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Moderation of Convex Bodies

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Everybody knows what an extreme point of a convex body is. Does this notion have an opposite? A most non-extreme point? Or: can we somehow say that a point is more or less extreme than another?

In this paper we show a way to do this.

We shall define the notion of moderation for points of a convex body. Then the points of largest moderation will be the moderate, those of smallest moderation the extreme points, as you would expect.

Introduction

A *convex body* $K \subset \mathbb{R}^d$ is, as usual, a compact convex set with nonempty interior. We always assume $d \geq 2$. A *cap* of K is a non-empty intersection of K with some closed half-space.

We define the *moderation number*, or simply the *moderation*, $m(x)$ of $x \in K$ as the infimum of the set of diameters of all caps of K containing x, divided by $\operatorname{diam} K$. In this paper we shall therefore always assume or scale K to have diameter 1.

Theorem 1. *A point $x \in K$ is an extreme point of K if and only if $m(x) = 0$.*

This reformulation of a theorem from Schneider's book [2] was at the origin of our approach in this paper. Also, we notice a certain relationship with the investigation contained in [1].

Let $m_K = \sup_{x \in K} m(x)$ be the *moderation* of K.

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Put $S_K(a) = \{x \in K : m(x) \geq a\}$. The points in $S_K(m_K)$ are called the *moderate points* of K, and the set $S_K(m_K)$ itself the *moderation set* of K.

We shall see (Theorem 3) that this set must be non-empty.

The concept of moderation allows seeing extremal point theory in another light. This is the main motivation for introducing it and for the initial investigation contained in this paper.

Clearly, balls are most moderate, i.e. have moderation 1. We shall establish that, while this property characterizes the balls for $d = 2$, in higher dimensions there are many more convex bodies with moderation 1.

The space of all compact sets in $\mathbb{R}^d$, as well as its subspaces $\mathcal{K}$, of all convex bodies, and $\mathcal{K}_1$, of all convex bodies of diameter 1, are equipped with the usual Pompeiu-Hausdorff metric.

Theorem 2. *The function $m : K \to [0,1]$ is upper semi-continuous.*

Proof. Let $x_n \to x$ and suppose that $m(x_n) \to k$ and $m(x) < k$. First let us notice that, for a fixed $u \in \mathbb{R}^d$, the cap

$$C_\lambda = \{y \in K : \langle y, u \rangle \leq \lambda\}$$

depends continuously on λ. Since $\operatorname{diam} C$ depends continuously on C, too, we see that $\operatorname{diam} C_\lambda$ is a continuous function of λ.

Now, there exists a cap $C \ni x$ with $\operatorname{diam} C < k$, because $m(k) < k$. For some u and λ, $C = C_\lambda$. By the above continuity property, for some $\lambda' > \lambda$, still $\operatorname{diam} C_{\lambda'} = k' < k$. But the new cap $C_{\lambda'}$ contains x in its interior, so it must contain all points x_n starting with some index $n = n_0$, whence $m(x_n) \leq k'$ for all indices $n \geq n_0$. This contradicts $m(x_n) \to k$, and our proof is finished.

Theorem 3. *For any convex body K and any number $\alpha \in [0, m_K]$ the set $S_K(\alpha)$ is nonempty and convex. In particular, the moderation set is non-empty and convex.*

Proof. Since the function m is by Theorem 2 upper semi-continuous, it realizes its supremum, so there exists a moderate point.

All sets $S_K(\alpha)$ are nonempty, for $0 \leq \alpha \leq m_K$, because they include the moderation set.

Let now $x, y \in S_K(\alpha)$ and z belong to the line-segment xy.

Suppose $m(z) < \alpha$. Let C be a cap verifying $z \in C$ and $\operatorname{diam} C < \alpha$. Then either $x \in C$ and $m(x) < \alpha$, or $y \in C$ and $m(y) < \alpha$, both in contradiction with the hypothesis.

Thus, $m(z) \geq \alpha$, and $S_K(\alpha)$ is convex.

About the moderation number

We now look at the moderation m_K of K. Obviously, the moderation number cannot be larger than 1. It is easily seen that every ball has moderation 1, but it will be a little surprising to conclude that $m_K = 1$ does not characterize the ball (in arbitrary dimension).

A little elementary plane geometry is needed to prove the following lemma.

Lemma. *Let $a, b, c, d \in \mathbb{R}^2$ be such that $ab \cap cd = \emptyset$ and $\|a-b\| = \|c-d\| = 1$. Then*

$$\max\{\|a-c\|, \|a-d\|, \|b-c\|, \|b-d\|\} > 1.$$

Theorem 4. *In the plane, $m_K = 1$ if and only if K is a ball.*

Proof. Let the convex body K have $m_K = 1$. Choose $x \in K$ such that $m(x) = 1$. We claim that each chord $ab \ni x$ of K has length $\|a-b\| = 1$.

Indeed, assume $a, b \in \operatorname{bd} K$, $ab \ni x$, and $\|a-b\| < 1$. The two caps of K with ab on their boundaries have diameter 1. The diameter is realized by two chords. If they are disjoint the Lemma yields the existence of a chord of length greater than 1. Thus, they must have an endpoint in common, which is a or b, say b. Hence, we found two points $a', a'' \in \operatorname{bd} K$ separated by the line through a, b, with $\|a'-b\| = \|a''-b\| = 1$. Now take any cap with x on its boundary, which contains a, but neither a' nor b. The diameter of this cap is 1 and is realized by a chord of K disjoint from $a'b$, which implies, by the Lemma, that $\operatorname{diam} K > 1$, a contradiction.

By Theorem 10, if $ab \subset \operatorname{bd} K$, then $m(x) \leq \frac{1}{2}$, which is false. So, there are arcs in $\operatorname{bd} K$ starting in a, respectively b, given in polar coordinates (θ, ρ) with origin x by $\rho = \rho(\theta)$, such that $\rho(0) = \|x-a\|$ and $\rho(\pi) = \|x-b\|$.

If ρ is constant, then $\rho = \frac{1}{2}$ and K is a circular disc.

If ρ is not constant, let ξ be maximal such that ρ is constant in $[0, \xi]$, and let a', b' be the points of polar coordinates $(\xi, \rho(0)), (\pi + \xi, \rho(\pi))$.

Let $a^* = (\xi^*, \rho(\xi^*)), b^* = (\pi + \xi^*, \rho(\pi + \xi^*))$ be such that ξ^* is slightly larger than ξ and $\rho(\xi^*) \neq \rho(0)$. Assume, to make a choice, that $\rho(\xi^*) < \rho(0)$. Then necessarily $\rho(\pi + \xi^*) > \rho(\pi)$.

The function ρ has at every θ a left derivative $\rho'_-(\theta)$ and a right derivative $\rho'_+(\theta)$, and $\rho'_-(\theta) \geq \rho'_+(\theta)$, because K is convex. But $\rho(\theta) = 1 - \rho(\pi + \theta)$. This yields $\rho'_-(\pi + \theta) \leq \rho'_+(\pi + \theta)$, whence $\rho'_- = \rho'_+ = \rho'$ everywhere.

Since $\rho(\xi) > \rho(\xi^*)$, there is a point $\eta \in]\xi, \xi^*[$ such that $\rho'(\eta) < 0$. Then $\rho'(\pi+\eta) > 0$.

Consider the points a_η and b_η, of polar coordinates $(\eta, \rho(\eta))$ and $(\pi + \eta, \rho(\pi + \eta))$, respectively. Because $\rho'(\eta) < 0$ and $\rho'(\pi + \eta) > 0$, the tangent lines at a_η, b_η to $\operatorname{bd} K$ and the chord $a_\eta b_\eta$ form an (isosceles) triangle Δ. Thus, any chord of K parallel and close enough to $a_\eta b_\eta$, but disjoint from Δ, is longer than $a_\eta b_\eta$. This and $\|a_\eta - b_\eta\| = 1$ contradict $\operatorname{diam} K = 1$.

In higher dimensions the situation changes.

Theorem 5. *Assume $M \subset S^{d-1}$ is symmetric with respect to $\mathbf{0}$ and meets each great $(d-2)$-dimensional subsphere of S^{d-1}. Then* $\operatorname{conv} M$ *has moderation* 1.

Proof. We prove that $m(\mathbf{0}) = 1$. Indeed, any cap C containing $\mathbf{0}$ must include $M \cap H$ for some hyperplane $H \ni \mathbf{0}$. Since $M \cap H$ is non-empty and symmetric with respect to $\mathbf{0}$, there must be two diametrically opposite points of S^{d-1} in C. Then $\operatorname{diam} C = 1$, and $m(\mathbf{0}) = 1$ too.

The next result presents a lower bound for m_K involving the inradius of K.

Theorem 6. *Let r be the inradius of K. Then*

$$m_K > \frac{\sqrt{1-2r+2r^2}}{1-r+\sqrt{1-2r+2r^2}}.$$

Proof. If K is a ball then $m_K = 1$, $r = 1/2$, and indeed $1 > 2-\sqrt{2}$.

Suppose now that K is not a ball.

Let $a, b \in K$ be at distance 1. Denote by x the mid-point of the line-segment ab.

Assume that $\|x-y\| = \varepsilon$ and $xy \perp ab$ for some point $y \in K$. Then the centre i of the circle inscribed in the triangle aby has moderation

$$(*) \qquad m(i) \geq \frac{\sqrt{1+4\varepsilon^2}}{1+\sqrt{1+4\varepsilon^2}}.$$

Indeed, let $j \in ay$ and $k \in ab$ satisfy $ij \parallel ab$ and $ai \perp ik$. Since $2i-k \in ay$, ij is half as long as ak.

Now, each cap of aby containing i and a (or i and b) has diameter at least $\|a-k\|$. Also, each cap of aby containing i but neither a nor b has diameter at least $2\|i-j\|$. Hence i has moderation $\|a-k\|$ in aby. Therefore $m(i) \geq \|a-k\|$, and elementary calculation yields $\|a-k\| = \sqrt{1+4\varepsilon^2}/(1+\sqrt{1+4\varepsilon^2})$.

Now let c be the center of a ball B of radius r included in K. Let $c_1 \in ab$ and $c_2 \in \operatorname{bd} B$ satisfy $c_1c_2 \perp ab$ and $c \in c_1c_2$. Since B cannot meet the hyperplane through a orthogonal to ab, $\|a-c_1\| \geq r$. Clearly, $\|c_1-c_2\| \geq r$ too.

Because K is not a ball, a and b cannot both belong to B, say $b \notin B$. For some point c' with $c_2 \in cc'$, bc' is tangent to $\operatorname{bd} B$. We take now $y \in bc'$ such that $xy \perp ab$. We have $y \in K$ and, remembering the notation $\|x-y\| = \varepsilon$,

$$\frac{\varepsilon}{\|c'-c_1\|} = \frac{1/2}{\|b-c_1\|}.$$

Since $\|c'-c_1\| > \|c_2-c_1\| \geq r$ and $\|b-c_1\| \leq 1-r$, we have

$$\varepsilon > \frac{1}{2} \cdot \frac{r}{1-r}.$$

This inequality and $(*)$ imply

$$m(i) > \frac{\sqrt{1-2r+2r^2}}{1-r+\sqrt{1-2r+2r^2}}.$$

Hence m_K must satisfy the same strict inequality.

Corollary 7. *For any convex body K, $m_K > \frac{1}{2}$.*

Proof. Indeed, putting $\varepsilon = r^2(1-r)^{-2}$, we have $\varepsilon > 0$ and

$$m_K > \frac{\sqrt{1-2r+2r^2}}{1-r+\sqrt{1-2r+2r^2}} = \frac{\sqrt{1+\varepsilon}}{1+\sqrt{1+\varepsilon}} > \frac{1}{2}.$$

Theorem 8. *Defined on $\mathcal{K}_1$, the function m_K is non-decreasing.*

Proof. Consider the convex bodies $K, L \in \mathcal{K}_1$, with $K \subset L$. Let x be a moderate point of K. Take a cap C of K containing x. Thus $C = K \cap \Pi$ for some closed half-space Π. Since $K \cap \Pi \subset L \cap \Pi$, we have $\operatorname{diam}(K \cap \Pi) \le \operatorname{diam}(L \cap \Pi)$ and

$$\inf_{\Pi \ni x} \operatorname{diam}(K \cap \Pi) \le \inf_{\Pi \ni x} \operatorname{diam}(L \cap \Pi),$$

i.e. the moderation m_K of x in K is not larger than the moderation of x in L. Hence $m_K \le m_L$.

Remark. The function m_K is not strictly increasing with respect to K.

Indeed, this is quite obvious for $d \ge 3$, because, for any convex body K with $S^1 \subset K \subset \operatorname{conv} S^2$,

$$m_K = m_{\operatorname{conv} S^2} = 1.$$

But, also for $d = 2$, the union E of the equilateral triangles abd and bcd, and the union R of the Reuleaux triangles abd and bcd (both E and R of diameter 1), have the same moderation, $3^{-1/2}$.

This Remark prompts the following.

Theorem 9. *Assume that the convex bodies $K, L \in \mathcal{K}_1$, with $K \subset L$, have the same moderation. Then the moderation set of K is included in the moderation set of L.*

We omit the proof, since it parallels that of Theorem 8.

Theorem 10. *If $d = 2$, $K \in \mathcal{K}_1$, and $x \in \operatorname{bd} K$, then $m(x) \le 1/2$.*

Proof. Indeed, if x is an extreme point, then $m(x) = 0$, by Theorem 1.

If not, let yz be the maximal line-segment included in $\operatorname{bd} K$, with $x \in yz$. We may assume $\|x-y\| \le \|x-z\|$. Since $\|y-z\| \le 1$, we have $\|x-y\| \le 1/2$. Then, clearly, there is a cap of K containing x, of diameter larger than $\|x-y\|$, but as close to $\|x-y\|$ as we wish. On the other hand, every cap containing x will also contain y or z and will therefore have diameter at least $\|x-y\|$. Thus, $m(x) = \|x-y\| \le 1/2$.

Theorem 10 does not hold in higher dimensions, because, for a half-ball $K \subset \mathbb{R}^d$ with $\operatorname{conv} S^{d-2}$ as a face, $\mathbf{0} \in \operatorname{bd} K$ and $m(\mathbf{0}) = 1$.

Thus, we contemplate the possibility for a convex body in dimension at least 3, to have moderate points in its boundary, while this is impossible in the plane: indeed, if $K \subset \mathbb{R}^2$, for $x \in \operatorname{bd} K$ we have $m(x) \leq 1/2$ by Theorem 10, and for $x \in S_K(m_K)$ we have $m(x) = m_K > 1/2$ by Corollary 7.

We present several examples of convex bodies with their moderation.

(i) The moderation number of an equilateral triangle equals 2/3.

(ii) The moderation number of a regular tetrahedron is 3/4.

(iii) The moderation number of a square equals $\sqrt{\frac{5}{8}}$.

(iv) The moderation number of a cube is $\sqrt{\frac{3}{4}}$.

For any convex body K and $x \in K$, let $M(x, K)$ denote in the next theorem the moderation of x in K.

Theorem 11. *Let K be a convex body in $\mathbb{R}^d$, $d \geq 3$, and $2 \leq k < d$. Let $x \in K$ and and $\mathcal{S}_x$ be the set of all k-dimensional sections of K through x. Then*

$$\min_{S \in \mathcal{S}_x} m(x, S) \leq m(x, K)$$

Proof. Let $C_0(x)$ be a cap of K containing x and attaining minimal diameter. Let $a, b \in K$ be at distance 1. Let S be a k-dimensional section through x containing a, b. It is clear that the diameter of the cap $C_0(x) \cap S$ of S is not larger than the diameter of $C_0(x)$. Thus we get $\min_{S \in \mathcal{S}_x} m(x, S) \leq m(x, K)$

Remark. The minimum in Theorem 11 is not necessarily attained. Indeed, consider the vertices $v_1, \ldots, v_4 \in S^2$ of a regular tetrahedron, join all pairs of vertices by minimizing geodesics (arcs of great circles) on S^2, and denote the union of these 6 arcs by E. Let M be a small open neighbourhood of E in S^2, and take $K = \operatorname{conv}(S^2 \setminus M)$. Every section S of K through $\mathbf{0}$ contains antipodal points of S^2, whence $m(\mathbf{0}, K) = 1$, but no such section S of K is a circular disk, and therefore $m(\mathbf{0}, S) < 1$ by Theorem 4.

About the moderation set

We investigate here the moderation set $S_K(m_K)$ of all moderate points of K. We shall see that, in general, the moderation set may have non-empty interior. This is however excluded if K is point-symmetric. Several examples will illustrate the situation.

Theorem 12. *If K is centrally symmetric, then the moderation set contains the centre and has empty interior.*

Proof. We may assume that $\mathbf{0}$ is the centre of K.

Let $x \in K \setminus \{\mathbf{0}\}$. For each cap $C \ni \mathbf{0}$, we have $x \in C$ or $x \in -C$. So the set of all diameters of caps containing $\mathbf{0}$ is included in the set of all diameters of caps containing x. Hence $m(x) \leq m(\mathbf{0})$, and $\mathbf{0} \in S_K(m_K)$.

Now, let Π be a hyperplane through $\mathbf{0}$, and Π^+ a half-space with boundary Π. Then

$$m(\mathbf{0}) = \inf_{\Pi^+} \operatorname{diam}(K \cap \Pi^+)$$

is realized for some half-space Π_0^+ of boundary Π_0.

Let $\Pi_1 \subset \Pi_0^+$ be a hyperplane parallel to and distinct from Π_0. The half-space $\Pi_1^+ \subset \Pi_0^+$ with boundary Π_1 determines a cap $C_1 = K \cap \Pi_1^+$. Let $a, b \in C_1$ verify $\|a-b\| = \operatorname{diam} C_1$. Assume, for example, that a is not closer than b to Π_1. The points a, b, the intersection c of $b(-a)$ with Π_0, and $d = a - b + c$ are the vertices of a parallelogram in $C_0 = K \cap \Pi_0^+$. We have

$$\begin{aligned}\operatorname{diam} C_1 = \|a-b\| &< \left\|a - \frac{a+c}{2}\right\| + \left\|\frac{b+d}{2} - b\right\| \\ &= \frac{1}{2}\|a-c\| + \frac{1}{2}\|d-b\| \le \frac{1}{2}\operatorname{diam} C_0 + \frac{1}{2}\operatorname{diam} C_0 = \operatorname{diam} C_0.\end{aligned}$$

Thus, every point $x \in K \setminus \Pi_0$ lies in a cap of diameter smaller than $\operatorname{diam} C_0 = m(\mathbf{0})$. Hence $m(x) < m(\mathbf{0})$, and $S_K(m_K) \subset \Pi_0$.

Theorem 13. *In any dimension, there are convex bodies for which the moderation set has non-empty interior.*

Proof. Let $\Gamma \in \mathbb{R}^2$ be a circle of centre $\mathbf{0}$ and diameter α, and take $u, v \in \Gamma$ with $\|u-v\| = \alpha$. Consider the points $u', v' \in \Gamma$ such that $u'v' \parallel uv$ and

$$0 < \angle u\mathbf{0}u' = \angle v\mathbf{0}v' < \arctan\frac{1}{2}.$$

The tangent lines at u' and v' to Γ meet at a point w. Then

$$\frac{\|u'\|}{\|u'-w\|} = \tan\angle u\mathbf{0}u' < \frac{1}{2},$$

which yields $\|u'-w\| > \alpha$.

Let $u'' \in u'w$, $v'' \in v'w$ satisfy

$$\|u'-v''\| = \|v'-u''\| = \alpha.$$

Put $K = \operatorname{conv}(\Gamma \cup \{u'', v''\})$, and choose α such that $\operatorname{diam} K = 1$.

Also, consider the midpoint x of $u'v'$, and the convex quadrilateral Q with vertices $\mathbf{0}$, $\mathbf{0}u' \cap xu$, x, $\mathbf{0}v' \cap xv$.

It is not difficult to verify that all points of Q are moderate in K, and $m_K = \alpha$. (In fact, the moderation set is larger, but strictly included in $\operatorname{conv}\{u, u', v, v'\}$.)

To produce a higher-dimensional example it suffices to rotate K around the axis through $\mathbf{0}$, x.

Theorem 14. *If a planar convex body is symmetric with respect to two lines, then their intersection is a moderate point.*

Proof. Let K be a planar convex body, symmetric with respect to the lines L_1 and L_2. We may assume that $L_1 \cap L_2 = \{\mathbf{0}\}$. Let R_i be the reflection with respect to

L_i ($i = 1, 2$). Note that $m(R_i(x)) = m(x)$. It is well known that $R = R_1 \circ R_2$ is the 2θ-rotation around $\mathbf{0}$, where θ is the angle between L_1 and L_2.

First we will show that $\mathbf{0}$ is a moderate point. Indeed, let x be a moderate point, the convex hull H of $\{R^i(x) : i \in \{0, 1, \cdots, \lceil \frac{\pi}{2\theta} \rceil\}\}$ contains $\mathbf{0}$. Since all $R^i(x)$ are moderate points and the moderation set is convex by Theorem 3, $\mathbf{0}$ is also a moderate point.

We now consider three examples. For the first two of them we leave the pleasure of verifying the calculation to the reader.

Example 15. Let $R(\alpha, \beta)$ be a rectangle with centre $\mathbf{0}$, diameter 1, and side-lengths α and β ($\alpha \le 2^{-1/2} \le \beta$, $\alpha^2 + \beta^2 = 1$).

(1) If $\alpha = \beta = 2^{-1/2}$, the moderation set is $\{\mathbf{0}\}$.

(2) If $1 < \frac{\beta}{\alpha} < 2\sqrt{\frac{13+4\sqrt{6}}{73}}$, the moderation set is the line-segment of length $\alpha - \sqrt{4\alpha^2 - 3\beta^2}$, with $\mathbf{0}$ as mid-point, parallel to the short side; it is included in $\mathrm{int}R(\alpha, \beta)$.

(3) If $2\sqrt{\frac{13+4\sqrt{6}}{73}} \le \frac{\beta}{\alpha} < 2$, the moderation set is the line-segment of length $\alpha + \beta - \sqrt{2\alpha^2 + \frac{\beta^2}{2}}$ with $\mathbf{0}$ as mid-point, parallel to the short side; it is included in $\mathrm{int}R(\alpha, \beta)$.

(4) If $\frac{\beta}{\alpha} \ge 2$, the moderation set is the line-segment joining the mid-points of the two long sides of $R(\alpha, \beta)$.

Example 16. Consider a rhombus with diagonal-lengths α and 1 ($\alpha < 1$).

(1) If $3^{-1/2} \le \alpha < 1$, then $\mathbf{0}$ is the unique moderate point.

(2) If $\alpha < 3^{-1/2}$, then the moderation set is the line-segment of length $\alpha - \frac{\alpha\sqrt{1+\alpha^2}}{2}$, with $\mathbf{0}$ as mid-point, and strictly included in the short diagonal.

Example 17. Let H be a half disk with diameter 1. The moderation set of H contains a single point interior to H.

Proof. Let D be the closed disk with centre $\mathbf{0}$ and diameter 1 and $H = D \cap \{(x, y) | x \ge 0\}$. Put $D(t) = H \cap \{(x, y) | x \ge t\}$ and $a = (0, 1/2)$. Let $T(t)$ be the subset of all points of H above the line l_t through $(t, 0)$ which intersects $\mathrm{bd}\, H$ at $b = (0, y_t)$ and $\mathrm{bd}\, D$ at c such that $\|a - b\| = \|a - c\|$. It is clear that the diameter of $D(t)$ is $f_D(t) := \sqrt{1 - 4t^2}$ and the diameter of $T(t)$ is $f_T(t) := \|a - b\| = 1/2 - y_t$. Note that $f_T(t)$ is increasing and $f_D(t)$ is decreasing (when t increases from 0 to $1/2$). Then the only one moderate point of H is the unique point $(t_0, 0)$, where $f_D(t_0) = f_T(t_0)$.

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