A direct application of the Hahn-Banach theorem yields $\widetilde{\gamma}(\Omega) \subset W$. There is an old result of Ionescu-Tulcea ([14], Theorem 3) saying that a weakly measurable function $\widetilde{\gamma}\colon \Omega \to X$ such that $x^*\widetilde{\gamma} = \rho(x^*\widetilde{\gamma})$, for every x^*, is strongly measurable. □

Notice that if $X = l_2[0,1]$ and $\Gamma(t) = \{e_t\}$, where $\{e_t\colon t \in [0,1]\}$ is the canonical orthonormal system, then Γ is weakly equivalent to zero, but it is not strongly measurable. Thus, it is not true that scalarly measurable selections of Γ in Proposition 2.12 have to be strongly measurable.

Theorem 2.13. *Let X be a Banach space not containing any isomorphic copy of c_0. If $\Gamma\colon \Omega \to \mathrm{c}(X)$ is scalarly integrable and determined by a WCG space, then Γ is Pettis integrable in* $\mathrm{cb}(X)$. *If $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ is scalarly integrable and determined by a WCG space, then Γ is Pettis integrable in* $\mathrm{cwk}(X)$.

Proof. Let $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ be a decomposition of Ω into pairwise disjoint sets of positive measure, such that Γ is scalarly bounded on each Ω_n. By Theorem 2.4 (or 2.5 in case of $\mathrm{cwk}(X)$) each multifunction $\Gamma|_{\Omega_n}$ is Pettis integrable in $\mathrm{cb}(X)$ (resp. $\mathrm{cwk}(X)$). Let

$$\mathrm{s}(x^*, M_n(E\cap\Omega_n)) := \int_{E\cap\Omega_n} \mathrm{s}(x^*, \Gamma)\, d\mu \quad \text{for every } E \in \Sigma \text{ and } x^* \in X^*.$$

Notice that for every $E \in \Sigma$, we have

$$\int_E \mathrm{s}(x^*, \Gamma)\, d\mu = \sum_{n=1}^{\infty} \int_{E\cap\Omega_n} \mathrm{s}(x^*, \Gamma)\, d\mu = \sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E\cap\Omega_n)), \tag{9}$$

what yields the absolute convergence of the series $\sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E\cap\Omega_n))$. But in fact we get much more.

Claim 1. If $\{E_k\colon k \in \mathbb{N}\}$ is a sequence of pairwise disjoint elements of Σ and x^* is arbitrary, then the double series $\sum_{k,n=1}^{\infty} \mathrm{s}(x^*, M_n(E_k\cap\Omega_n))$ is absolutely convergent. In particular

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mathrm{s}(x^*, M_n(E_k\cap\Omega_n)) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E_k\cap\Omega_n)). \tag{10}$$

Proof. We have the following sequence of inequalities:

$$\begin{aligned}\infty > \int_{\bigcup_k E_k} |\,\mathrm{s}(x^*, \Gamma)|\, d\mu &= \sum_{k=1}^{\infty} \int_{E_k} |\,\mathrm{s}(x^*, \Gamma)|\, d\mu \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \int_{E_k\cap\Omega_n} |\,\mathrm{s}(x^*, \Gamma)|\, d\mu \ge \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\,\mathrm{s}(x^*, M_n(E_k\cap\Omega_n))|\end{aligned}$$

It follows that the double series $\sum_{k,n=1}^{\infty} \mathrm{s}(x^*, M_n(E_k \cap \Omega_n))$ is absolutely convergent and the equality (10) is valid.

Below, by a series $\sum_{n=1}^{\infty} C_n$, where every C_n is closed and convex, we understand the set

$$\left\{\sum_{n=1}^{\infty} c_n : \forall\, n \in \mathbb{N}\ c_n \in C_n \quad \& \quad \sum_{n=1}^{\infty} c_n \text{ is unconditionally convergent}\right\}.$$

Claim 2. Given any $E \in \Sigma$, the series $\sum_{n=1}^{\infty} M_n(E \cap \Omega_n)$ is unconditionally convergent, that is if $x_n \in M_n(E \cap \Omega_n)$ is quite arbitrary, then the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent.

Proof. Let $E \in \Sigma_\mu^+$ and x^* be fixed. We have then for each $n \in \mathbb{N}$

$$-\,\mathrm{s}(-x^*, M_n(E \cap \Omega_n)) \le x^*(x_n) \le \mathrm{s}(x^*, M_n(E \cap \Omega_n))$$

and then

$$|x^*(x_n)| \le |\,\mathrm{s}(-x^*, M_n(E \cap \Omega_n))| + |\,\mathrm{s}(x^*, M_n(E \cap \Omega_n))|,$$

what implies

$$\sum_{n=1}^{\infty} |x^*(x_n)| \le \sum_{n=1}^{\infty} |\,\mathrm{s}(-x^*, M_n(E \cap \Omega_n))| + \sum_{n=1}^{\infty} |\,\mathrm{s}(x^*, M_n(E \cap \Omega_n))| < \infty. \tag{11}$$

As $c_0 \not\subseteq X$, the well known result of Bessaga and Pełczyński [2] constrains the unconditional convergence of $\sum_{n=1}^{\infty} x_n$. □

The formula

$$M(E) := \overline{\sum_{n=1}^{\infty} M_n(E \cap \Omega_n)} = \overline{\mathrm{conv}} \left\{\sum_{n=1}^{\infty} x_n \colon x_n \in M_n(E \cap \Omega_n),\ n \in \mathbb{N}\right\} \tag{12}$$

defines now a bounded (due to (11)) closed convex subset of X. In case of the $\mathrm{cwk}(X)$-valued multifunction, we have

$$M(E) = \sum_{n=1}^{\infty} M_n(E \cap \Omega_n) = \left\{\sum_{n=1}^{\infty} x_n \colon x_n \in M_n(E \cap \Omega_n),\ n \in \mathbb{N}\right\} \tag{13}$$

and $M(E)$ is a weakly compact convex set.

Claim 3. If $E \in \Sigma$ and x^* is arbitrary, then

$$\mathrm{s}(x^*, M(E)) = \sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E \cap \Omega_n)). \tag{14}$$

Proof. We have

$$\alpha := \mathrm{s}(x^*, M(E)) = \sup\left\{\sum_{n=1}^{\infty} x^*(x_n)\colon x_n \in M_n(E\cap\Omega_n),\ n\in\mathbb{N}\right\}.$$

If $\delta > 0$ is arbitrary, then there is a sequence $\langle x_n^\delta\rangle$ with all $x_n^\delta \in M_n(E\cap\Omega_n)$ such that $\sum_{n=1}^{\infty} x^*(x_n^\delta) > \alpha - \delta$. Hence, there is also $k\in\mathbb{N}$ with $\sum_{n=1}^{k} x^*(x_n^\delta) > \alpha-\delta$. It follows that

$$\sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E\cap\Omega_n)) \ge \sum_{n=1}^{k} \mathrm{s}(x^*, M_n(E\cap\Omega_n)) \ge \sum_{n=1}^{k} x^*(x_n^\delta) > \alpha-\delta$$

and so $\sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E\cap\Omega_n)) \ge \mathrm{s}(x^*, M(E))$.

To prove the reverse inequality let us fix $\delta > 0$ and fix for each $n\in\mathbb{N}$ a point $x_n \in M_n(E\cap\Omega_n)$ with $\mathrm{s}(x^*, M_n(E\cap\Omega_n)) < x^*(x_n) + \delta/2^n$. Then

$$\sum_{n=1}^{\infty} \mathrm{s}(x^*, M_n(E\cap\Omega_n)) \le \sum_{n=1}^{\infty} x^*(x_n) + \delta < \mathrm{s}(x^*, M(E)) + \delta.$$

□

Claim 4. $M\colon \Sigma \to \mathrm{cb}(X)$ is a weak multimeasure and $M\colon\Sigma\to \mathrm{cwk}(X)$ is an h-multimeasure.

Proof. Let $\{E_k\colon k\in\mathbb{N}\}$ be a sequence of pairwise disjoint elements of Σ and let $E=\bigcup_{k=1}^{\infty} E_k$. If x^* is arbitrary, then the equality (14) (applied to E and to each E_k separately) and (10) yield

$$\begin{aligned}
&\mathrm{s}(x^*, M(E))\\
=\ &\mathrm{s}\left(x^*, \left\{\sum_{n=1}^{\infty} x_n\colon x_n\in M_n(E\cap\Omega_n),\ n\in\mathbb{N}\right\}\right) = \sum_{n=1}^{\infty}\mathrm{s}(x^*, M_n(E\cap\Omega_n))\\
=\ &\sum_{n=1}^{\infty}\mathrm{s}\left(x^*, M_n\left(\bigcup_k E_k\cap\Omega_n\right)\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{s}(x^*, M_n(E_k\cap\Omega_n))\\
=\ &\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{s}(x^*, M_n(E_k\cap\Omega_n)) = \sum_{k=1}^{\infty}\mathrm{s}(x^*, M(E_k)).
\end{aligned}$$

The above shows that M is a weak multimeasure, but if its values are weakly compact and convex it is a $\mathrm{cwk}(X)$-valued h-multimeasure.

The equalities (9) and (14) prove the Pettis integrability of Γ in $\mathrm{cb}(X)$ (resp. in $\mathrm{cwk}(X)$). This completes the proof of the whole theorem. □

Theorem 2.13 is an interesting generalization of a well known result of Diestel [8] and Dimitrov [9] saying that if X is a separable Banach space and $c_0 \nsubseteq X$, then each X-valued scalarly integrable function is Pettis integrable. It is known that separability of X is essential (see [11]). In particular, the reverse theorem to Theorem 2.13 is also valid: If $c_0 \subset X$ isomorphically, then there exists a scalarly integrable multifunction (in fact a function) with values in $\mathrm{ck}(X)$ that is not Pettis integrable in $\mathrm{cb}(X)$.

Theorem 2.14. *Let X be an arbitrary Banach space not containing any isomorphic copy of c_0. Then each scalarly integrable function $f\colon \Omega \to X$ that is determined by a WCG subspace of X is Pettis integrable.*

3. Pettis integrability in $\mathrm{ck}(X)$.

It is our aim to find conditions guaranteeing the integrability in $\mathrm{ck}(X)$ of $\mathrm{ck}(X)$-valued multifunctions. Unfortunately, there is an essential asymmetry between integrability in $\mathrm{cwk}(X)$ and $\mathrm{ck}(X)$. This is presented by Theorem 3.3, where we show that the compactness of the operator T_Γ and determination of Γ by a WCG (or even separable) space guarantees the integrability of $\Gamma\colon \Sigma \to \mathrm{ck}(X)$ in $\mathrm{ck}(X)$ but in general, this assumption is too strong. As a consequence, we have to apply a different technique to obtain the required results.

Given a scalarly integrable multifunction $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ determined by a space $Y \subset X$, we define an operator $T_{\Gamma Y}\colon Y^* \to L_1(\mu)$ by setting

$$T_{\Gamma Y}(y^*) := T_\Gamma(y^*_e),$$

where y^*_e is an arbitrary extension of y^* to the whole X. One can easily see that the definition of $T_{\Gamma Y}$ is correct and $T_{\Gamma Y}$ is bounded.

The next two lemmata have sequential and general forms, but as we need the sequential ones, only these proofs are presented.

Lemma 3.1. *Let $\Gamma\colon \Omega \to \mathrm{ck}(X)$ be a scalarly integrable multifunction determined by a space $Y \subset X$. Then $T_{\Gamma Y}$ is [sequentially] $\sigma(Y^*,Y)$-weakly continuous if and only if T_Γ is [sequentially] $\sigma(X^*,Y)$-weakly continuous on $B(X^*)$.*

Proof. Assume that T_Γ is sequentially $\sigma(X^*,Y) - \sigma(L_1,L_\infty)$ continuous and take a sequence $y^*_n \overset{\sigma(Y^*,Y)}{\longrightarrow} 0$. If $x^*_n \in X^*$ is an extension of y^*_n, then $x^*_n \overset{\sigma(X^*,Y)}{\longrightarrow} 0$ and so, by the assumption, $T_{\Gamma Y}(y^*_n) = T_\Gamma(x^*_n) \to 0$, weakly in $L_1(\mu)$.

Assume now that $T_{\Gamma Y}$ is sequentially $\sigma(Y^*,Y) - \sigma(L_1,L_\infty)$ weak continuous on $B(Y^*)$ and take $x^*_n \overset{\sigma(X^*,Y)}{\longrightarrow} 0$. If $y^*_n = x^*_n \mid Y$, then $y^*_n \overset{\sigma(Y^*,Y)}{\longrightarrow} 0$ and so $T_\Gamma(x^*_n) = T_{\Gamma Y}(y^*_n) \to 0$ weakly in $L_1(\mu)$. □

Lemma 3.2. *Let $\Gamma\colon \Omega \to \mathrm{ck}(X)$ be scalarly integrable and determined by a space $Y \subseteq X$. If T_Γ is compact, then T_Γ is [sequentially] $\sigma(X^*,Y)$-norm continuous on $B(X^*)$.*

Proof. Let $(x^*_n) \subset B(X^*)$ be a sequence that is $\sigma(X^*,Y)$-convergent to zero. By the compactness assumption, we may assume that given subsequence of (x^*_n), there is a function $h \in L_1(\mu)$ and a subsequence $(x^*_{n_k})$ of the subsequence such that $T_\Gamma(x^*_{n_k}) \to h$ in $L_1(\mu)$ and almost everywhere. We are going to show that $h = 0$ almost everywhere.

Let (x_α^*) be a subnet of $(x_{n_k}^*)$ that is $\sigma(X^*,X)$ converging to a point $\overline{x}^*$. Clearly $\overline{x}^* \in Y^\perp$. Since $\mathit{\Gamma}$ is compact valued, we have then for every $\omega \in \mathit{\Omega}$

$$T_\mathit{\Gamma}(x_\alpha^*)(\omega) = \mathrm{s}(x_\alpha^*, \mathit{\Gamma}(\omega)) \to \mathrm{s}(\overline{x}^*, \mathit{\Gamma}(\omega)) = T_\mathit{\Gamma}(\overline{x}^*)(\omega).$$

Clearly $\mathrm{s}(\overline{x}^*, \mathit{\Gamma}) = T_\mathit{\Gamma}(\overline{x}^*) = h$ a.e. and so $h = 0$ a.e., because $\overline{x}^* \in Y^\perp$ yields $\mathrm{s}(\overline{x}^*, \mathit{\Gamma}) = 0$ a.e.

Thus, we have proven that every subsequence of $(T_\mathit{\Gamma}(x_n))$ contains a subsequence that is norm convergent to zero in $L_1(\mu)$. It follows that $(T_\mathit{\Gamma}(x_n))$ itself is norm convergent to zero in $L_1(\mu)$. This proves the required sequential $\sigma(X^*,Y)$-norm continuity of $T_\mathit{\Gamma}$ on $B(X^*)$. □

Theorem 3.3. *If $\mathit{\Gamma}\colon \mathit{\Omega} \to \mathrm{ck}(X)$ is a scalarly integrable multifunction, then the following conditions are equivalent:*

(i) *$\mathit{\Gamma}$ is Pettis integrable in $\mathrm{ck}(X)$ and has a Pettis integrable (quasi) selection with norm relatively compact range of its Pettis integral;*

(ii) *$T_\mathit{\Gamma}\colon X^* \to L_1(\mu)$ is compact and $\mathit{\Gamma}$ is determined by a WCG space $Y \subseteq X$;*

(iii) *$\mathit{\Gamma}$ is Pettis integrable in $\mathrm{ck}(X)$ and each Pettis integrable (quasi) selection of $\mathit{\Gamma}$ has norm relatively compact range of its Pettis integral.*

(iv) *$\mathit{\Gamma}$ is Pettis integrable in $\mathrm{ck}(X)$ and $M_\mathit{\Gamma}(\mathit{\Sigma}) = \bigcup_{E\in\mathit{\Sigma}} M_\mathit{\Gamma}(E)$ is norm relatively compact.*

Proof. The version with selections. $(i) \Rightarrow (ii)$ Assume Pettis integrability of $\mathit{\Gamma}\colon \mathit{\Omega} \to \mathrm{ck}(X)$ in $\mathrm{ck}(X)$ and let $M_\mathit{\Gamma}(E) = (P)\int_E \mathit{\Gamma}\,d\mu$, for every $E \in \mathit{\Sigma}$. By Proposition 2.2 there is a WCG space $Y \subset X$ determining $\mathit{\Gamma}$. Let $f\colon \mathit{\Omega} \to X$ be a Pettis integrable selection of $\mathit{\Gamma}$ with norm relatively compact range of its integral. It is well known that T_f is a compact operator. Define $G\colon \mathit{\Sigma} \to \mathrm{ck}(X)$ by $G(\omega) = \mathit{\Gamma}(\omega) - f(\omega)$. Now, let $\langle x_n^*\rangle \subset B(X^*)$ be an arbitrary sequence. Let $\langle x_{n_k}^*\rangle$ be a subsequence such that $y_k^* := x_{n_k}^*|Y \xrightarrow{k} \overline{y}^* \in B(Y^*)$ weak* in Y^*. We have then

$$\lim_k \int_E \mathrm{s}(x_{n_k}^* - \overline{y}_e^*, G)\,d\mu = \lim_k \mathrm{s}(y_k^* - \overline{y}^*, M_G(E)) = 0 \ \text{ for every } E \in \mathit{\Sigma},$$

and the convergence of the sequence $\langle \mathrm{s}(y_k^* - \overline{y}^*, M_G(E))\rangle$ is uniform on $\mathit{\Sigma}$, because $M_G(E) \subseteq M_G(\mathit{\Omega})$, for every $E \in \mathit{\Sigma}$. Thus, the sequence $\langle \mathrm{s}(x_{n_k}^*, G)\rangle$ is convergent in $L_1(\mu)$ to $\mathrm{s}(\overline{y}_e^*, G)$ (cf. [22, Proposition II.5.3]). It follows that T_G is compact. Hence $T_\mathit{\Gamma} = T_G + T_f$ is also compact.

$(i) \Rightarrow (iv)$ With the above notation $M_\mathit{\Gamma}(\mathit{\Sigma}) \subset M_G(\mathit{\Omega}) + M_f(\mathit{\Sigma})$.

$(ii) \Rightarrow (iii)$ Assume that $\mathit{\Gamma}\colon \mathit{\Omega} \to \mathrm{ck}(X)$ fulfills the condition (ii). Assume that Y determines $\mathit{\Gamma}$ and for an arbitrary but fixed $A \in \mathit{\Sigma}$ define a sublinear function $\varphi\colon Y^* \to (-\infty, +\infty)$ by the formula

$$\varphi(y^*) := \int_A \mathrm{s}(y_e^*, \Gamma)\,d\mu. \tag{15}$$

Since Y determines $\mathit{\Gamma}$, Lemma 3.2 yields the sequential $\sigma(X^*,Y) - \sigma(L_1, L_\infty)$ continuity of $T_\mathit{\Gamma}$ on $B(X^*)$, what is equivalent, according to Lemma 3.1, to the sequential

$\sigma(Y^*,Y) - \sigma(L_1, L_\infty)$ continuity of $T_{\Gamma Y}$. This means that if $Y^* \ni y_n^* \overset{\sigma(Y^*,Y)}{\to} 0$, then $(y_n^*)_e \overset{\sigma(X^*,Y)}{\to} 0$ and consequently, $\varphi(y_n^*) = \int_E \mathrm{s}((y_n^*)_e, \Gamma)\, d\mu \to 0$, for every $E \in \Sigma$. It follows that φ is sequentially $\sigma(Y^*,Y)$ continuous.

We want to show now that for each $A \in \Sigma$ there exists $M(A) \in \mathrm{ck}(X)$ such that $\int_A \mathrm{s}(x^*, \Gamma)\, d\mu = \mathrm{s}(x^*, M(A))$ for all functionals x^*. We shall prove first that φ is w^*–lower semicontinuous, i.e. that for each real α the set

$$Q(\alpha) := \{y^* \in Y^* \colon \varphi(y^*) \le \alpha\}$$

is w^*–closed. According to the Krein-Šmulian theorem it suffices to show that $Q(\alpha) \cap B(Y^*)$ is w^*–closed. But $B(Y^*)$ is weak*-sequentially compact and so $Q(\alpha) \cap B(Y^*)$ is w^*–closed if and only if it is weak*-sequentially closed. The weak*-sequential closeness of $Q(\alpha) \cap B(Y^*)$ is, however, a direct consequence of the sequential $\sigma(Y^*,Y)$-continuity of φ.

Consequently, the function φ is w^*–lower semicontinuous and so, according to Proposition 1.3, there exists a closed convex set $M(A) \subset Y$ such that $\varphi(y^*) = \mathrm{s}(y^*, M(A))$, for every $y^* \in Y^*$.

I am going to prove that $M(A) \in \mathrm{ck}(Y)$. To achieve that, first notice that if $B(Y^*) \ni y_n^* \overset{\sigma(Y^*,Y)}{\to} 0$, then the convergence is uniform on $M(A)$. Indeed, we have for each $y \in M(A)$

$$-\varphi(-y_n^*) = -\mathrm{s}(-y_n^*, M(A)) \le y_n^*(y) \le \mathrm{s}(y_n^*, M(A)) = \varphi(y_n^*)$$

with the boundary expressions tending to zero. Thus $M(A)$ is a closed limited subset of a WCG space Y, what means that $M(A)$ is compact.

As Y^* is linearly isometric to the quotient Banach space $X^*/Y^\perp$, we get in this way, for each $A \in \Sigma$, the equality

$$\mathrm{s}(x^*, M(A)) = \int_A \mathrm{s}(x^*, \Gamma)\, d\mu, \tag{16}$$

what proves the integrability of Γ in $\mathrm{ck}(X)$.

If f is a Pettis integrable selection of Γ, then let $G(\omega) := \Gamma(\omega) - f(\omega)$, for every ω. Notice that G is $\mathrm{ck}(X)$-valued and Pettis integrable in $\mathrm{ck}(X)$ and the zero function is its selection. Applying the implication $(i) \Rightarrow (ii)$ to G we obtain the compactness of T_G. It follows that $T_f(B(X^*)) \subset T_\Gamma(B(X^*)) - T_G(B(X^*))$, and the right hand set is norm relatively compact in $L_1(\mu)$. Thus, T_f is compact. But $M_f(\Sigma) = \{T_f^*(\chi_E) \colon E \in \Sigma\}$, and so the range of the Pettis integral of f is norm relatively compact.

$(iii) \Rightarrow (i)$ is obvious, because in virtue of [5] there is at least one Pettis integrable selection of Γ.

$(iv) \Rightarrow (i)$ If f is a Pettis integrable selection of Γ, then $M_f(\Sigma) \subset M_\Gamma(\Sigma)$ and the right hand side set is, by the assumption, norm relatively compact. □

Corollary 3.4. *Let* $\Gamma : \Omega \to \mathrm{ck}(X)$ *be scalarly integrable and such that for each* $x^* \in X^*$ *we have* $\mathrm{s}(x^*, \Gamma) \ge 0$, *a.e. Then,* Γ *is Pettis integrable in* $\mathrm{ck}(X)$ *if and only if* $T_\Gamma \colon X^* \to L_1(\mu)$ *is compact and* Γ *is determined by a WCG space* $Y \subseteq X$.

Proof. The zero function is a quasi selection of Γ. $\square$

Remark 3.5. One should observe that if *(ii)* of Theorem 3.3 is fulfilled but Γ is not $\mathrm{ck}(X)$-valued, then Γ may be not Pettis integrable in $\mathrm{ck}(X)$. To see this, it is enough to take a constant Γ such that $\Gamma(\omega) = W$, where $W \in \mathrm{cwk}(X)$ but $W \notin \mathrm{ck}(X)$.

There exist Pettis integrable functions with not norm relatively compact range of their integrals (cf. [11, 2D], where an example of an l_∞-valued function is described. That function is determined by the separable space c_0.). Each such a function f is an example of a $\mathrm{ck}(X)$-valued multifunction that is $\mathrm{ck}(X)$-integrable, but does not have any Pettis integrable selection with norm relatively compact range of its integral and its operator T_f is not compact.

Let $\Gamma(\omega) := \mathrm{conv}\{0, f(\omega)\}$, where f is the just mentioned l_∞-valued function. As f is scalarly bounded, Γ is also scalarly bounded and so T_Γ is weakly compact. It follows from Theorem 2.5 that Γ is Pettis integrable in $\mathrm{cwk}(l_\infty)$. So this is an example of a $\mathrm{ck}(l_\infty)$-valued multifunction possesseing a Pettis integrable selection with norm relatively compact range of its integral, that is not Pettis integrable in $\mathrm{ck}(l_\infty)$. $\square$

The next theorem shows that the general integration in $\mathrm{ck}(X)$ can be reduced to that of Theorem 3.3.

Theorem 3.6. *If $\Gamma\colon \Omega \to \mathrm{c}(X)$ is Pettis integrable in $\mathrm{ck}(X)$ and has a Pettis integrable (quasi) selection g, then Γ can be represented in the form $\Gamma = G + g$, where $G\colon \Omega \to \mathrm{c}(X)$ is Pettis integrable in $\mathrm{ck}(X)$ and $M_G(\Sigma)$ is norm relatively compact. In particular, each $\mathrm{cwk}(X)$-valued multifunction which is Pettis integrable in $\mathrm{ck}(X)$ has such a decomposition.*

Proof. We have to observe that in the proof of *(i)* $\Rightarrow$ *(ii)* and *(i)* $\Rightarrow$ *(iv)* we do not use the fact that Γ is $\mathrm{ck}(X)$-valued. We use only Pettis integrability of Γ in $\mathrm{ck}(X)$. Consequently, it follows from Theorem 3.3 that $M_G(\Sigma)$ is norm relatively compact. If $\Gamma\colon \Omega \to \mathrm{cwk}(X)$, then in virtue of [5], Γ has a Pettis integrable selection g. If $G = \Gamma - g$, then G satisfies the conditions of Theorem 3.3. $\square$

If we strengthen the assumptions about μ or about X, then stronger results can be obtained. The first point of the next theorem is a generalization of Stegal's theorem [11, Proposition 3J] who proved that a Pettis integrable function with perfect domain has norm relatively compact range of its integral. The second point generalizes the result of Talagrand [24], who proved, assuming validity of Martin's axiom, that if l_∞ is not a quotient of a Banach space X, then the Pettis integral of each X-valued Pettis integrable function is norm relatively compact. The third point generalizes Proposition 11.1 from [20] and Theorem 6.7 from [21].

Theorem 3.7. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a multifunction that is Pettis integrable in $\mathrm{ck}(X)$ and assume that one of the following conditions is satisfied:*

(i) *μ be a perfect measure;*
(ii) *Martin's axiom holds true and l_∞ is not a quotient of X;*
(iii) *Each X-valued Pettis integrable function has norm relatively compact range of its integral (spaces with the weak RNP or with the weak**RNP possess such a*

property (see [20], Proposition 11.1*)).*

Then $M_\Gamma(\Sigma)$ is norm relatively compact.

Proof. (i) According to [5] Γ has a Pettis integrable selection $f\colon \Omega \to X$. But according to [11] the range of the Pettis integral of f is norm relatively compact. Now we have to observe that in the proof of the implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iv)$ of Theorem 3.3 we do not use the fact that Γ is ck(X)-valued. We use only Pettis integrability of Γ in ck(X). Consequently, it follows from Theorem 3.3 that $M_\Gamma(\Sigma)$ is norm relatively compact.

(ii) The proof is similar to that of (i) but now, instead of [11], we apply [24].

(iii) We apply now Proposition 11.1 from [20] or Theorem 6.7 from [21]. □

Definition 3.8. Let Y be a linear subspace of X. We say that the pair (X, Y) has the subsequential w*-lifting property, if for every weak*-convergent sequence $\langle y_n^*\rangle$ of functionals on Y there exists a weak*-convergent sequence $\langle x_{n_k}^*\rangle$ of functionals on X such that $x_{n_k}^*|Y = y_{n_k}^*$, for every $k \in \mathbb{N}$. If (X, Y) has the subsequential w*-lifting property for every WCG subspace Y of X, then we say that X has the subsequential w*-lifting property. □

It is obvious that each space X possessing the weak*-sequentially compact unit ball of X^* has the subsequential w*-lifting property.

Lemma 3.9. *Let $\Gamma\colon \Omega \to$ ck(X) be scalarly integrable and determined by a WCG space $Y \subseteq X$. If (X, Y) has the subsequential w*-lifting property and T_Γ is weakly compact, then T_Γ is compact.*

Proof. Let $\langle x_n^*\rangle \subset B(X^*)$ be an arbitrary sequence and define $y_n^* := x_n^*|Y$, for every $n \in \mathbb{N}$. As the unit ball of Y^* is weak*-sequentially compact, there is a weak*-convergent subsequence $\langle y_{n_k}^*\rangle$. Due to the lifting assumption, there is a $\sigma(X^*, X)$ convergent sequence $\langle z_{n_{k_m}}^*\rangle \subset B(X^*)$ such that $z_{n_{k_m}}^*|Y = y_{n_{k_m}}^*$, for every $m \in \mathbb{N}$, and $\lim_{m\to\infty} z_{n_{k_m}}^* = z^*$ in $\sigma(X^*, X)$. Since Γ is ck(X) valued, we have

$$T_\Gamma(z_{n_{k_m}}^*) = \mathrm{s}(z_{n_{k_m}}^*, \Gamma) \to \mathrm{s}(z^*, \Gamma) \quad \text{pointwise.}$$

Moreover, due to the weak compactness of T_Γ, a subsequence of $\langle T_\Gamma(z_{n_{k_m}}^*)\rangle$ is weakly convergent in $L_1(\mu)$. Applying the fact that Y determines Γ, we obtain for every $m \in \mathbb{N}$ the equality

$$T_\Gamma(z_{n_{k_m}}^*) = T_\Gamma(x_{n_{k_m}}^*) \quad \text{a.e.}$$

It follows that $\langle T_\Gamma(x_n^*)\rangle$ has an $L_1(\mu)$ convergent subsequence, and so T_Γ is compact. □

Remark 3.10. It is an easy consequence of the above lemma that the subsequential weak*-lifting property is not shared by all Banach spaces. In fact, if there is an X-valued Pettis integrable function f such that its Pettis integral in not norm relatively compact, then T_f is weakly compact but not compact. In particular such an X is without the subsequential weak*-lifting property. l_∞ is an example of a Banach space without the subsequential weak*-lifting property. In fact, in [11] there is an example of an l_∞-valued Pettis integrable function with non-relatively compact range of its integral.

Proposition 3.11. *Let $\Gamma\colon \Omega \to \mathrm{ck}(X)$ be a scalarly integrable multifunction with weakly compact T_Γ (so for instance this holds true if Γ is Pettis integrable in $\mathrm{cwk}(X)$). If Γ is Pettis integrable in $\mathrm{cb}(X)$, then it is Pettis integrable in the family of limited subsets of X. Consequently, such a Γ is Pettis integrable in $\mathrm{ck}(X)$, when any of the conditions given below is satisfied:*

(a) *X is a Gelfand-Phillips space (that is each limited subset of X is relatively compact);*
(b) *X has the subsequential w^*-lifting property;*
(c) *$B(X^*)$ is weak*-sequentially compact;*
(d) *X^* is weak*-angelic.*

Proof. We are going to show that each set $M_\Gamma(E)$ is limited in X. So let $x_n^* \to 0$ in $\sigma(X^*, X)$. By the assumption the operator T_Γ is weakly compact, and so the sequence $\langle \mathrm{s}(x_n^*, \Gamma)\rangle$ is uniformly integrable. Moreover, $\mathrm{s}(x_n^*, \Gamma(\omega)) \to 0$, because Γ is $\mathrm{ck}(X)$-valued. Thus, applying the Vitali convergence theorem, we have

$$\lim_n \mathrm{s}(x_n^*, M_\Gamma(E)) = \lim_n \int_E \mathrm{s}(x_n^*, \Gamma)\, d\mu = 0.$$

In a similar way, we obtain $\lim_n \mathrm{s}(-x_n^*, M_\Gamma(E)) = 0$. Now, if $x \in M_\Gamma(E)$, then

$$-\mathrm{s}(-x_n^*, M_\Gamma(E)) \le x_n^*(x) \le \mathrm{s}(x_n^*, M_\Gamma(E))$$

and this ends the proof.

If X has the Gelfand-Phillips property, then clearly Γ is integrable in $\mathrm{ck}(X)$. If the condition (b) is satisfied, then Lemma 3.9 yields the compactness of T_Γ and we may apply Theorem 3.3. If (c) or (d) is fulfilled, then X has the subsequential w^*-lifting property. □

Corollary 3.12. *Let $\Gamma\colon \Omega \to \mathrm{ck}(X)$ be a scalarly integrable multifunction with weakly compact T_Γ. If Γ is Pettis integrable in $\mathrm{cb}(X)$ and X does not contain any isomorphic copy of l_1 or X is weakly sequentially complete, then Γ is Pettis integrable in $\mathrm{cwk}(X)$.*

Proof. According to Proposition 3.11 Γ is Pettis integrable in the family of limited subsets of X. But as it has been proven in [3], if $l_1 \not\subseteq X$, then limited subsets of X are relatively weakly compact. Similarly in case of the weak sequential completeness, because limited sets are always conditionally weakly complete. □

Another consequence of Lemma 3.9 is the following

Proposition 3.13. *Assume that X has the subsequential weak* lifting property. Then, a scalarly integrable multifunction $\Gamma\colon \Omega \to \mathrm{ck}(X)$ is Pettis integrable in $\mathrm{ck}(X)$ if and only if $T_\Gamma\colon X^* \to L_1(\mu)$ is (weakly) compact and Γ is determined by a WCG subspace of X.*

Proof. The proof is similar to that of Theorem 3.3, but now we apply Lemma 3.9 and Lemma 3.2. □

I would like to obtain a generalization of Proposition 3.11 to cwk(X) integrable multifunctions. To achieve that I need a new notion.

Definition 3.14. We say that a set $L \subset X$ is Mackey limited if each sequence $\langle x_n^* \rangle$ that is $\tau(X^*, X)$ convergent to zero is uniformly convergent on L.

Obviously, each limited set is Mackey limited and each relatively weakly compact set is Mackey limited. In case of a WCG space limited sets coincide with norm relatively compact sets and so the family of Mackey limited sets is in general larger.

To prove the first part of the next result one has to repeat the first part of the proof of Proposition 3.11, having in mind that now every set $\Gamma(\omega)$ is weakly compact. □

Proposition 3.15. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ (or $\Gamma\colon \Omega \to \mathrm{ck}(X)$, resp.) be a scalarly integrable multifunction with weakly compact T_Γ. If Γ is Pettis integrable in* cb(X), *then it is Pettis integrable in the family of Mackey limited subsets of X. If Γ is Pettis integrable in* cwk(X), *then $M_\Gamma(\Sigma)$ is Mackey limited (limited, resp.).*

Proof. If $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ (or $\Gamma\colon \Omega \to \mathrm{ck}(X)$) is Pettis integrable in cwk(X), then according to [5], Γ has a Pettis integrable selection f. Let $G := \Gamma - f$. Then G is monotone, that is $E \subset F$ yields $M_G(E) \subset M_G(F)$. In particular, $M_G(\Sigma) = M_G(\Omega)$. In virtue of the validity of the first part of the Proposition the set $M_G(\Sigma)$ is Mackey limited (or limited, in virtue of Proposition 3.11). But it is well known that the set $M_f(\Sigma)$ is also limited. Consequently, the set $M_\Gamma(\Sigma)$ is Mackey limited (or limited, resp.). □

We finish with an analogue of Theorem 1.10 in case of ck(X) integration. Its proof follows the same way as that of Theorem 1.10, but one has to use the weak*-topology instead of the corresponding Mackey topology.

Theorem 3.16. *Let $\Gamma : \Omega \to \mathrm{ck}(X)$ be scalarly integrable. If $\mathcal{Z}_\Gamma$ is stable and uniformly integrable, then Γ is Pettis integrable in* ck(X) *and T_Γ is compact.*

4. Core characterization of integrability.

It is now my aim to prove a multivalued version of the spectacular result of Talagrand [25, Theorem 5-2-2] characterizing Pettis integrability by the so called core property of the integrand. Throughout this section, if Γ is a multifunction, then $\mathcal{Z}_\Gamma := \{\mathrm{s}(x^*, \Gamma)\colon \|x^*\| \le 1\}$ is the family of support functions of Γ (not of the equivalence classes).

Lemma 4.1. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction. Assume that the operator T_Γ is weakly compact and there exist real numbers $\alpha < \beta$, a set $E \in \Sigma_\mu^+$ and a functional $x_0^* \in B(X^*)$ such that*

$$\beta \le \int_E \mathrm{s}(x_0^*, \Gamma)\, d\mu \tag{17}$$

and x_0^* is in the $\tau(X^*,X)$-closure of the set

$$R := \left\{ x^* \in B(X^*) \colon \int_E \mathrm{s}(x^*, \varGamma)\, d\mu \le \alpha \right\}. \tag{18}$$

Then there exists a function $h \in L_1(\mu)$ such that

$$\mu\{\omega \in E \colon h(\omega) < \mathrm{s}(x_0^*, \varGamma(\omega))\} > 0,$$

and x_0^* is in the $\tau(X^*,X)$-closure of the set

$$S := \{x^* \in B(X^*) \colon \mathrm{s}(x^*, \varGamma) \le h \text{ a.e. on } E\}.$$

If $\varGamma$ is $\mathrm{ck}(X)$-valued and x_0^* is in the weak*-closure of R, then S may be replaced by the set

$$S' := \{x^* \in B(X^*) \colon \mathrm{s}(x^*, \varGamma) = h \text{ a.e. on } E\},$$

with x_0^* being in the weak*-closure of S'. Moreover, there exists a functional $\overline{x}^* \in B(X^*)$ such that $h = \mathrm{s}(\overline{x}^*, \varGamma)$ a.e.

Proof. We assume for the simplicity that $E = \varOmega$. First notice that $\sigma(L_1(\mu), L_\infty(\mu))$-closure of the set $\mathrm{conv}\,[\mathcal{Z}_\varGamma]$ is a weakly compact convex subset of $L_1(\mu)$ ($[\mathcal{Z}_\varGamma]$ is the range of $\mathcal{Z}_\varGamma$ via $T_\varGamma$ in $L_1(\mu)$). Define then for every set $W \in \mathrm{cwk}(X)$ and $\varepsilon > 0$ the set

$$U_{W,\varepsilon} := \{x^* \in B(X^*) \colon |\,\mathrm{s}(x^* - x_0^*, W)| \le \varepsilon \;\;\&\;\; |\,\mathrm{s}(x_0^* - x^*, W)| \le \varepsilon\}$$

and let

$$V_{W,\varepsilon} := R \cap U_{W,\varepsilon}.$$

It can be easily verified that R and $U_{W,\varepsilon}$ are convex. Moreover, $U_{W,\varepsilon}$ is a $\tau(X^*,X)$-closed neighborhood of x_0^*. As x_0^* is in the $\tau(X^*,X)$-closure of R, we have $V_{W,\varepsilon} \neq \emptyset$ and the collection $\{V_{W,\varepsilon} \colon W \in \mathrm{cwk}(X), \varepsilon > 0\}$ has the finite intersection property.

Let $H_{W,\varepsilon}$ be the closure of $T_\varGamma(V_{W,\varepsilon})$ in $L_1(\mu)$. Since $T_\varGamma$ is weakly compact, the set $H_{W,\varepsilon}$, being weakly closed, is also weakly compact and the collection $\{H_{W,\varepsilon} \colon W \in \mathrm{cwk}(X), \varepsilon > 0\}$ has the finite intersection property. This yields now the existence of a function $h \in \bigcap_{(W,\varepsilon)} H_{W,\varepsilon}$. Let us fix (W,ε). Then there exists a sequence of functionals $x_n^* \in V_{W,\varepsilon}$ such that

$$\int_\varOmega |\,\mathrm{s}(x_n^*, \varGamma) - h|\, d\mu < 1/2^n \tag{19}$$

and consequently $\mathrm{s}(x_n^*, \varGamma) \longrightarrow h$ a.e.

Claim 1. $\mu\{\omega \in \varOmega \colon h(\omega) < \mathrm{s}(x_0^*, \varGamma(\omega))\} > 0.$

Proof. Suppose that $h \ge \mathrm{s}(x_0^*, \varGamma)$ a.e. It follows then from (19) that

$$\alpha \ge \int_\varOmega \mathrm{s}(x_n^*, \varGamma)\, d\mu \ge \int_\varOmega h\, d\mu - 1/2^n \ge \int_\varOmega \mathrm{s}(x_0^*, \varGamma)\, d\mu - 1/2^n \ge \beta - 1/2^n\,,$$

what is certainly false for sufficiently large $n \in \mathbb{N}$. The above contradiction proves the claim. □

Claim 2. If $x^*_{W,\varepsilon}$ is a weak* cluster point of $\langle x^*_n\rangle$, then $\mathrm{s}(x^*_{W,\varepsilon}, \Gamma) \leq h$ a.e.

Proof. Denote by Ω_0 the set $\{\omega\colon \mathrm{s}(x^*_n, \Gamma(\omega)) \nrightarrow h(\omega))\}$ and let $\eta > 0$ be arbitrary. For each $\omega \notin \Omega_0$, let $n_\omega \in \mathbb{N}$ be such that $|\,\mathrm{s}(x^*_n, \Gamma(\omega)) - h(\omega)| < \eta$, for every $n > n_\omega$. Then let $\langle z^*_\delta(\omega,\eta)\rangle_{\delta\in\Delta(\omega)}$ be a net composed of convex combinations of the set $\{x^*_k\colon k > n_\omega\}$ and $\tau(X^*, X)$ convergent to $x^*_{W,\varepsilon}$. If $z^*_\delta(\omega,\eta) = \sum_{n>n_\omega} a_{n,\omega} x^*_n$, then

$$\mathrm{s}(z^*_\delta(\omega,\eta), \Gamma(\omega)) \leq \sum_{n>n_\omega} a_{n,\omega}\, \mathrm{s}(x^*_n, \Gamma(\omega)) < \sum_{n>n_\omega} a_{n,\omega}(h(\omega)+\eta) \leq h(\omega)+\eta.$$

Consequently, $\mathrm{s}(x^*_{W,\varepsilon}, \Gamma(\omega)) \leq h(\omega)+\eta$, what proves the claim, because η is arbitrary. $\square$

Let us now fix for each (W,ε) a point $x^*_{W,\varepsilon}$ as above. Since $U_{W,\varepsilon}$ is convex and $\tau(X^*, X)$ closed we have $x^*_{W,\varepsilon} \in U_{W,\varepsilon}$.

Let us order partially the set $\{(W,\varepsilon) : W \in \mathrm{cwk}(X), \varepsilon > 0\}$ by setting $(W_1, \varepsilon_1) \prec (W_2, \varepsilon_2)$ if $W_1 \subset W_2$ and $\varepsilon_2 < \varepsilon_1$. If $(V,\delta) \in \mathrm{cwk}(X) \times (0,\infty)$ is fixed, then for each $(W,\varepsilon) \succ (V,\delta)$ the inequalities $|\,\mathrm{s}(x^*_{(W,\varepsilon)} - x^*_0, W)| < \varepsilon < \delta$ and $|\,\mathrm{s}(x^*_0 - x^*_{(W,\varepsilon)}, W)| < \varepsilon < \delta$ hold true. It follows that $x^*_{(W,\varepsilon)} \to x^*_0$ in $\tau(X^*, X)$ and so x^*_0 is a $\tau(X^*, X)$ cluster point of the set $\{x^*_{W,\varepsilon}\colon W \in \mathrm{cwk}(X), \varepsilon > 0\}$.

Together with *Claim 2* this proves that x^*_0 is a $\tau(X^*, X)$ cluster point of the set $\{x^* \in B(X^*)\colon\ \mathrm{s}(x^*, \Gamma) \leq h \text{ a.e.}\}$, as required.

Assume now that Γ is $\mathrm{ck}(X)$-valued. Then, instead of weakly compact sets W, we choose finite sets $F \subset \Omega$ and put in the above proof $W = \bigcup\{F(\omega)\colon \omega \in F\}$. Let $x^*_{(F,\varepsilon)} = w^*{-}\lim_\delta x^*_\delta$, where $\langle x^*_\delta\rangle_{\delta\in\Delta}$ is a subnet of $\langle x^*_n\rangle$ and $x^*_n \in V_{W,\varepsilon}$, $n \in \mathbb{N}$. Since for every ω the set $\Gamma(\omega)$ is compact, the function $x^* \longrightarrow \mathrm{s}(x^*, \Gamma(\omega))$ is weak*-continuous on $B(X^*)$ and so

$$\mathrm{s}(x^*_{(F,\varepsilon)}, \Gamma(\omega)) = \lim_\delta \mathrm{s}(x^*_\delta, \Gamma(\omega)) = h(\omega),$$

where the last equality holds a.e. We may take as $\overline{x}^*$ an arbitrary $x^*_{(F,\varepsilon)}$. $\square$

Lemma 4.2. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction. Assume that the operator T_Γ is weakly compact. If for an $E \in \Sigma^+_\mu$ the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Gamma)\, d\mu$ is not weak* lower semicontinuous, then there exist a functional $x^*_0 \in B(X^*)$, a function $h \in L_1(\mu)$ and a scalarly measurable multifunction $G\colon \Sigma \to \mathrm{cwk}(X)$ dominated by Γ such that*

$$\mu\{\omega \in E\colon h(\omega) < -\mathrm{s}(-x^*_0, G(\omega)) = \mathrm{s}(x^*_0, G(\omega)) = \mathrm{s}(x^*_0, \Gamma(\omega))\} > 0, \tag{20}$$

and x^*_0 is in the $\tau(X^*, X)$-closure of the set

$$\{x^* \in B(X^*)\colon\ \mathrm{s}(x^*, G) \leq h \ \textit{a.e. on } E\}.$$

If Γ is $\mathrm{ck}(X)$-valued, then the last set may be replaced by

$$\{x^* \in B(X^*)\colon\ \mathrm{s}(x^*, G) = h \ \textit{a.e. on } E\}$$

and x^*_0 is in its weak*-closure.

Proof. If $x^* \longrightarrow \int_E s(x^*, \Gamma)\, d\mu$ is not weak* lower semicontinuous, then there exists a point $x_0^* \in B(X^*)$ and real numbers $\alpha < \beta$ such that the relations (17) and (18) are fulfilled. Let $G\colon \Omega \to \mathrm{cwk}(X)$ be defined by

$$G(\omega) := \{x \in \Gamma(\omega) \colon x_0^*(x) = s(x_0^*, \Gamma(\omega))\}.$$

Due to [26, Lemma 3] the multifunction G is scalarly measurable. Moreover, the multifunction G satisfies the assumptions of Lemma 4.1 because $s(x^*, G(\omega)) \le s(x^*, \Gamma(\omega))$ for every x^* and $s(x_0^*, G(\omega)) = s(x_0^*, \Gamma(\omega))$, for every $\omega \in \Omega$. Consequently, there exists a function $h \in L_1(\mu)$ such that

$$\mu\{\omega \in E \colon h(\omega)) < s(x_0^*, G(\omega))\} > 0,$$

and x_0^* is in the $\tau(X^*, X)$-closure of the set

$$\{x^* \in B(X^*) \colon\ s(x^*, G) \le h \text{ a.e. on } E\}.$$

As x_0^* is constant on each $G(\omega)$ the condition (20) is satisfied. If G is $\mathrm{ck}(X)$-valued, then the last set may be replaced by

$$\{x^* \in B(X^*) \colon\ s(x^*, G) = h \text{ a.e. on } E\}$$

and x_0^* is in its weak*-closure. □

Definition 4.3. Let $\Gamma\colon \Omega \to \mathrm{c}(X)$ be a multifunction. For each $E \in \Sigma$ we define the core of Γ on E by the formula

$$\mathrm{cor}_\Gamma(E) := \bigcap_{\mu(N)=0} \overline{\mathrm{conv}}\, \Gamma(E \setminus N) = \bigcap_{\mu(N)=0} \overline{\mathrm{conv}} \left(\bigcup_{\omega \in E \setminus N} \Gamma(\omega) \right). \tag{21}$$

It is easily seen that $A \subset B$ implies $\mathrm{cor}_\Gamma(A) \subset \mathrm{cor}_\Gamma(B)$.

Lemma 4.4. *For each $(x^*, a) \in X^* \times \mathbb{R}$ let us set $G(x^*, a) := \{x \in X \colon x^*(x) \le a\}$ and $H(x^*, a) := \{x \in X \colon x^*(x) \ge a\}$. If $\Gamma\colon \Omega \to \mathrm{c}(X)$ is a multifunction and $E \in \Sigma$, then*

$$\mathrm{cor}_\Gamma(E) = \bigcap \{G(x^*, a) \colon\ s(x^*, \Gamma) \le a \text{ a.e. on } E\} \tag{22}$$

and

$$\mathrm{cor}_\Gamma(E) = \bigcap \{H(x^*, a) \colon\ -s(-x^*, \Gamma) \ge a \text{ a.e. on } E\}. \tag{23}$$

Moreover,

$$\bigcap \{H(x^*, a) \colon\ s(x^*, \Gamma) \ge a \text{ a.e. on } E\} \subseteq \mathrm{cor}_\Gamma(E). \tag{24}$$

and

$$\bigcap \{G(x^*, a) \colon\ -s(-x^*, \Gamma) \le a \text{ a.e. on } E\} \subseteq \mathrm{cor}_\Gamma(E). \tag{25}$$

Proof, (22). Take an arbitrary $x_0 \notin \mathrm{cor}_\Gamma(E)$. Then, there is $N \in \Sigma_0$ such that $x_0 \notin \overline{\mathrm{conv}}\,\Gamma(E \setminus N)$. Hence, there is $x_0^* \in X^*$ and $a \in \mathbb{R}$ such that

$$\begin{aligned} x_0^*(x_0) > a &\geq \mathrm{s}(x_0^*, \overline{\mathrm{conv}}\,\Gamma(E \setminus N)) \geq \mathrm{s}(x_0^*, \Gamma(E \setminus N)) \\ &= \sup\{\mathrm{s}(x_0^*, \Gamma(\omega))\colon \omega \in E \setminus N\}. \end{aligned}$$

It follows that $\mathrm{s}(x_0^*, \Gamma(\omega)) \leq a$ a.e. on E but $x_0^*(x_0) > a$ and so $x_0 \notin G(x_0^*, a)$. This proves one inclusion of (22).

To prove (25) notice that we have also $-\mathrm{s}(-x_0^*, \Gamma(\omega)) \leq a$ a.e. on E, but $x_0^*(x_0) > a$ and so $x_0 \notin G(x_0^*, a)$.

Take now an arbitrary $x_0 \in \mathrm{cor}_\Gamma(E)$ and let (x^*, a) be such that $\mathrm{s}(x^*, \Gamma(\omega)) \leq a$ for every $\omega \in E \setminus N$, where $\mu(N) = 0$. Then we have also $\mathrm{s}(x^*, \overline{\mathrm{conv}}\,\Gamma(\omega)) \leq a$ for every $\omega \in E \setminus N$. But we have $\mathrm{cor}_\Gamma(E) \subset \overline{\mathrm{conv}}\,\Gamma(E \setminus N)$ and so $\mathrm{s}(x^*, \mathrm{cor}_\Gamma(E)) \leq a$. In particular $x^*(x_0) \leq a$. This proves that $x_0 \in G(x^*, a)$ and so the formula (22) holds true.

In a similar way (23) and (24) can be proved. □

Corollary 4.5. *If $E \in \Sigma$ and $x_0 \in \mathrm{cor}_\Gamma(E)$, then*

$$\operatorname*{ess\,inf}_E [-\mathrm{s}(-x^*, \Gamma))] \leq x^*(x_0) \leq \operatorname*{ess\,sup}_E \mathrm{s}(x^*, \Gamma) \quad \textit{for every } x^* \in X^*.$$

Theorem 4.6. *A scalarly integrable multifunction $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ is Pettis integrable in $\mathrm{cwk}(X)$ if and only if it has the following properties:*

(WC) *$T_\Gamma\colon X^* \to L_1(\mu)$ is weakly compact;*
(CC) *If $\Delta\colon \Omega \to \mathrm{cwk}(X)$ is a scalarly measurable multifunction that is dominated by Γ, then $\mathrm{cor}_\Delta(E) \neq \emptyset$, for every $E \in \Sigma_\mu^+$.*

Proof. Assume first the Pettis integrability of Γ in $\mathrm{cwk}(X)$ and let $M_\Gamma\colon \Sigma \to \mathrm{cwk}(X)$ be the resulting h-multimeasure. The weak compactness of T_Γ is then a consequence of Proposition 2.2. If $\Delta\colon \Omega \to \mathrm{cwk}(X)$ is scalarly measurable and dominated by Γ, then it is Pettis integrable in $\mathrm{cwk}(X)$ (Corollary 1.5). If $E \in \Sigma_\mu^+$ and x^* are fixed, then

$$\mathrm{s}\left(x^*, \frac{M_\Delta(E)}{\mu(E)}\right) = \frac{1}{\mu(E)} \int_E \mathrm{s}(x^*, \Delta)\, d\mu \leq \mathrm{s}(x^*, \overline{\mathrm{conv}}\ \Delta(E)),$$

where the right hand side may be infinity and the set E may be replaced by any set $E \setminus N$ with $\mu(N) = 0$. The Hahn-Banach Theorem yields now $\frac{M_\Delta(E)}{\mu(E)} \subseteq \overline{\mathrm{conv}}\ \Delta(E \setminus N)$ and so $\mathrm{cor}_\Delta(E) \neq \emptyset$.

Assume now that the conditions (WC) and (CC) are fulfilled and Δ is dominated by Γ. Then, T_Δ is weakly compact. We are going to prove first that Δ is Pettis integrable in $\mathrm{cb}(X)$. According to Proposition 1.3, we have to show that for each $E \in \Sigma_\mu^+$ the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Delta)\, d\mu$ is weak* lower semicontinuous. So suppose that for some $E \in \Sigma_\mu^+$ the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Delta)\, d\mu$ is not weak* lower semicontinuous. Then, according to Lemma 4.2, there exist a scalarly measurable multifunction $G\colon \Sigma \to \mathrm{cwk}(X)$, a functional $x_0^* \in B(X^*)$ and $h \in L_1(\mu)$ such that

$$\mu\{\omega \in E\colon h(\omega) < -\,\mathrm{s}(-x_0^*, G(\omega)) = \mathrm{s}(x_0^*, G(\omega)) = \mathrm{s}(x_0^*, \Delta(\omega))\} > 0,$$

and x_0^* is in the $\tau(X^*, X)$-closure of the set

$$\{x^* \in B(X^*)\colon\ \mathrm{s}(x^*, G) \le h \text{ a.e. on } E\}.$$

Notice that $\mathrm{cor}_G(F) \neq \emptyset$, for each $F \in \Sigma_\mu^+$, because of (CC). Let us fix $\gamma, \delta \in \mathbb{R}$ such that

$$\mu\{\omega \in E\colon h(\omega) < \gamma < \delta < -\,\mathrm{s}(-x_0^*, G(\omega)) = \mathrm{s}(x_0^*, G(\omega)) = \mathrm{s}(x_0^*, \Delta(\omega))\} > 0.$$

Let

$$A := \{\omega \in E\colon h(\omega) < \gamma < \delta < -\,\mathrm{s}(-x_0^*, G(\omega)) = \mathrm{s}(x_0^*, G(\omega)) = \mathrm{s}(x_0^*, \Delta(\omega))\}$$

and, let $\langle x_\alpha^* \rangle$ be a net of functionals from $\{x^* \in B(X^*)\colon\ \mathrm{s}(x^*, G) \le h \text{ a.e. on } E\}$ that is $\tau(X^*, X)$-convergent to x_0^*. We have then $\mathrm{s}(x_\alpha^*, G) \le \gamma$ a.e. on A and so, it follows from Lemma 4.4(22) that

$$x_\alpha^*|_{\mathrm{cor}_G(A)} \le \gamma.$$

Consequently,

$$x_0^*|_{\mathrm{cor}_G(A)} \le \gamma.$$

On the other hand, as $-\,\mathrm{s}(-x_0^*, G(\omega)) \ge \delta$ whenever $\omega \in A$, it follows from Lemma 4.4(23) that

$$x_0^*|_{\mathrm{cor}_G(A)} \ge \delta.$$

This contradiction shows that the functional $x^* \longrightarrow \int_E \mathrm{s}(x^*, \Delta)\, d\mu$ is weak* lower semicontinuous, for every $E \in \Sigma_\mu^+$. Consequently, the multifunction Δ is Pettis integrable in $\mathrm{c}(X)$ and then in $\mathrm{cb}(X)$, due to its scalar integrability.

Now, we may apply Proposition 1.7 (We need here its weaker version: If every $\mathrm{cwk}(X)$-valued scalarly integrable multifunction dominated by Γ is Pettis integrable in $\mathrm{cb}(X)$, then Γ is Pettis integrable in $\mathrm{cwk}(X)$.). □

More careful analysis of the above proof shows that the following result also holds true.

Theorem 4.7. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction.*

If Γ is Pettis integrable in $\mathrm{cb}(X)$, then $\mathrm{cor}_\Gamma(E) \neq \emptyset$, for every $E \in \Sigma_\mu^+$.

If T_Γ is weakly compact and $\mathrm{cor}_\Delta(E) \neq \emptyset$, for every $E \in \Sigma_\mu^+$ and every extremal face Δ of Γ, then Γ is Pettis integrable in $\mathrm{cwk}(X)$. □

Theorem 4.6 allows us to generalize Proposition 1.7. We obtain in this way the result proved earlier in [6], by a different method.

Theorem 4.8. *Let $\Gamma\colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction with weakly compact T_Γ. If each scalarly measurable selection of Γ is Pettis integrable, then Γ is Pettis integrable in $\mathrm{cwk}(X)$.*

Proof. If $\varDelta\colon \varOmega \to \mathrm{cwk}(X)$ is a scalarly measurable multifunction dominated by $\varGamma$, then – according to [6] – it has a scalarly measurable selection. By our assumption the selection is Pettis integrable and so $\mathrm{cor}_{\varDelta}(E) \neq \emptyset$, for every $E \in \varSigma_\mu^+$. Consequently, in virtue of Theorem 4.6, $\varGamma$ is Pettis integrable in $\mathrm{cwk}(X)$. □

Remark 4.9. Let $\varGamma\colon \varOmega \to \mathrm{ck}(X)$ be a scalarly integrable multifunction. Even if $T_\varGamma$ is weakly compact and $\mathrm{cor}_\varGamma(E) \neq \emptyset$ for every $E \in \varSigma_\mu^+$, it may happen that $\varGamma$ is not Pettis integrable even in $\mathrm{cb}(X)$.

Let $(\varOmega, \varSigma, \mu)$ be the measure space considered in [11, Theorem 2B] and let $f\colon \varOmega \to l_\infty(\varOmega)$ be the bounded and scalarly measurable function considered there, that is not Pettis integrable with respect to μ. Define $\varGamma\colon \varOmega \to \mathrm{cwk}(l_\infty(\varOmega))$ by the formula $\varGamma(\omega) := \mathrm{conv}\{0, f(\omega)\}$. Then $\varGamma$ is scalarly integrable and the zero function is its Pettis integrable selection. Consequently, $\mathrm{cor}_\varGamma(E) \neq \emptyset$, for every $E \in \varSigma_\mu^+$. If ν_f is the Gelfand integral of f, then the boundedness of f yields the weak relative compactness of $\nu_f(\varSigma)$. Suppose that $\varGamma$ is Pettis integrable in $\mathrm{cb}(l_\infty)$ and let $M_\varGamma$ be its Pettis integral. Due to the specific form of $\varGamma$, one can easily check that $M_\varGamma(\varSigma) \subset \overline{\mathrm{conv}}\,\{0, \nu_f(\varSigma)\}$, and the last set is weakly compact. But this means that $\varGamma$ is Pettis integrable in $\mathrm{cwk}(l_\infty)$. On the other hand, f is a non-Pettis integrable selection of $\varGamma$ and so, by Corollary 1.5, $\varGamma$ cannot be Pettis integrable in $\mathrm{cwk}(l_\infty(\varOmega))$. □

5. Convergence theorems.

The theorem I am going to present is a generalization of the Vitali convergence theorem for the Pettis integrable functions proved by Musiał (cf. [19] or [21]). The assumptions of this theorem guarantee that for each $x^* \in X^*$ and $E \in \varSigma$ the sequence $\left\{\int_E \mathrm{s}(x^*, \varGamma_n)\, d\mu \colon n \in \mathbb{N}\right\}$ is convergent to $\int_E \mathrm{s}(x^*, \varGamma)\, d\mu$, and that the set $\{\mathrm{s}(x^*, \varGamma)\colon x^* \in B(X^*)\}$ is weakly relatively compact in $L_1(\mu)$. They may be replaced by any others guaranteeing the above weak compactness and the convergence of the appropriate sequences of scalar integrals.

Theorem 5.1. *Let $\varGamma\colon \varOmega \to \mathrm{cwk}(X)$ be scalarly integrable and let $\{\varGamma_n\colon \varOmega \to \mathrm{cwk}(X)\colon n \in \mathbb{N}\}$ be a sequence of multifunctions Pettis integrable in $\mathrm{cwk}(X)$ and satisfying the following two conditions:*

(*a*) *the set $\{\mathrm{s}(x^*, \varGamma_n)\colon \|x^*\| \le 1,\ n \in \mathbb{N}\}$ is uniformly integrable;*
(*b*) $\lim_n \mathrm{s}(x^*, \varGamma_n) = \mathrm{s}(x^*, \varGamma)$ *in μ–measure, for each $x^* \in X^*$.*

Then $\varGamma$ is Pettis integrable in $\mathrm{cwk}(X)$ and,

(*SC*)

$$\lim_n s\left(x^*, \int_E \varGamma_n\, d\mu\right) = \mathrm{s}\left(x^*, \int_E \varGamma\, d\mu\right)$$

for every $x^* \in X^*$ and $E \in \varSigma$.

If $\varGamma$ and $\varGamma_n$'s are assumed to be $\mathrm{c}(X)$-valued and determined by a WCG space and, $\varGamma_n$'s are Pettis integrable in $\mathrm{cb}(X)$, then $\varGamma$ is Pettis integrable in $\mathrm{cb}(X)$ and (SC) holds true.

Proof. [Integrability in $\mathrm{cwk}(X)$] Given $n \in \mathbb{N}$, let Y_n be a WCG space generated by a weakly compact convex set $W_n \subset B(X)$ and determining the multifunction $\mathit{\Gamma}_n$. Then the set $\sum_n \frac{1}{2^n} W_n$ is a weakly compact set generating a space Y. It follows from (b) that $\mathit{\Gamma}$ is determined by Y.

The set $\{\mathrm{s}(x^*, \mathit{\Gamma}_n) \colon \|x^*\| \le 1,\ n \in \mathbb{N}\}$ is weakly relatively compact in $L_1(\mu)$ and so, due to (b), $T_{\mathit{\Gamma}}$ is weakly compact. It follows then from Theorem 2.5 that $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cwk}(X)$. The required scalar convergence is a direct consequence of the Vitali convergence theorem for real valued functions.

[Integrability in $\mathrm{cb}(X)$] We apply Theorem 2.4 instead of 2.5. □

One may ask if the scalar convergence (SC) can be replaced by a stronger one. In general the answer is negative even for Pettis integrable functions (see the next remark) but assuming stronger convergence of the multifunctions one obtains also a stronger convergence of the corresponding integrals.

Theorem 5.2. *Let $\mathit{\Gamma} \colon \mathit{\Omega} \to \mathrm{cwk}(X)$ be scalarly integrable and let $\{\mathit{\Gamma}_n \colon \mathit{\Omega} \to \mathrm{cwk}(X) \colon n \in \mathbb{N}\}$ be a sequence of multifunctions Pettis integrable in $\mathrm{cwk}(X)$ and satisfying the following two conditions:*

(a) *the set $\{\mathrm{s}(x^*, \mathit{\Gamma}_n) \colon \|x^*\| \le 1,\ n \in \mathbb{N}\}$ is uniformly integrable;*
(b) *$\lim_n d_H(\mathit{\Gamma}_n, \mathit{\Gamma}) = 0$ a.e. (d_H is the Hausdorff distance).*

Then $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cwk}(X)$ and,

$$\lim_n \sup_{\|x^*\| \le 1} \int_{\mathit{\Omega}} |\,\mathrm{s}(x^*, \mathit{\Gamma}_n) - \mathrm{s}(x^*, \mathit{\Gamma})|\, d\mu = 0.$$

for every $x^* \in X^*$ and $E \in \mathit{\Sigma}$.

In particular

$$\int_E \mathit{\Gamma}_n \, d\mu \longrightarrow \int_E \mathit{\Gamma} \, d\mu$$

in the Hausdorff metric, uniformly on $\mathit{\Sigma}$.

If $\mathit{\Gamma}$ and $\mathit{\Gamma}_n$'s are assumed to be $\mathrm{cb}(X)$-valued and determined by a WCG space and, $\mathit{\Gamma}_n$'s are Pettis integrable in $\mathrm{cb}(X)$, then $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cb}(X)$ and the above two convergences hold true.

Proof. It follows from Theorem 5.1 that $\mathit{\Gamma}$ is Pettis integrable in $\mathrm{cwk}(X)$ (or in $\mathrm{cb}(X)$). Together with the condition (a) this yields the uniform integrability of the family $\{\mathrm{s}(x^*, \mathit{\Gamma}_n) - \mathrm{s}(x^*, \mathit{\Gamma}) \colon \|x^*\| \le 1,\ n \in \mathbb{N}\}$. Let us fix an arbitrary $\varepsilon > 0$ and let $\delta > 0$ be such that $\mu(E) < \delta$ implies $\int_E |\,\mathrm{s}(x^*, \mathit{\Gamma}_n) - \mathrm{s}(x^*, \mathit{\Gamma})|\, d\mu < \varepsilon$, for every $x^* \in B(X^*)$ and $E \in \mathit{\Sigma}$. Since for each $\omega \in \mathit{\Omega}$, $x^* \in B(X^*)$ and $n \in \mathbb{N}$ the inequality $|\,\mathrm{s}(x^*, \mathit{\Gamma}_n(\omega)) - \mathrm{s}(x^*, \mathit{\Gamma}(\omega))| \le d_H(\mathit{\Gamma}_n(\omega), \mathit{\Gamma}(\omega))$ holds true, we can, as in the proof of Proposition 1.2 find a measurable non-negative function f_n such that

$$\forall\, x^* \in B(X^*)\ |\,\mathrm{s}(x^*, \mathit{\Gamma}_n) - \mathrm{s}(x^*, \mathit{\Gamma})| \le f_n \le d_H(\mathit{\Gamma}_n, \mathit{\Gamma}) \ \text{ a.e.}$$

By the assumption (b) the sequence $\langle f_n \rangle$ is a.e. convergent to zero. In particular, there is a set $\widetilde{\mathit{\Omega}} \in \mathit{\Sigma}$ such that $\mu(\mathit{\Omega} \setminus \widetilde{\mathit{\Omega}}) < \delta$ and $\langle f_n|_{\widetilde{\mathit{\Omega}}} \rangle$ is uniformly convergent to zero. Thus, if $E \in \Sigma$, then

$$\begin{aligned} d_H(M_{\Gamma_n}(E), M_\Gamma(E)) &= \sup_{\|x^*\|\le 1} |\mathrm{s}(x^*, M_{\Gamma_n}(E)) - \mathrm{s}(x^*, M_\Gamma(E))| \\ &\le \sup_{\|x^*\|\le 1} \int_E |\,\mathrm{s}(x^*, \Gamma_n) - \mathrm{s}(x^*, \Gamma)|\, d\mu \\ &\le \sup_{\|x^*\|\le 1} \int_{E\cap\widetilde{\Omega}} |\,\mathrm{s}(x^*, \Gamma_n) - \mathrm{s}(x^*, \Gamma)|\, d\mu \\ &\qquad + \sup_{\|x^*\|\le 1} \int_{E\setminus\widetilde{\Omega}} |\,\mathrm{s}(x^*, \Gamma_n) - \mathrm{s}(x^*, \Gamma)|\, d\mu \\ &\le \int_{\widetilde{\Omega}} f_n\, d\mu + \sup_{\|x^*\|\le 1} \int_{\Omega\setminus\widetilde{\Omega}} |\,\mathrm{s}(x^*, \Gamma_n) - \mathrm{s}(x^*, \Gamma)|\, d\mu < 2\varepsilon \end{aligned}$$

for sufficiently large n's, because $\mu(\Omega \setminus \widetilde{\Omega}) < \delta$. □

The following corollary strengthens the result of Rodriguez [23, Theorem 2.8] for sequences of Pettis integrable functions that are pointwise norm convergent.

Corollary 5.3. *Let $\langle f_n \rangle$ be a sequence of X-valued Pettis integrable functions. Assume that the set $\{x^* f_n \colon \|x^*\| \le 1,\ n \in \mathbb{N}\}$ is uniformly integrable and there is a function $f\colon \Omega \to X$ such that $\|f_n - f\| \to 0$ a.e.*

Then f is Pettis integrable and $\|f_n - f\|_P \to 0$. In particular,

$$\left\| \int_E f_n\, d\mu - \int_E f\, d\mu \right\| \to 0$$

uniformly on Σ.

Remark 5.4. If X is infinite dimensional, then there is a sequence $\langle f_n \rangle$ of Pettis integrable functions satisfying (a), (b) and (SC) with a Pettis integrable f as Γ and such that the convergence

$$\left\| \int_E f_n\, d\mu - \int_E f\, d\mu \right\| \to 0 \tag{26}$$

fails for every $E \in \Sigma_\mu^+$. Indeed, let $\langle x_n \rangle \subset B(X)$ be a sequence that is weakly, but not norm, convergent to zero. Let $\Gamma_n = f_n \equiv x_n$, for every n. Then f_n's are Pettis integrable functions and the conditions (a), (b) are fulfilled with $\Gamma = 0$. Notice that $M_0(\Sigma) = \{0\}$ is a compact set.

Consider now an arbitrary sequence $\langle f_n \rangle$ of Pettis integrable simple functions that is convergent to a Pettis integrable $f\colon \Omega \to X$ in the sense

$$\forall\, x^*\ \forall\, E \in \Sigma\ \int_E x^* f_n\, d\mu \longrightarrow \int_E x^* f\, d\mu.$$

If the set $M_f(\Sigma)$ is not norm relatively compact, then there are even no functions $g_n \in \mathrm{conv}\{f_k \colon k \ge n\}$ such that $\||g_n - f|\|_P \to 0$. Indeed, if there was such a sequence

$\langle g_n \rangle$, then for a fixed $\varepsilon > 0$ there would exist $n_0 \in \mathbb{N}$ such that $|||g_{n_0} - f|||_P < \varepsilon$. If $\{\int_{E_i} g_{n_0}\,d\mu \colon i \le p\}$ is an ε-mesh for $M_{g_{n_0}}(\Sigma)$ then, for each $E \in \Sigma$ there is $i \le p$ such that

$$\left\|\int_{E_i} g_{n_0}\,d\mu - \int_E f\,d\mu\right\| \le \left\|\int_E g_{n_0}\,d\mu - \int_E f\,d\mu\right\| + \left\|\int_E g_{n_0}\,d\mu - \int_{E_i} g_{n_0}\,d\mu\right\| < 2\varepsilon.$$

Consequently, $\{\int_{E_i} g_{n_0}\,d\mu \colon i \le p\}$ is a 2ε-mesh for $M_f(\Sigma)$ and so $M_f(\Sigma)$ is norm relatively compact. □

We get from the above considerations the following interesting result:

Corollary 5.5. *Let $\mathbb{P}(\mu, X)$ be the space of all X-valued Pettis integrable functions furnished with the norm $\|\cdot\|_P$. If there is a function $f \in \mathbb{P}(\mu, X)$ such that $M_f(\Sigma)$ is separable but not norm relatively compact, then the topology on $\mathbb{P}(\mu, X)$ induced by the duality $(\mathbb{P}(\mu, X), L_\infty(\mu) \otimes X^*)$ is strictly weaker, then the weak topology of $\mathbb{P}(\mu, X)$.*

Proof. Clearly $\sigma(\mathbb{P}(\mu, X), L_\infty(\mu) \otimes X^*) \subset \sigma(\mathbb{P}(\mu, X), \mathbb{P}(\mu, X)^*)$. In case of equality we could apply theorem of Mazur to each sequence $\langle f_k \rangle$ convergent in the topology $\sigma(\mathbb{P}(\mu, X), L_\infty(\mu) \otimes X^*)$ to get a sequence of convex combinations converging to the same limit in the norm topology of $\mathbb{P}(\mu, X)$, contradicting the earlier considerations. The existence of a sequence $\langle f_k \rangle$ of simple functions converging to f in the topology $\sigma(\mathbb{P}(\mu, X), L_\infty(\mu) \otimes X^*)$ is a consequence of [19, Theorem 3] (see also [20, Theorem 10.1], [21, Theorem 5.3] and [25, Theorem 5-3-2]). □

Theorem 5.6. *Let X be a Banach space not containing any isomorphic copy of c_0 and let $\Gamma \colon \Omega \to \mathrm{cwk}(X)$ be a scalarly integrable multifunction. If $\{\Gamma_n \colon \Omega \to \mathrm{cwk}(X) \colon n \in \mathbb{N}\}$ is a sequence of multifunctions Pettis integrable in $\mathrm{cwk}(X)$ and satisfying the condition*

$$\lim_n \int_E \mathrm{s}(x^*, \Gamma_n)\,d\mu = \int_E \mathrm{s}(x^*, \Gamma)\,d\mu \quad \textit{for all } E \in \Sigma \textit{ and all } x^* \in X^*, \tag{27}$$

then Γ is Pettis integrable in $\mathrm{cwk}(X)$ and (SC) holds true for every $x^* \in X^*$ and $E \in \Sigma$.

If Γ and Γ_n's are assumed to be $\mathrm{c}(X)$-valued and determined by a WCG space and, Γ_n's are Pettis integrable in $\mathrm{cb}(X)$, then Γ is Pettis integrable in $\mathrm{cb}(X)$ and (SC) holds true.

Proof. Due to Theorem 2.13 we have to check only if Γ is determined by a WCG space but this is a direct consequence of (27). □

I am going to formulate now two consequences of Theorem 5.1. The first one is the Lebesgue Domination Convergence Theorem generalizing the corresponding theorem of Geitz and Musiał [20, Theorem 8.2].

Theorem 5.7. *Let $\Gamma \colon \Omega \to \mathrm{cwk}(X)$ be scalarly integrable and let $\{\Gamma_n \colon \Omega \to \mathrm{cwk}(X) \colon n \in \mathbb{N}\}$, be a sequence of multifunctions that are Pettis integrable in $\mathrm{cwk}(X)$ and satisfy the following conditions:*

(*a'*) *There exists a function $h \in L_1(\mu)$ such that for every $x^* \in B(X^*)$ and every $n \in \mathbb{N}$*

$$|\,\mathrm{s}(x^*, \varGamma_n)| \leq h \quad \text{a.e.},$$

(*b*) $\lim_n \mathrm{s}(x^*, \varGamma_n) = \mathrm{s}(x^*, \varGamma)$ *in μ–measure, for each $x^* \in X^*$,*

then $\varGamma$ is Pettis integrable in cwk(X) *and* (*SC*) *holds true for every $x^* \in X^*$ and $E \in \varSigma$.*

If $\varGamma$ and $\varGamma_n$'s are assumed to be c(X)*-valued and determined by a WCG space and, $\varGamma_n$'s are Pettis integrable in* cb(X)*, then $\varGamma$ is Pettis integrable in* cb(X) *and* (*SC*) *holds true.*

The next result was first formulated in case of Pettis integrable functions in [19, Theorem 2].

Theorem 5.8. *Let $\varGamma, \varGamma_n \colon \varOmega \to \mathrm{c}(X), n \in \mathbb{N}$, be scalarly integrable multifunctions such that*

$\lim_n \mathrm{s}(x^*, \varGamma_n) = \mathrm{s}(x^*, \varGamma)$ *in μ–measure, for each $x^* \in X^*$. Let $\varDelta \colon \varSigma \to \mathrm{cwk}(X)$ be a multifunction Pettis integrable in* cwk(X) *and satisfying one of the following conditions:*

(α) *for every $x^* \in B(X^*)$ and every $n \in \mathbb{N}$* $|\,\mathrm{s}(x^*, \varGamma_n)| \leq |\,\mathrm{s}(x^*, \varDelta)|$ a.e., *or*
(β) *for every $x^* \in B(X^*)$ and every $n \in \mathbb{N}$* $\mathrm{s}(x^*, \varGamma_n) \leq \mathrm{s}(x^*, \varDelta)$ a.e.

Then $\varGamma$ is Pettis integrable in cb(X) *and* (*SC*) *holds true for every $x^* \in X^*$ and $E \in \varSigma$.*

If $\varGamma, \varGamma_n \colon \varOmega \to \mathrm{cwk}(X), n \in \mathbb{N}$, then $\varGamma$ is Pettis integrable in cwk(X)*.*

Proof. In case of (α) the integrability of each $\varGamma_n$ and $\varGamma$ is a simple consequence of Theorem 2.6, because $\varDelta$ is Pettis integrable in cwk(X). In case of (β) it is a consequence of Corollary 1.5. The weak compactness of $T_\varDelta$ yields then the uniform integrability of the set $\{\mathrm{s}(x^*, \varGamma_n) \colon \|x^*\| \leq 1,\ n \in \mathbb{N}\}$ and so the equality (*SC*) follows directly from Theorem 5.1. □

Remark 5.9. The assumption (β) of Theorem 5.8 is satisfied in particular if $\varGamma_n(\omega) \subseteq \varGamma(\omega)$, for almost all ω. In case of Pettis integrable functions such an assumption makes the result completely trivial but in case of multifunctions this makes sense. □

In case of ck(X)-valued multifunctions taking their values in a separable Banach space X, the next result can be found in Amrani [1].

Theorem 5.10. *Let $\varGamma, \varGamma_n \colon \varOmega \to \mathrm{c}(X), n \in \mathbb{N}$ be scalarly integrable and such that $\lim_n \mathrm{s}(x^*, \varGamma_n) = \mathrm{s}(x^*, \varGamma)$ in μ–measure, for each $x^* \in X^*$. Let $\varDelta \colon \varSigma \to \mathrm{ck}(X)$ be a multifunction Pettis integrable in* ck(X) *and such that for every $x^* \in B(X^*)$ and every $n \in \mathbb{N}$*

(β) $\mathrm{s}(x^*, \varGamma_n) \leq \mathrm{s}(x^*, \varDelta)$ a.e.

Then Γ is Pettis integrable in $\mathrm{ck}(X)$ *and,*

$$\int_E \Gamma_n \, d\mu \longrightarrow \int_E \Gamma \, d\mu$$

in the Hausdorff metric, for every $E \in \Sigma$.

Proof. The integrability of Γ_n's in $\mathrm{cwk}(X)$ is a consequence of Theorem 5.8. Then (β) implies the inclusions $M_{\Gamma_n}(E) \subset M_\Delta(E)$ and $M_\Gamma(E) \subset M_\Delta(E)$, for every $E \in \Sigma$ and every $n \in \mathbb{N}$. This forces the compactness of the integrals. Moreover, applying the integrability of Δ in $\mathrm{ck}(X)$, it is easy to see that the functions $\mathrm{s}(\cdot, \int_E \Gamma_n \, d\mu), n \in \mathbb{N}$, are equicontinuous in the weak* topology of $B(X^*)$. Indeed, let $E \in \Sigma$, $\delta > 0$ be fixed and let $F_n(x^*) := \mathrm{s}(x^*, \int_E \Gamma_n \, d\mu)$, $n \in \mathbb{N}$. By the assumption, all functions F_n are $\sigma(X^*, X)$-continuous on $B(X^*)$. Moreover, it follows from (β) that $\int_E \Gamma_n \, d\mu \subset \int_E \Gamma \, d\mu$, for every n. Then, let $\{x_1, \ldots, x_k\} \subset \int_E \Gamma \, d\mu$ be a δ-mesh of $\int_E \Gamma \, d\mu$ and $U := \{x^* \colon |x^*(x_i)| < \delta, i = 1, \ldots, k\}$ be a weak* neighborhood of zero. If $x_1^*, x_2^* \in U \cap B(X^*)$, then for every n let $a_n, b_n \in \int_E \Gamma_n \, d\mu$ be such that $\langle x_1^* - x_2^*, a_n \rangle = \mathrm{s}(x_1^* - x_2^*, \int_E \Gamma_n \, d\mu)$ and $\langle x_2^* - x_1^*, b_n \rangle = \mathrm{s}(x_2^* - x_1^*, \int_E \Gamma_n \, d\mu)$. We have then

$$\begin{aligned}
|F_n(x_1^*) - F_n(x_2^*)| &= \left| \mathrm{s}\left(x_1^*, \int_E \Gamma_n \, d\mu\right) - \mathrm{s}\left(x_2^*, \int_E \Gamma_n \, d\mu\right) \right| \\
&\le \left| \mathrm{s}\left(x_1^* - x_2^*, \int_E \Gamma_n \, d\mu\right) \right| + \left| \mathrm{s}\left(x_2^* - x_1^*, \int_E \Gamma_n \, d\mu\right) \right| \\
&= |\langle x_1^* - x_2^*, a_n \rangle| + |\langle x_2^* - x_1^*, b_n \rangle| \\
&\le |\langle x_1^* - x_2^*, a_n - x_j \rangle| + |\langle x_1^* - x_2^*, x_j \rangle| \\
&\qquad + |\langle x_2^* - x_1^*, b_n - x_k \rangle| + |\langle x_2^* - x_1^*, x_k \rangle| \le 6\delta,
\end{aligned}$$

where x_j, x_k are chosen in such a way that $\|a_n - x_j\| < \delta$ and $|b_n - x_k| < \delta$. As δ is arbitrary, the sequence $\langle F_n \rangle$ is equicontinuous on $(B(X^*), w^*)$. Now the Ascoli theorem yields the existence of a subsequence $\langle F_{n_k} \rangle$ that is uniformly convergent on $B(X^*)$. Equivalently, it is convergent in the Hausdorff metric. □

6. Fatou type lemmata.

Fatou type inclusions in the theory of multifunctions form quite an abundant topic, which in case of non-separable Banach spaces deserves further deep investigation. I present here one example of such an inclusion in case of the Kuratowski sequential upper limit for the weak topology.

Definition 6.1. If $\langle H_n \rangle$ is a sequence of nonvoid subsets of X, then the Kuratowski sequential upper limit of $\langle H_n \rangle$ relatively to $\sigma(X, X^*)$ is denoted by $w - Ls(H_n)$ and defined by the formula

$$w - Ls(H_n) := \left\{ x \in X \colon \exists\, n_1 < n_2 < \ldots \ \forall\, k\ \exists\, x_k \in H_{n_k}\ x_k \xrightarrow{\sigma(X, X^*)} x \right\}.$$

□

The following lemma has been proven in [12, Proposition 3.10] for a separable Banach space, but the proof remains valid in the general case.

Proposition 6.2. *Let $\langle H_n \rangle$ be a sequence of nonvoid subsets of X contained in the same weakly compact set. Then,*

$$\overline{\text{conv}}[w - Ls(H_n)] = \bigcap_{n=1}^{\infty} \overline{\text{conv}}\Big[\bigcup_{k \geq n} H_n\Big] \tag{28}$$

and

$$\text{s}(x^*, \overline{\text{conv}}[w - Ls(H_n)] = \limsup_n \text{s}(x^*, H_n) \quad \textit{for every } x^* \in X^*. \tag{29}$$

□

The subsequent proposition is a multivalued version of the Fatou lemma for Kuratowski's upper limit relatively to the topology $\sigma(X, X^*)$, in case of a sequence of Pettis integrable multifunctions. The result has been proven by Ziat [27, Corollary 4.3] for separable Banach spaces. His proof heavily depends on the separability of X.

Proposition 6.3. *Let $\varGamma_n \colon \varSigma \to \text{cwk}(X)$ be a sequence of multifunctions that are Pettis integrable in $\text{cwk}(X)$ and are determined by a WCG space $Y \subset X$. Assume that there is a multifunction $\varGamma \colon \varSigma \to \text{cwk}(X)$ such that:*

(i) $\varGamma_n(\omega) \subseteq \varGamma(\omega)$, *for all $n \in \mathbb{N}$ and all $\omega \in \varOmega$;*
(ii) *The collection $\{\text{s}(x^*, \varGamma_n) \colon \|x^*\| \leq 1, n \in \mathbb{N}\}$ is uniformly integrable.*

Then

(Ks) $w - Ls\left[\int_E \varGamma_n \, d\mu\right] \subseteq \int_E \overline{\text{conv}}[w - Ls(\varGamma_n(\omega))] \, d\mu(\omega)$, *for every $E \in \varSigma$.*

Proof. Let $\varXi(\omega) := w - Ls(\varGamma_n(\omega))$, for every ω. The weak measurability of $\varXi$ follows from Proposition 6.2 and its scalar quasi-integrability is a consequence of Proposition 1.14. It is my aim now to prove the Pettis integrability of $\overline{\text{conv}}\,\varXi$ in $c(Y)$. As usual, I am going to show that the sublinear functional

$$\varphi(y^*) := \int_\varOmega \text{s}(y^*_e, \varXi) \, d\mu.$$

is weak* lower semicontinuous on Y^*. That is given $\alpha \in \mathbb{R}$ we have to prove that the set

$$Q_\alpha := \left\{ y^* \in B(Y^*) \colon \int_\varOmega \text{s}(y^*_e, \varXi) \, d\mu \leq \alpha \right\}$$

is weak*-closed. So let $\langle y^*_k \rangle$ be a weak*-convergent sequence of points in Q_α, converging to $\overline{y}^*$, and let x^*_k be a sequence in $B(X^*)$, such that $x^*_k|Y = y^*_k$ for every $k \in \mathbb{N}$. Then pick out a Pettis integrable selection f_n of $\varGamma_n$, $n = 1, 2 \ldots$ By the assumption (ii) the collection $H := \{x^* f_n \colon n \in \mathbb{N}, \|x^*\| \leq 1\}$ is weakly relatively compact and so for each k there exists a subsequence of $\langle x^*_k f_n \rangle_n$ that is weakly convergent to a function $h_k \in L_1(\mu)$. Then a convex combination of such subsequence is convergent to h_k in $L_1(\mu)$ and a.e. An appropriate application of the diagonal method yields the existence of $g_m \in \mathrm{conv}\{f_n \colon n \geq m\}$ such that

$$\forall\, k \in \mathbb{N}\ x_k^* g_m \longrightarrow h_k \quad \text{a.e.}$$

Then, according to (i) for each ω the sequence $\langle g_m(\omega)\rangle$ has a weak cluster point $g(\omega) \in \varXi(\omega)$. In particular,

$$\mathrm{s}(x^*, \varXi(\omega) - g(\omega)) \geq 0 \quad \text{for every } x^* \text{ and, } h_k = x_k^* g \text{ a.e. for every } k.$$

As $\{h_k \colon k \in \mathbb{N}\} \subset \overline{\mathrm{conv}}\, H$ and the last set is weakly compact in $L_1(\mu)$, there exist a function $h \in L_1(\mu)$ and a sequence $\langle h_{k_n}\rangle$ such that

$$x_{k_n}^* g = h_{k_n} \longrightarrow h, \quad \text{weakly in } L_1(\mu).$$

Then, due to Mazur's Theorem, there is an increasing sequence $\langle p_n\rangle$ of integers and nonnegative reals $\{a_{ni} \colon p_n < i \leq p_{n+1},\ n \in \mathbb{N}\}$ such that $\sum_{i=p_n+1}^{p_{n+1}} a_{ni} = 1,\ n \in \mathbb{N}$, and

$$w_n := \sum_{i=p_n+1}^{p_{n+1}} a_{ni} h_{k_i} \longrightarrow h \quad \text{a.e. and in } L_1(\mu), \text{ when } n \to \infty.$$

If

$$z_n^* := \sum_{i=p_n+1}^{p_{n+1}} a_{ni} x_{k_i}^*, \quad n \in \mathbb{N},$$

then $z_n^*|Y \in Q_\alpha$, for every $n \in \mathbb{N}$ and,

$$w_n = z_n^* g \longrightarrow h \quad \text{a.e. and in } L_1(\mu). \tag{30}$$

If $\overline{z}^*$ is a weak* cluster point of $\langle z_n^*\rangle$, then a subnet of $\langle z_n^* g\rangle$ is pointwise converging to $\overline{z}^* g$. Consequently,

$$h = \overline{z}^* g \quad \text{a.e.}, \quad \overline{z}^*|Y = \overline{y}^* \quad \text{and} \quad \lim_n \int_\Omega z_n^* g\, d\mu = \int_\Omega \overline{z}^* g\, d\mu. \tag{31}$$

In particular, the sequence $\langle \int_\Omega z_n^* g\, d\mu\rangle$ is bounded and so there is $\beta \in \mathbb{R}$ such that

$$\int_\Omega \mathrm{s}\left(z_n^*, \Xi - g\right)\, d\mu \leq \beta.$$

Applying [4] one can find a measurable function v, an increasing sequence $\langle q_n\rangle$ of integers and nonnegative reals $\{b_{ni} \colon q_n < i \leq q_{n+1},\ n \in \mathbb{N}\}$ such that $\sum_{i=q_n+1}^{q_{n+1}} b_{ni} = 1$, for every $n \in \mathbb{N}$ and,

$$v_n := \sum_{i=q_n+1}^{q_{n+1}} b_{ni}\, \mathrm{s}(z_i^*, \Xi - g) \longrightarrow v \quad \text{a.e.} \tag{32}$$

Let for each $n \in \mathbb{N}$

$$r_n^* := \sum_{i=q_n+1}^{q_{n+1}} b_{ni} z_i^*$$

and let $\widetilde{r}^* \in X^*$ be a weak* cluster point of the sequence $\langle r_n^* \rangle$. Notice that $\widetilde{r}^*|Y = \overline{y}^*$. Due to (30) and (31) we have $z_n^* g \to \overline{z}^* g$ a.e. and in $L_1(\mu)$. Hence, we have also $r_n^* g \to \overline{z}^* g$ a.e. and in $L_1(\mu)$. If $\langle r_\gamma^* \rangle$ is a subnet of $\langle r_n^* \rangle$ w*-converging to $\widetilde{r}^*$, then clearly $r_\gamma^* g \to \widetilde{r}^* g$ a.e. and so $\widetilde{r}^* g = \overline{z}^* g$ a.e. It follows that $\widetilde{r}^* g$ is measurable and

$$r_n^* g \to \widetilde{r}^* g \ \text{ in } L_1(\mu). \tag{33}$$

By the assumption each set $\varXi(\omega)$ is closed and so $\mathrm{s}(\bullet, \varXi(\omega) - g(\omega))$ is weak* lower semicontinuous: if $\langle r_\gamma^* \rangle$ is the subnet of $\langle r_n^* \rangle$ $\sigma(X^*, X)$-converging to $\widetilde{r}^*$, then

$$\mathrm{s}(\widetilde{r}^*, \varXi(\omega) - g(\omega)) \leq \liminf_\gamma \mathrm{s}(r_\gamma^*, \varXi(\omega) - g(\omega)).$$

Taking into account the subadditivity of support functions and the equality (32) we have for almost all ω

$$\mathrm{s}(\widetilde{r}^*, \varXi(\omega) - g(\omega)) \leq \liminf_\gamma \mathrm{s}(r_\gamma^*, \varXi(\omega) - g(\omega)) \leq \liminf_\gamma v_\gamma(\omega) = \lim_n v_n(\omega).$$

Consequently, applying Fatou's Lemma

$$\begin{aligned} &\int_\varOmega \mathrm{s}(\widetilde{r}^*, \varXi)\, d\mu - \int_\varOmega \widetilde{r}^* g \, d\mu \\ &= \int_\varOmega \mathrm{s}(\widetilde{r}^*, \varXi - g)\, d\mu \leq \int_\varOmega \lim_n v_n \, d\mu \leq \liminf_n \int_\varOmega v_n \, d\mu \\ &= \liminf_n \int_\varOmega \sum_{i=q_n+1}^{q_{n+1}} b_{ni}\, \mathrm{s}(z_i^*, \Xi)\, d\mu - \lim_n \int_\varOmega r_n^* g \, d\mu. \end{aligned} \tag{34}$$

Applying (33) to (34), we obtain

$$\varphi(\overline{y}^*) = \int_\varOmega \mathrm{s}(\widetilde{r}^*, \varXi)\, d\mu \leq \liminf_n \int_\varOmega \sum_{i=q_n+1}^{q_{n+1}} b_{ni}\, \mathrm{s}(z_i^*, \Xi)\, d\mu \leq \alpha.$$

This proves the weak*-closeness of Q_α and the Pettis integrability of $\omega \longrightarrow \overline{\mathrm{conv}}[w - Ls(\varGamma_n(\omega))]$ in $c(Y) \subset \mathrm{c}(X)$. It follows that the multifunction is also Pettis integrable in $\mathrm{c}(X)$.

In order to prove the required inclusion (Ks), according to the Hahn-Banach theorem, it is enough to prove that the following inequality holds true for every $x^* \in X^*$:

$$\mathrm{s}\left(x^*, \overline{\mathrm{conv}}\left(w - Ls\left[\int_E \varGamma_n \, d\mu\right]\right)\right) \leq \mathrm{s}\left(x^*, \int_E \overline{\mathrm{conv}}[w - Ls(\varGamma_n(\omega))]\, d\mu\right).$$

But, applying Propositions 6.2 and 1.14 we obtain

$$\begin{aligned}
& \mathrm{s}\left(x^*, \overline{\mathrm{conv}}\left(w - Ls\left[\int_E \Gamma_n \, d\mu\right]\right)\right) \\
= \ & \limsup_n \mathrm{s}\left(x^*, \int_E \Gamma_n \, d\mu\right) = \limsup_n \int_E \mathrm{s}(x^*, \Gamma_n) \, d\mu \\
\leq \ & \int_E \limsup_n \mathrm{s}(x^*, \Gamma_n(\omega)) \, d\mu \\
= \ & \int_E \mathrm{s}(x^*, [w - Ls(\Gamma_n(\omega))]) \, d\mu \\
= \ & \int_E \mathrm{s}(x^*, \overline{\mathrm{conv}}[w - Ls(\Gamma_n(\omega))]) \, d\mu \\
= \ & \mathrm{s}\left(x^*, \int_E \overline{\mathrm{conv}}[w - Ls(\Gamma_n(\omega))] \, d\mu\right)
\end{aligned}$$

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Remark 6.4. In the proof of Proposition 6.3 one could use theorem of Komlos [16] instead of the result of Bukhvalov and Lozanovskij [4].

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Added in proof: D. Cichon has informed me that M. Valadier introduced in 1974 [On the Strassen Theorem, in: Lecture Notes in Economics and Math. Systems 102, 203–215] a notion of a pseudo-selection, that is more general than the quasi-selection proposed here. Each quasi-selection is a pseudo-selection, but not conversely.