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Contingent Epiderivatives of Functions on Time Scales

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We characterize functions defined on times scales by the contingent epiderivative. Relations between delta and nabla derivatives and the contingent epiderivative of functions defined on time scales are investigated. We formulate two notions that are exploited in optimal control theory, namely the Fermat Rule and pseudo-convexity. Appropriate illustrative examples are presented.

1. Introduction

The time scales calculus was introduced by Bernd Aulbach and Stefan Hilger in 1988 [7]. The calculus unifies two areas of the classical analysis: discrete and continuous. It is known that applications of the calculus on time scales can be found in economics, engineering, biology, finance and physics [1, 4, 15]. In general, the differentiation tools that are usually used for functions defined on time scales are delta and nabla derivatives. On the other hand, the contingent epiderivative that was introduced in [6] is widely used in optimal control theory. The contingent epiderivative is a derivative that involves a notion of set-valued maps (maps that have sets as their values). Namely, we associate with a function f the set-valued map $F_\uparrow$ defined by $F_\uparrow(x) := [f(x), +\infty)$, whose graph is the epigraph of f. The graphs of the derivatives of such set-valued maps $F_\uparrow$ are the epigraphs of functions which are called *epiderivatives*. There are tools of nonsmooth analysis that allow to differentiate maps (single-valued) that are not differentiable in the classical sense.

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The present work is dedicated to the study of relations between nabla, delta derivatives and the contingent epiderivative. We apply the results to the Fermat Rule and pseudo-convexity. Results in this paper are motivated by the following ([2, 3, 5, 6, 7, 8, 13, 14, 16]). We use the contingent epiderivative in order to characterize functions defined on time scales and it seems to be very fresh and new approach. It turns out that in some cases, as it is shown in Example 5.8, the contingent epiderivative is a better tool than Δ and ∇ derivatives: it is possible to compute the epiderivative at a right and left dense point, while nabla or delta derivatives do not exist at such a point. Moreover, applying the contingent epiderivative to functions defined on time scales we get that the Fermat Rule can be extended to any function defined on any time scale.

The paper is organized as follows. In Section 2 we overview basic facts about time scales and give some preliminary information. In Section 3 some ideas of epigraphs and convexity for functions defined on time scales are presented. In Section 4 we characterize the relations between the contingent epiderivative and delta and nabla derivatives while in Section 5 the Fermat Rule and a converse of the Fermat Rule for pseudo-convex functions are formulated. We finish the paper with some examples illustrating the relations between the contingent epiderivative and nabla, delta derivatives.

2. Preliminaries

An introduction to time scales can be found in [9, 10, 11]. Here we recall only some basic facts. By a time scale, denoted here by $\mathbb{T}$, we mean a nonempty closed subset of $\mathbb{R}$. Throughout the text we assume $\mathbb{T}$ to be a time scale with at least two points and I to be an arbitrary interval of $\mathbb{R}$. As the theory of time scales gives the way to unify continuous and discrete analysis, the standard cases of time scales are the following: $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=c\mathbb{Z}$, $c>0$ or $\mathbb{T}=\overline{q^{\mathbb{Z}}} := \{q^k \,|\, k\in\mathbb{Z}\wedge q>1\}\cup\{0\}$.

For $t\in\mathbb{T}$ we define *the forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ and *the forward graininess function* $\mu:\mathbb{T}\to[0,+\infty)$ by

(i) $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ and $\sigma(\sup\mathbb{T})=\sup\mathbb{T}$ if $\sup\mathbb{T}<+\infty$;
(ii) $\mu(t)=\sigma(t)-t$.

One can also define *the backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ and *the backward graininess function* $\nu:\mathbb{T}\to[0,+\infty)$ by

(i) $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}$ and $\rho(\inf\mathbb{T})=\inf\mathbb{T}$ if $\inf\mathbb{T}>-\infty$;
(ii) $\nu(t)=t-\rho(t)$.

A point t is called *left-scattered (right-scattered,* respectively*)* if $\rho(t)<t$, ($\sigma(t)>t$, respectively). A point t is called *left-dense (right-dense,* respectively*)* if $\rho(t)=t$, ($\sigma(t)=t$, respectively). In the continuous-time case, when $\mathbb{T}=\mathbb{R}$, we have that for all $t\in\mathbb{R}:\sigma(t)=\rho(t)=t$ and $\mu(t)=\nu(t)=0$. In the discrete-time case, for each $t\in\mathbb{T}=c\mathbb{Z}:\sigma(t)=t+c,\ \rho(t)=t-c,\ \mu(t)=\nu(t)=c$. For the composition between a function $f:\mathbb{T}\to\mathbb{R}$ with functions $\sigma:\mathbb{T}\to\mathbb{T}$ and $\rho:\mathbb{T}\to\mathbb{T}$ we use the following abbreviations: $f^\sigma(t)=f(\sigma(t))$ and $f^\rho(t)=f(\rho(t))$. Additionally we define the following sets:

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T}^\kappa = \mathbb{T} & \text{if } \sup\mathbb{T} = \infty; \end{cases}$$

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T})) & \text{if } |\inf\mathbb{T}| < \infty \\ \mathbb{T}_\kappa = \mathbb{T} & \text{if } \inf\mathbb{T} = -\infty; \end{cases}$$

$$\mathbb{T}^\kappa_\kappa := \mathbb{T}^\kappa \cap \mathbb{T}_\kappa.$$

Definition 2.1. For a function $f : \mathbb{T} \to \mathbb{R}$ we define the delta derivative of f at $t \in \mathbb{T}^\kappa$, denoted by $f^\Delta(t)$, to be the number, if it exists, with the property that for all $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of $t \in \mathbb{T}^\kappa$ such that for all $s \in U$ the following holds

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|.$$

Moreover, we say that a function f is Δ – differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2.2. For a function $f : \mathbb{T} \to \mathbb{R}$ we define the nabla derivative of f at $t \in \mathbb{T}_\kappa$, denoted by $f^\nabla(t)$. It is the number, if it exists, such that for all $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that for all $s \in U$ holds

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|.$$

Moreover, we say that a function f is ∇ – differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Example 2.3. (a) Let $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$ and f is Δ and ∇ – differentiable iff it is differentiable in the ordinary sense.

(b) Let $\mathbb{T} = c\mathbb{Z}$, $c > 0$ then $f^\Delta(t) = \frac{f(t+c)-f(t)}{c}$ and $f^\nabla(t) = \frac{f(t)-f(t-c)}{c}$ and they always exist.

(c) Let $\mathbb{T} = \overline{q^\mathbb{Z}}$, $q > 1$ then $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$ and $f^\nabla(t) = q \cdot \frac{f(t)-f(\frac{t}{q})}{(q-1)t}$ for $t \neq 0$, while for $t = 0$ we get $f^\Delta(0) = f^\nabla(0) = \lim_{s\to 0} \frac{f(s)-f(0)}{s}$.

3. Epigraphs and convexity

Although such property like convexity is defined for functions on real intervals, the following definition of convexity was proposed in [16]. Here we state the definition of a convex function defined on any subset $X \subset \mathbb{T}$ such that X consists of at least two different points. Moreover, we do not demand here f to be continuous on X.

Definition 3.1. Let $X \subset \mathbb{T}$ and X consists of at least two points. A function f defined on X is called *convex on* X if for any $t_1, t_2, t \in X$ such that $t_1 < t_2$ and $t_1 \leq t \leq t_2$ holds

$$(t_2 - t)f(t_1) + (t_1 - t_2)f(t) + (t - t_1)f(t_2) \geq 0. \tag{1}$$

Similarly, f is said to be *concave on* X if we substitute inequality "$\geq$" by "$\leq$" in (1).

Remark 3.2. (a) Note that $X \subset \mathbb{T}$ may be closed or open, finite or infinite.

(b) For $\mathbb{T} = \mathbb{R}$, Definition 3.1 agrees with the standard definition of convexity and concavity of a function. Indeed, let $t_1, t_2 \in X$ and $t_1 < t_2$. If $t \in X$ and $t_1 \le t \le t_2$, then $t = \alpha t_1 + (1-\alpha)t_2$ with $\alpha = \frac{t_2 - t}{t_2 - t_1}$, $1 - \alpha = \frac{t - t_1}{t_2 - t_1}$. Thus, we define convexity using convex combinations of points from X. Then condition (1) can be rewritten as

$$f(t) = f(\alpha t_1 + (1-\alpha)t_2) \le \alpha f(t_1) + (1-\alpha) f(t_2).$$

(c) Let I be any interval of $\mathbb{R}$. Then if $f : I \to \mathbb{R}$ is convex/concave on I (in usual sense for a function on real interval) then also the function $f|_{\mathbb{T}} : X \to \mathbb{R}$ is convex/concave on $X = I \cap \mathbb{T}$.

We can also make another interpretation of convexity of a function defined on a time scale. We need the following general definition.

Definition 3.3. Let X be any nonempty subset of $\mathbb{R}$. By *the epigraph of* $f : X \to \mathbb{R}$, denoted by $\mathrm{Epi}(f)$, we mean the following set:

$$\mathrm{Epi}(f) := \{(t, \lambda) \in X \times \mathbb{R} : f(t) \le \lambda\}. \tag{2}$$

The hypograph of $g : X \to \mathbb{R}$ is defined in the symmetric way:

$$\mathrm{Hyp}(g) := \{(t, \lambda) \in X \times \mathbb{R} : g(t) \ge \lambda\}. \tag{3}$$

Note that $\mathrm{Epi}(f) \cap \mathrm{Hyp}(f) = \mathrm{Graph}(f)$.

If $X = \mathbb{T}$ is a time scale then we can rewrite the same definition of epigraph and introduce the following extension of the epigraph of a function $f : \mathbb{T} \to \mathbb{R}$. By $G(f)$ we denote the following set:

$$G(f) = \bigcup_{t \in \mathbb{T}} \{\alpha(t, y) + \beta(\sigma(t), z) :\ y \ge f(t), z \ge f^{\sigma}(t), \alpha + \beta = 1, \alpha, \beta \ge 0\}. \tag{4}$$

Remark 3.4. It is easy to see that $G(f) \subset \overline{\mathrm{conv}}\,\mathrm{Epi}(f)$, where $\overline{\mathrm{conv}}\,\mathrm{Epi}(f)$ is the closure of the convex hull of $\mathrm{Epi}(f)$.

Let $X = I \cap \mathbb{T} \neq \emptyset$, where I is an arbitrary interval of $\mathbb{R}$. Using the formulation of $G(f)$ we can assign to any $f : X \to \mathbb{R}$ the function $\overline{f} : I \to \mathbb{R}$ defined by the formula

$$\mathrm{Epi}(\overline{f}) = G(f). \tag{5}$$

Let us notice that for $f, g : X \to \mathbb{R}$ and $a, b \in \mathbb{R}$ holds: $a\overline{f} + b\overline{g} = \overline{af + bg}$.

Proposition 3.5. *Let $f : X \to \mathbb{R}$, where $X = I \cap \mathbb{T}$ and I be an arbitrary interval of $\mathbb{R}$. Then the following statements are equivalent.*

a) *The set $G(f)$ is convex.*
b) *$\overline{f}$ is convex on I.*
c) *f is convex on X.*

Proof. Directly from the definition of $\overline{f}$, items *a*) and *b*) are equivalent. And it is easy to see that *b*) implies *c*). So we only need to prove that *c*) implies *a*). Let us assume, on the contrary, that $G(f)$ is not convex. Then there are two points $p = (t_1, y_1), q = (t_2, y_2) \in G(f)$ such that $t_1, t_2 \in X$ and $w = \alpha p + \beta q \notin G(f)$ for some $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Moreover, there exists $t \in X$ such that $t = \alpha t_1 + \beta t_2$ and $f(t) > \alpha y_1 + \beta y_2 \geq \alpha f(t_1) + \beta f(t_2)$ what is in contradiction with the convexity of f. □

Remark 3.6. Note that function f defined by formula (5) can be presented in the following way (see e.g. [12])

$$\overline{f}(t) = \begin{cases} f(t), & \text{if } t \in \mathbb{T} \\ f(s) + \frac{f(\sigma(s)) - f(s)}{\mu(s)}(t - s), & \text{if } t \in (s, \sigma(s)),\ s \in \mathbb{T},\ s \text{ is right-scattered.} \end{cases}$$

Moreover, if we set $\sigma(s) = s_1$ and $s = \rho(s_1)$ then for $t \in (\rho(s_1), s_1)$, $s_1 \in \mathbb{T}$, s_1 being left-scattered one can write $\overline{f}(t) = f(s_1) + \frac{f(s_1) - f(\rho(s_1))}{\nu(s_1)}(t - s_1)$.

4. Contingent epiderivatives

In this section we introduce the notion of the contingent epiderivative using the idea of a set-valued map (or a multifunction) from $\mathbb{R}$ to $\mathbb{R}$. The general definition, for a multifunction between metric spaces, one can find in [5, 6].

Definition 4.1. A *set-valued map* (*multifunction*) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is a map that has sets as its values and we denote it simply by

$$F : \mathbb{R} \twoheadrightarrow \mathbb{R}.$$

Every set-valued map is characterized by its *graph*, Graph(F), as the subset of $\mathbb{R}^2$ defined by

$$\text{Graph}(F) := \{(x, y) \in \mathbb{R}^2 : y \in F(x)\}. \tag{6}$$

We shall say that $F(x)$ is the *image* or the *value* of F at x. A set-valued map is said to be *nontrivial* if its graph is not empty, i.e. if there exists an element $x \in \mathbb{R}$ such that $F(x)$ is not empty. The *domain* of F is the set of elements $x \in \mathbb{R}$ such that $F(x)$ is not empty: $\text{Dom}(F) = \{x \in \mathbb{R} : F(x) \neq \emptyset\}$. The *image* of F is the union of the images (or values) $F(x)$, when x ranges over $\mathbb{R}$: $\text{Im}(F) = \bigcup_{x \in \mathbb{R}} F(x)$.

Definition 4.2 ([6]). We say that a set-valued map $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ is *Lipschitz around* $x \in \mathbb{R}$ if there exists a positive constant l and a neighborhood $U \subset \text{Dom}(F)$ of x such that

$$\forall\ x_1,\ x_2 \in U, \quad F(x_1) \subset F(x_2) + |x_1 - x_2|[-l, l]. \tag{7}$$

In this case F is also called Lipschitz (or l-Lipschitz on U).

We recall a general definition of contingent cones to a subset of $\mathbb{R}^2$. This theory is presented for a subset of any normed space X in [6].

Definition 4.3 ([6]). Let $K \subset \mathbb{R}^2$ and a point $p \in \overline{K}$ belong to the closure of K. The contingent cone $T_K(p)$ is defined by

$$T_K(p) := \left\{ v : \liminf_{h \to 0+} d(p + hv, K)/h = 0 \right\}, \tag{8}$$

where $d(q, K) = \inf_{k \in K} d(q, k)$ is the distance between point $q = p + hv$ and the set K.

Remark 4.4 ([6]). $T_K(p)$ is a closed set.

It is very convenient to have the following characterization of the contingent cone in terms of sequences:

$$v \in T_K(p) \iff \exists\, h_n \to 0^+ \text{ and } \exists\, v_n \to v \;\text{ such that } \forall\, n,\, p + h_n v_n \in K. \tag{9}$$

Remark 4.5 ([6]). If $p \in \operatorname{Int}(K)$, then $T_K(p) = \mathbb{R}^2$.

Definition 4.6 ([6]). Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map. The *contingent derivative of F at* $p \in \operatorname{Graph}(F)$, denoted by $DF(p)$, is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ defined by:

$$\operatorname{Graph}(DF(p)) := T_{\operatorname{Graph}(F)}(p).$$

For $F := f$ a single-valued function, we set $Df(x) := Df(x, f(x))$.

Let us point out that

$$\operatorname{Graph}(DF(p)) = \{u = (u_1, u_2) : u_2 \in DF(p)(u_1)\} \;\text{ and }\; u \in T_{\operatorname{Graph}(F)}(p). \tag{10}$$

In [6] one can find the following characterization of the contingent derivative (based on [6, Proposition 5.1.4, p. 186]):

Proposition 4.7 ([6]). *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map and let $p = (x, y) \in \operatorname{Graph}(F)$. Then*

$$v \in DF(p)(u) \iff \liminf_{h \to 0+, u' \to u} d\left(v, \frac{F(x + hu') - y}{h}\right) = 0.$$

If $x \in \operatorname{Int}(\operatorname{Dom}(F))$ and F is Lipschitz around x, then

$$v \in DF(p)(u) \iff \liminf_{h \to 0+} d\left(v, \frac{F(x + hu) - y}{h}\right) = 0.$$

Let $X \subset \mathbb{R}$, possibly a time scale, and let us consider an extended function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ whose domain is $\operatorname{Dom}(f) = \{t \in X : f(t) \neq \pm\infty\}$. We call an extended function *nontrivial* if $\operatorname{Dom}(f) \neq \emptyset$. To introduce elements of the theory of contingent epiderivatives for functions on time scales we define two set-valued maps $F_\uparrow : X \twoheadrightarrow \mathbb{R}$ and $F_\downarrow : X \twoheadrightarrow \mathbb{R}$, both corresponding to $f : X \to \mathbb{R}$, in the following way (see [6]):

Definition 4.8. Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be an extended function with nonempty domain $\operatorname{Dom}(f) \neq \emptyset$. Then

a) $$F_{\uparrow}(t) = \begin{cases} f(t) + \mathbb{R}_+ & \text{if } t \in \mathrm{Dom}(f) \\ \emptyset & \text{if } f(t) = +\infty \\ \mathbb{R} & \text{if } f(t) = -\infty, \end{cases}$$

b) $$F_{\downarrow}(t) = \begin{cases} f(t) - \mathbb{R}_+ & \text{if } t \in \mathrm{Dom}(f) \\ \emptyset & \text{if } f(t) = -\infty \\ \mathbb{R} & \text{if } f(t) = +\infty, \end{cases}$$

where $f(t) + \mathbb{R}_+ = \{\lambda : f(t) \leq \lambda\}$ and $f(t) - \mathbb{R}_+ = \{\lambda : f(t) \geq \lambda\}$.

Remark 4.9. It is easy to notice that $\mathrm{Graph}(F_{\uparrow}) = \mathrm{Epi}(f) = \bigcup_t \{(t, y) : y \in F_{\uparrow}(t)\}$ and $\mathrm{Graph}(F_{\downarrow}) = \mathrm{Hyp}(f)$.

With set-valued functions $F_{\uparrow}$ and $F_{\downarrow}$ one can associate their contingent derivatives. We naturally have that values of the contingent derivative of $F_{\uparrow} : \mathbb{R} \twoheadrightarrow \mathbb{R}$ are half lines in the sense that: $\forall\, \lambda \geq f(t)$, $\forall\, u \in \mathrm{Dom}\,(DF_{\uparrow}(t, \lambda))$, holds:

$$DF_{\uparrow}(t, \lambda)(u) = DF_{\uparrow}(t, \lambda)(u) + \mathbb{R}_+.$$

Definition 4.10 ([6]). Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function with nonempty domain and let $t \in \mathrm{Dom}(f)$. We say that the extended function $D_{\uparrow} f(t) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$D_{\uparrow} f(t)(u) := \inf\{v : v \in DF_{\uparrow}(t, f(t))(u)\}$$

where $F_{\uparrow}$ is from Definition 4.8 and $u \in \mathbb{R}$, is the *contingent epiderivative of f at t in the direction u*. The function f is said to be *contingently epidifferentiable at t* if its contingent epiderivative never takes the value $-\infty$. If $DF_{\uparrow}(t, f(t))(u) = \emptyset$ then we set $D_{\uparrow} f(t)(u) = +\infty$.

Remark 4.11 ([6]). Notice that for a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ with nonempty domain the following conditions are equivalent:

(i) f is contingently epidifferentiable at $t \in \mathrm{Dom}(f)$;
(ii) $D_{\uparrow} f(t)(0) = 0$.

In [5] one can find the following characterization of the contingent epiderivative for some particular situations:

a) $DF_{\uparrow}(t, f(t))(u) = \mathbb{R}$ iff $D_{\uparrow} f(t)(u) = -\infty$;
b) $DF_{\uparrow}(t, f(t))(u) = [v_0, +\infty)$ iff $D_{\uparrow} f(t)(u) = v_0$.

Now let us consider $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$. Then with $f : X \to \mathbb{R} \cup \{\pm\infty\}$ we associate the function $\overline{f} : I \to \mathbb{R} \cup \{\pm\infty\}$ defined by formula (5). Let $\overline{F}_{\uparrow}$ denote the corresponding multifunction for $\overline{f}$ according to Definition 4.8.

Proposition 4.12. *For all $p \in \mathrm{Graph}\,(F_{\uparrow})$ and for each $u \in \mathbb{R}$ holds*

$$DF_{\uparrow}(p)(u) \subset D\overline{F}_{\uparrow}(p)(u). \tag{11}$$

Proof. Since $\mathrm{Graph}(F_{\uparrow}) \subset \mathrm{Graph}(\overline{F}_{\uparrow})$ and $\mathrm{Graph}\,(DF_{\uparrow}(p)) = T_{\mathrm{Graph}(F_{\uparrow})}(p)$, it follows that $\mathrm{Graph}(DF_{\uparrow}(p)) \subset \mathrm{Graph}(D\overline{F}_{\uparrow}(p))$. Let $v \in DF_{\uparrow}(p)(u)$. Then $(u, v) \in \mathrm{Graph}\,(DF_{\uparrow}(p)) \subset \mathrm{Graph}\,\big(D\overline{F}_{\uparrow}(p)\big)$ and we get that $v \in D\overline{F}_{\uparrow}(p)(u)$. $\square$

Remark 4.13. Inclusion (11) is also satisfied when both sets are empty. It means then that the contingent epiderivative $D_{\uparrow}f(t)(u) = +\infty$.

Remark 4.14. Directly from definitions of $\overline{F}_{\uparrow}$ and its contingent derivative we have the following:

1. for $t \in \mathbb{T}$ being right-dense and $u \geq 0$: $DF_{\uparrow}(t, f(t))(u) = D\overline{F}_{\uparrow}(t, f(t))(u)$;
2. for $t \in \mathbb{T}$ being left-dense and $u \leq 0$: $DF_{\uparrow}(t, f(t))(u) = D\overline{F}_{\uparrow}(t, f(t))(u)$;
3. for $t \in \mathbb{T}$ being two sided-scattered and $u \neq 0$: $DF_{\uparrow}(t, f(t))(u) = \emptyset$.

We recall the following useful proposition from [6] that shows that the contingent epiderivative can be characterized as a limit of differential quotients:

Proposition 4.15 ([6]). *Let $g : I \to \mathbb{R} \cup \{\pm\infty\}$ be a nontrivial extended function and t belong to its domain. Then*

$$D_{\uparrow}g(t)(u) = \liminf_{h \to 0+, u' \to u} \frac{g(t + hu') - g(t)}{h}. \tag{12}$$

The function g is contingently epidifferentiable at t if and only if $D_{\uparrow}g(t)(0) = 0$.

Corollary 4.16. *Let $f : \mathbb{T} \to \mathbb{R} \cup \{\pm\infty\}$ and $t \in \mathrm{Dom}(f)$. Then for $u \in \mathbb{R}$*

$$D_{\uparrow}f(t)(u) \geq \liminf_{h \to 0+, u' \to u} \frac{\overline{f}(t + hu') - \overline{f}(t)}{h} = D_{\uparrow}\overline{f}(t)(u). \tag{13}$$

Proof. From Proposition 4.12 and Definition 4.10 we have that $D_{\uparrow}f(t)(u) \geq D_{\uparrow}\overline{f}(t)(u)$. The equality follows from Proposition 4.15. □

We can state the following relations between delta and nabla derivatives of f at point t (if they exist) and the contingent epiderivative of the corresponding function $\overline{f}$ defined by formula (5).

Proposition 4.17. *Let $t \in \mathbb{T}^{\kappa}_{\kappa}$ and $f : X \to \mathbb{R} \cup \{\pm\infty\}$, where $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$.*

a) *If $f^{\Delta}(t)$ exists then for $u \geq 0$:*

$$D_{\uparrow}\overline{f}(t)(u) = uf^{\Delta}(t). \tag{14}$$

b) *If $f^{\nabla}(t)$ exists then for $u \leq 0$:*

$$D_{\uparrow}\overline{f}(t)(u) = uf^{\nabla}(t). \tag{15}$$

Proof. We give the proof for the part *a)* while the part *b)* one can prove analogously. First let us observe that if f is delta-differentiable at t and $\overline{f}$ is contingently epidifferentiable then for $u = 0$ we have true formula (14). Thus let us consider $u > 0$ and let t be right-scattered. Then for all $s \in [t, \sigma(t)]$ (by Theorem 1.16 from [10]) we have that $\overline{f}(s) = f^{\Delta}(t)(s - t) + f(t)$, while for $t \in \mathbb{T}$ we have that $f(t) = \overline{f}(t)$. Moreover,

$$D_{\uparrow}\overline{f}(t)(u) = \liminf_{h \to 0+, u' \to u} \frac{f^{\Delta}(t)(t + hu' - t) + f(t) - \overline{f}(t)}{h} = uf^{\Delta}(t).$$

Let t be right-dense. Then for $u > 0$

$$uf^{\Delta}(t) = u \lim_{t_n \to t} \frac{f(t_n) - f(t)}{t_n - t}.$$

As t is right-dense there exist sequences (h_n) and (u_n) such that $t_n = t + h_n u_n$ and with n going to infinity we have that h_n tends to zero, u_n tends to u and, consequently, $t_n \in \mathbb{T}$ tends to t. Hence $uf^{\Delta}(t) = \lim_{n\to\infty} \frac{f(t+h_n u_n) - f(t)}{h_n u_n} = D_{\uparrow}\overline{f}(t)(u)$.
Now we consider $u \leq 0$. The case when $u = 0$ is obvious and formula (15) holds. Thus let us take $u < 0$ and let t be left-scattered. Then for all $s \in [\rho(t), t]$ we get

$$\overline{f}(s) = f^{\nabla}(t)(s - t) + f(t)$$

where $\overline{f}(t) = f(t)$, for $t \in \mathbb{T}$. Moreover,

$$D_{\uparrow}\overline{f}(t)(u) = \liminf_{h \to 0+, u' \to u} \frac{f(t) + f^{\nabla}(t)(t + hu' - t) - \overline{f}(t)}{h} = uf^{\nabla}(t).$$

To finish the proof one has to consider t left-dense but this can be done analogously. □

We can characterize values of the contingent epiderivative of $f : \mathbb{T} \to \mathbb{R} \cup \{\pm\infty\}$ by the contingent epiderivative of $\overline{f}$. Directly from definitions of $\overline{f}$ and its contingent epiderivative we have the following:

Proposition 4.18. *Let* $f : X \to \mathbb{R}\cup\{\pm\infty\}$, $\overline{f} : I \to \mathbb{R}\cup\{\pm\infty\}$ *for* $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$ *and* $\mathrm{Epi}(\overline{f}) = G(f)$. *Then*

1. f *is contingently epidifferentiable iff* $\overline{f}$ *is contingently epidifferentiable;*
2. *for* $t \in \mathbb{T}$ *being right–dense and* $u \geq 0 : D_{\uparrow}f(t)(u) = D_{\uparrow}\overline{f}(t)(u)$*;*
3. *for* $t \in \mathbb{T}$ *being left–dense and* $u \leq 0 : D_{\uparrow}f(t)(u) = D_{\uparrow}\overline{f}(t)(u)$*;*
4. *for* $t \in \mathbb{T}$ *being two sided–scattered and* $u \neq 0 : D_{\uparrow}f(t)(u) = +\infty$.

5. Properties and examples

In [11] one can find the definition of local right–maximum and local right–minimum of a function on $\mathbb{T}$. Following this idea we propose definitions of local right–minimizer, local left–minimizer and local minimizer.

Definition 5.1. We call a point $t_0 \in \mathbb{T}^{\kappa}$ a *local right–minimizer of a function* $f : \mathbb{T} \to \mathbb{R}$ provided

(i) if t_0 is right–scattered, then $f(\sigma(t_0)) \geq f(t_0)$;
(ii) if t_0 is right–dense, then there is a neighborhood U of t_0 such that $f(t) \geq f(t_0)$ for all $t \in U \cap \mathbb{T}$ with $t > t_0$.

We call a point $t_0 \in \mathbb{T}_{\kappa}$ a *local left–minimizer of a function* $f : \mathbb{T} \to \mathbb{R}$ provided

(i) if t_0 is left–scattered, then $f(\rho(t_0)) \geq f(t_0)$;
(ii) if t_0 is left–dense, then there is a neighborhood U of t_0 such that $f(t) \geq f(t_0)$ for all $t \in U \cap \mathbb{T}$ with $t < t_0$.

We call a point $t_0 \in \mathbb{T}^\kappa_\kappa$ a *local minimizer of a function* $f : \mathbb{T} \to \mathbb{R}$ if it is local right–minimizer and local left–minimizer of f.

The definition of $\overline{f}$ for a function f implies the following equivalence.

Proposition 5.2. *Let* $f : X \to \mathbb{R}$, $\overline{f} : I \to \mathbb{R}$, *where* $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$ *and* $\overline{f}$ *is defined by formula* (5). *Then* $t_0 \in \mathbb{T}$ *is a local minimizer of* $f : X \to \mathbb{R}$ *if and only if* t_0 *is a local minimizer of corresponding* $\overline{f}$.

Proof. If we assume that t_0 is a local minimizer of $\overline{f}$, then by the definition of $\overline{f}$, it is obvious that t_0 is a local minimizer of f. Thus, let us assume that t_0 is a local minimizer of $f : X \to \mathbb{R}$. In order to show that t_0 is a local minimizer of $\overline{f} : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ and $X = I \cap \mathbb{T}$, we have to prove that t_0 is a local right-minimizer and a local left-minimizer of $\overline{f}$. We start with the proof of the statement that t_0 is a local left-minimizer of $\overline{f}$.

Firstly, we assume that t_0 is left–scattered. It is clear that $\overline{f}(\rho(t_0)) \geq \overline{f}(t_0)$ since $\overline{f}(t) = f(t)$ for $t \in \mathbb{T}$. Thus we consider $s \in (\rho(t_0), t_0)$ and then one can write

$$\overline{f}(s) = a(s - t_0) + \overline{f}(t_0)$$

where $a = \frac{f(t_0) - f(\rho(t_0))}{t_0 - \rho(t_0)}$. Since $t_0 > \rho(t_0)$ and, by the assumption of minimality, $f(t_0) \leq f(\rho(t_0))$ we get $a \leq 0$ and $a(s - t_0) \geq 0$. Therefore,

$$a(s - t_0) + \overline{f}(t_0) \geq \overline{f}(t_0)$$

what implies that

$$\overline{f}(s) \geq \overline{f}(t_0)$$

for $s \in (\rho(t_0), t_0)$.

Nextly we assume that t_0 is left-dense. Then there exists $U = (t_0 - \delta, t_0]$, $\delta > 0$ such that for every $t \in U \cap \mathbb{T}$, $f(t) \geq f(t_0)$ and f is non-increasing on $(t_0 - \delta, t_0] \cap \mathbb{T}$. As before, for $t \in \mathbb{T}$ the proof is obvious, thus we consider $s \in U \setminus \mathbb{T}$. Then there exists $t \in U \cap \mathbb{T}$ such that $s \in (\rho(t), t)$ and

$$\overline{f}(s) = f(t) + \frac{f(t) - f(\rho(t))}{\nu(t)}(s - t).$$

Since for $t \in \mathbb{T}$, $\overline{f}(t) = f(t)$, $s < t$ and (by monotonicity) $f(\rho(t)) \geq f(t)$, we get

$$\overline{f}(s) = f(t) + \frac{f(t) - f(\rho(t))}{\nu(t)}(s - t) \geq f(t) \geq f(t_0) = \overline{f}(t_0).$$

Therefore $\overline{f}(s) \geq \overline{f}(t_0)$ for $s \in (t_0 - \sigma, t_0]$, what implies that t_0 is a local left-minimizer of $\overline{f}$.

Now we prove that t_0 is a local right-minimizer of $\overline{f}$. Firstly we assume t_0 to be right-scattered. Since $\overline{f}(\tau) = f(\tau)$ for $\tau \in \mathbb{T}$, $\overline{f}(\sigma(t_0)) \geq \overline{f}(t_0)$. If $s \in (t_0, \sigma(t_0))$, then

$$\overline{f}(s) = f(t_0) + \frac{f(\sigma(t_0)) - f(t_0)}{\mu(t_0)}(s - t_0).$$

By the assumption we get $f(t_0) \le f(\sigma(t_0))$, so $\frac{f(\sigma(t_0))-f(t_0)}{\mu(t_0)} \ge 0$ and $\frac{f(\sigma(t_0))-f(t_0)}{\mu(t_0)}(s - t_0) \ge 0$. Therefore,

$$\overline{f}(s) = f(t_0) + \frac{f(\sigma(t_0)) - f(t_0)}{\mu(t_0)}(s - t_0) \ge f(t_0) = \overline{f}(t_0).$$

Thus $\overline{f}(s) \ge \overline{f}(t_0)$ for $s \in [t_0, \sigma(t_0))$.

Assume now that t_0 is right-dense. Then there exists a neighborhood $U = [t_0, t_0+\delta)$, $\delta > 0$ of t_0 such that $f(t) \ge f(t_0)$ for all $t \in U \cap \mathbb{T}$ and f is non-decreasing on $U \cap \mathbb{T}$. As before, $\overline{f}(t) = f(t) \ge f(t_0) = \overline{f}(t_0)$ for all $t \in \mathbb{T}$. Thus we consider $s \in [t_0, t_0+\delta) \setminus \mathbb{T}$. Then there exists $t \in U \cap \mathbb{T}$ such that $s \in (t, \sigma(t))$. By Remark 3.6, we can write

$$\overline{f}(s) = f(t) + \frac{f(\sigma(t)) - f(t)}{\mu(t)}(s - t).$$

Since for $t \in \mathbb{T}$, $\overline{f}(t) = f(t)$, $s > t$ and (by monotonicity) $f(\sigma(t)) \ge f(t)$, we get $\overline{f}(s) \ge f(t) \ge \overline{f}(t_0)$ for $s \in [t_0, t_0 + \delta)$. Therefore t_0 is a local right-minimizer of f and the proof is complete. □

Theorem 5.3 (Fermat Rule). *Let $f : \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ be a nontrivial* ($\mathrm{Dom}(f) \ne \emptyset$) *extended function and $t \in \mathrm{Dom}(f)$ be a local minimizer of f. Then t is a solution to the variational inequalities:*

$$\forall u \in \mathbb{R}, \quad D_{\uparrow} f(t)(u) \ge 0. \tag{16}$$

Proof. By Proposition 5.2, if t is a local minimizer of f then it is also a local minimizer of $\overline{f}$. Hence the Fermat Rule is also true for $\overline{f}$. Based on [6] we can write that for all $u' \in \mathbb{R}$ and all $h > 0$, $\frac{\overline{f}(t+hu')-\overline{f}(t)}{h} \ge 0$. Next taking the lim inf when h tends to 0 and u' tends to u we get inequality (16) true for $\overline{f}$ and all $u \in \mathbb{R}$. By (13) we have that $D_{\uparrow} f(t)(u) \ge D_{\uparrow}\overline{f}(t)(u)$ for all u, so the thesis is true for f. □

In the sequel we need the following.

Definition 5.4. Let $f : \mathbb{T} \to \mathbb{R} \cup \{\pm\infty\}$. We say that f is *pseudo-convex at* $t_0 \in \mathrm{Dom}(f)$ if

$$\forall t \in \mathbb{T}, \quad D_{\uparrow} f(t_0)(t - t_0) \le f(t) - f(t_0).$$

Definition 5.4 gives similar characterization like one can find for global minimum in [11].

Proposition 5.5. *Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$, where $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$, $t_0 \in \mathrm{Dom}(f)$ and for all $u \in \mathbb{R}$ the corresponding function $\overline{f}$ satisfy $D_{\uparrow}\overline{f}(t_0)(u) \ge 0$. If f is pseudo-convex at t_0 then f achieves its global minimum at t_0.*

Proof. Let $u = t - t_0$. Then

$$0 \le D_{\uparrow}\overline{f}(t_0)(u) \le D_{\uparrow} f(t_0)(u) \le f(t) - f(t_0).$$

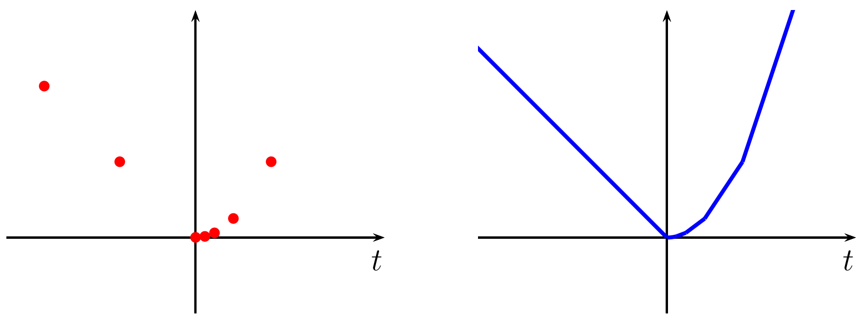
Hence $f(t) - f(t_0) \ge 0$. So $f(t) \ge f(t_0)$ and function f achieves global minimum at t_0. □

Remark 5.6. Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$, where $I \subset \mathbb{R}$, $X = I \cap \mathbb{T}$, be a convex function. For such a function Proposition 5.5 is still valid.

Example 5.7. Let $\mathbb{T} = \{q^{\mathbb{Z}}\} \cup \mathbb{Z}$, $q > 1$. Then $t_0 = 0 \in \mathbb{T}$. Let us consider $f(t) = t^2$ for $t \geq 0$ and $f(t) = -t$ for $t < 0$. Then

$$f^{\Delta}(t) = \begin{cases} t + \sigma(t), & \text{if } t \geq 0 \\ -1, & \text{if } t < 0 \end{cases} \quad \text{and} \quad f^{\nabla}(t) = \begin{cases} t + \rho(t), & \text{if } t \geq 0 \\ -1, & \text{if } t < 0. \end{cases}$$

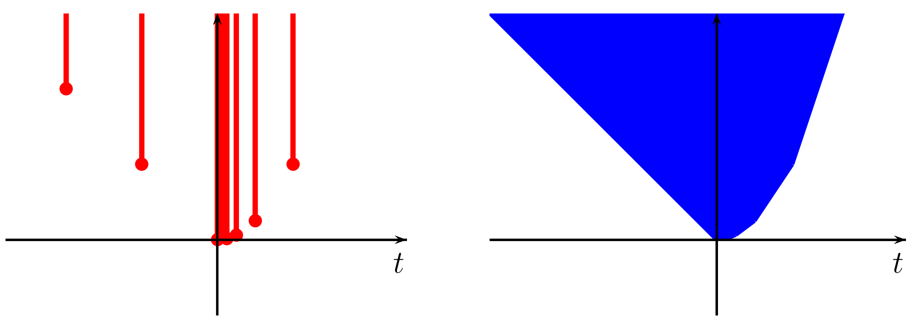
Therefore $f^{\Delta}(0) = 0$ and $f^{\nabla}(0) = -1$. The graphs of the function $f : \mathbb{T} \to \mathbb{R}$ and its corresponding function $\overline{f} : \mathbb{R} \to \mathbb{R}$ are given by Figures 5.1(a), 5.1(b).



(a) the graph of f (b) the graph of $\overline{f}$

Figure 5.1: The graphs of the functions f and $\overline{f}$

According to Definition 5.3 the multifunctions that correspond to f and $\overline{f}$, have the graphs given by Figures 5.2(a) and 5.2(b).



(a) the graph of $F_{\uparrow}$ (b) the graph of $\overline{F}_{\uparrow}$

Figure 5.2: Graph$(F_{\uparrow})$ and Graph$(\overline{F}_{\uparrow})$.

Then the contingent derivative of $\overline{F}_{\uparrow}$ at $(0, 0) \in$ Graph$(\overline{F}_{\uparrow})$ is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ with the graph given by Figure 5.3.

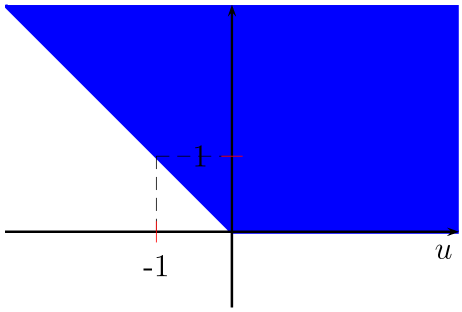
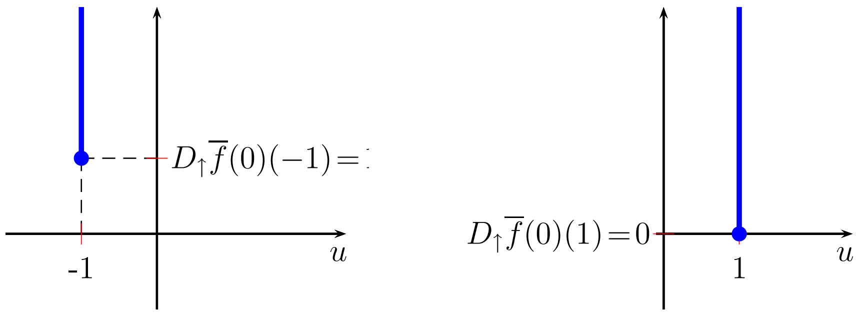


Figure 5.3: $D\overline{F}_{\uparrow}(0,0)$.

Hence the contingent derivative of $\overline{F}_{\uparrow}$ at $(0,0)$ in u is given by Figures 5.6(a) and 5.6(b).



(a) for $u=-1$ (b) for $u=1$

Figure 5.4: $D\overline{F}_{\uparrow}(0,0)(u)$.

Then

$$D_{\uparrow}\overline{f}(0)(u)=\begin{cases}-u, & \text{for } u<0\\ 0, & \text{for } u\geqslant 0\end{cases}$$

and additionally (see Proposition 4.17)

$$D_{\uparrow}\overline{f}(0)(u)=\begin{cases}u\cdot f^{\nabla}(0), & \text{for } u<0\\ u\cdot f^{\Delta}(0), & \text{for } u\geqslant 0.\end{cases}$$

Example 5.8. Let $\mathbb{T}=\{2^{\mathbb{Z}}\}\cup\left\{-2^{\mathbb{Z}}\right\}\cup\{0\}$ and $t_0=0\in\mathbb{T}$. The function that is taken under our consideration is again the same as in Example 5.7, namely $f(t)=t^2$ for $t\geq 0$ and $f(t)=-t$ for $t<0$. In this case Δ and ∇ derivatives at $t_0=0$ do not exist. It turns out that the contingent epiderivative of function f exists at $t_0=0$. Indeed, the contingent derivative of $\overline{F}_{\uparrow}$ at $(0,0)\in\text{Graph}(\overline{F}_{\uparrow})$ is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ with the graph given by Figure 5.5

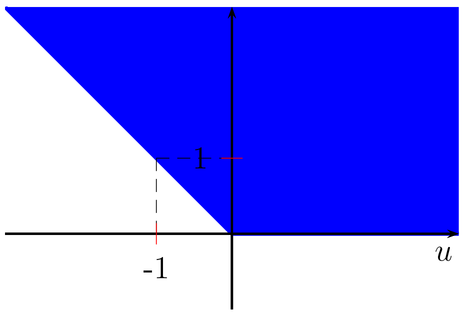
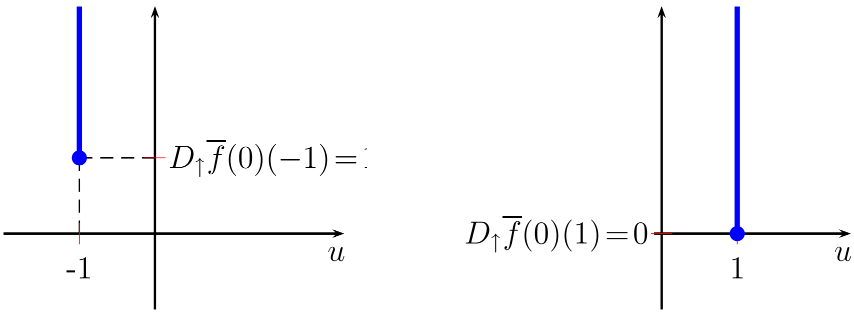


Figure 5.5: $D\overline{F}_{\uparrow}(0,0)$.

So, again the contingent derivative of $\overline{F}_{\uparrow}$ at $(0,0)$ in u is given by Figures 5.6(a) and 5.6(b).



(a) for $u = -1$

(b) for $u = 1$

Figure 5.6: $D\overline{F}_{\uparrow}(0,0)(u)$.

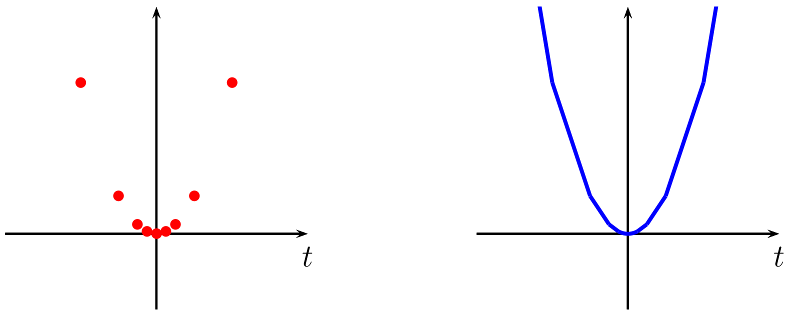
Now it is easy to see that

$$D_{\uparrow}\overline{f}(0)(u) = \begin{cases} -u, & \text{for } u < 0 \\ 0, & \text{for } u \geqslant 0. \end{cases}$$

Example 5.9. Let $\mathbb{T} = \{2^{\mathbb{Z}}\} \cup \left\{-2^{\mathbb{Z}}\right\} \cup \{0\}$ and let us consider $f(t) = t^2$. Then

$$f^{\Delta}(t) = t + \sigma(t) \quad \text{and} \quad f^{\nabla}(t) = t + \rho(t)$$

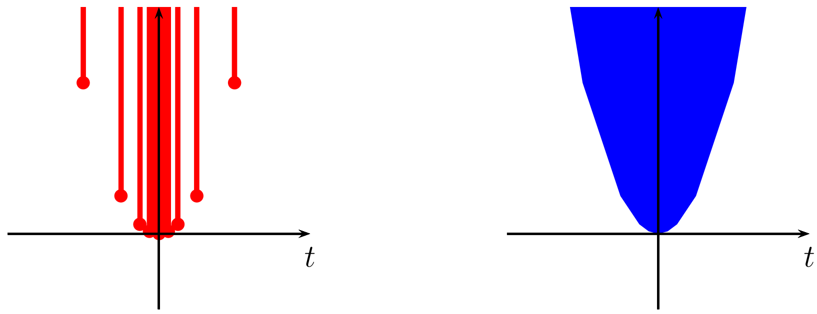
Therefore $f^{\Delta}(0) = f^{\nabla}(0) = 0$ and $f^{\Delta}(1) = 3$ and $f^{\nabla}(1) = \frac{3}{2}$. The graphs of function $f : \mathbb{T} \to \mathbb{R}$ and its corresponding function $\overline{f} : \mathbb{R} \to \mathbb{R}$ are given by Figures 5.7(a) and 5.7(b).



(a) the graph of f (b) the graph of $\overline{f}$

Figure 5.7: The graph of the functions f and $\overline{f}$

According to Definition 5.3 the multifunctions that correspond to f and $\overline{f}$, have the graphs given by Figures 5.8(a) and 5.8(b).



(a) the graph of $F_{\uparrow}$ (b) the graph of $\overline{F}_{\uparrow}$

Figure 5.8: Graph($F_{\uparrow}$) and Graph($\overline{F}_{\uparrow}$).

Then the contingent derivative of $\overline{F}_{\uparrow}$ at $(0,0) \in \text{Graph}(\overline{F}_{\uparrow})$ is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ with the graph given by Figure 5.9.

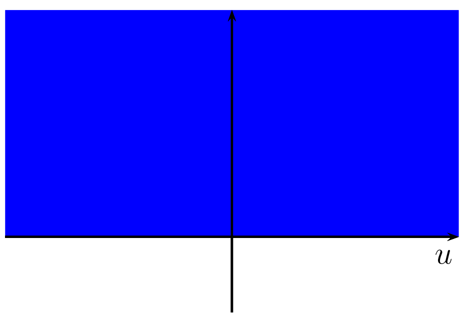
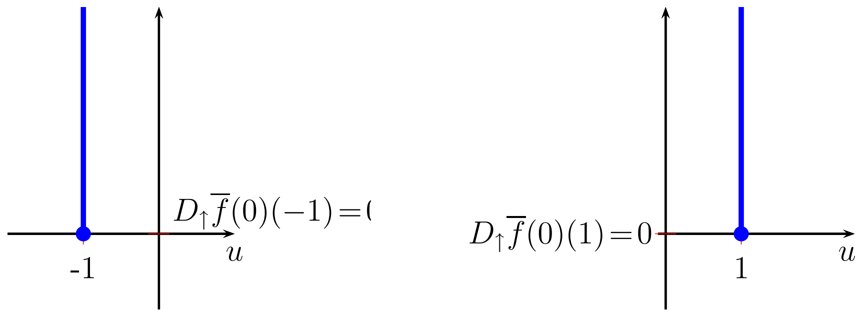


Figure 5.9: $D\overline{F}_{\uparrow}(0,0)$.

Hence the contingent derivative of $\overline{F}_{\uparrow}$ at $(0,0)$ in u is given by 5.10(a) and 5.10(b).



(a) for $u=-1$ (b) for $u=1$

Figure 5.10: $D\overline{F}_\uparrow(0,0)(u)$.

Then

$$D_\uparrow\overline{f}(0)(u)=0$$

and additionally (see Proposition 4.17)

$$D_\uparrow\overline{f}(0)(u)=\begin{cases} u\cdot f^\nabla(0), & \text{for } u<0\\ u\cdot f^\Delta(0), & \text{for } u\geqslant 0.\end{cases}$$

Moreover the contingent derivative of $\overline{F}_\uparrow$ at $(1,1)\in \mathrm{Graph}(\overline{F}_\uparrow)$ is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ with the graph given by Figure 5.11

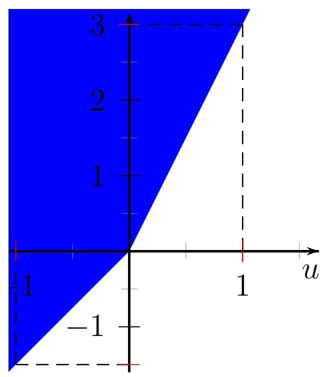
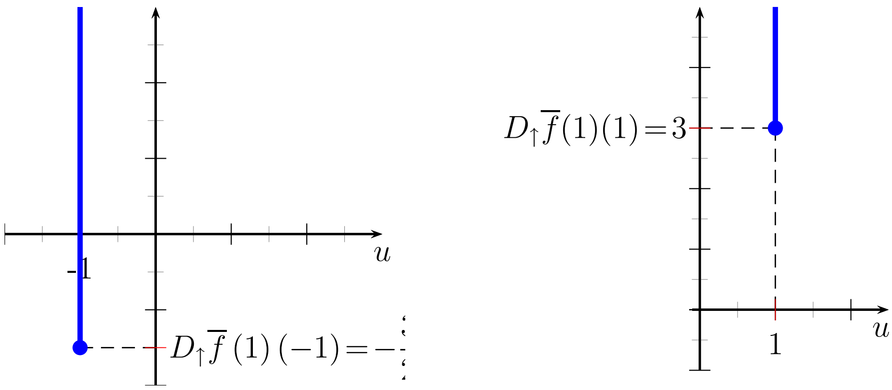


Figure 5.11: $D\overline{F}_\uparrow(1,1)$.

Hence the contingent derivative of $\overline{F}_\uparrow$ at $(1,1)$ in u is given by Figures 5.12(a) and 5.12(b).



(a) for $u = -1$ (b) for $u = 1$

Figure 5.12: $D\overline{F}_{\uparrow}(1,1)(u)$.

Then

$$D_{\uparrow}\overline{f}\,(1)\,(u) = \begin{cases} \frac{3}{2} \cdot u, & \text{for } u < 0 \\ 3 \cdot u, & \text{for } u \geqslant 0. \end{cases}$$

and additionally (see Proposition 4.17)

$$D_{\uparrow}\overline{f}\,(1)\,(u) = \begin{cases} f^{\nabla}(1) \cdot u, & \text{for } u < 0 \\ f^{\Delta}(1) \cdot u, & \text{for } u \geqslant 0. \end{cases}$$

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