Subgradients of Minimal Time Functions Under Minimal Requirements

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This paper concerns the study of a broad class of minimal time functions corresponding to control problems with constant convex dynamics and closed target sets in arbitrary Banach spaces. In contrast to other publications, we do not impose any nonempty interior and/or calmness assumptions on the initial data and deal with generally non-Lipschitzian minimal time functions. The major results present refined formulas for computing various subgradients of minimal time functions under minimal requirements in both cases of convex and nonconvex targets. Our technique is based on advanced tools of variational analysis and generalized differentiation.

1. Introduction

Consider the *minimal time problem* with constant dynamics given by

$$\text{minimize } t \geq 0 \text{ subject to } (x + tF) \cap \Omega \neq \emptyset, \quad x \in X, \tag{1}$$

where X is an arbitrary Banach space of state variables, $\Omega \subset X$ is a closed *target* set, and $F \subset X$ is a closed, bounded, and convex set describing the *constant dynamics* $\dot{x} \in F$ to attain the target set Ω from the state $x \in X$. We refer the reader to [1, 3, 6, 8, 14, 20, 22] and the bibliographies therein for various results and discussions on the minimal time problems and their applications, particularly to control and optimization.

The main attention of this paper is paid to the optimal value function in problem (1)

known as the *minimal time function* and defined by

$$T_\Omega^F(x) := \inf \left\{ t \geq 0 \big| \; \Omega \cap (x + tF) \neq \emptyset \right\}. \tag{2}$$

The requirements on the initial data (X, Ω, F) of (1) imposed above are our *standing assumptions* in this paper. Observe that we do *not* assume the standard *interiority condition* $0 \in \operatorname{int} F$, which is a conventional requirement on F in the study of the minimal time function (2) ensuring, in particular, the Lipschitz continuity of (2) as well as of the corresponding *Minkowski gauge*

$$\rho_F(u) := \inf \left\{ t \geq 0 \big| \; u \in tF \right\}, \quad u \in X, \tag{3}$$

generating (2) under the interiority condition by

$$T_\Omega^F(x) = \inf_{w \in \Omega} \rho_F(w - x), \quad x \in X, \tag{4}$$

where $\rho_F(u) = \inf\{t > 0 \big| \; t^{-1}u \in F\}$ in this case. Representation (4) with the Lipschitz continuous gauge (3) relates the minimal time function $T_\Omega^F(x)$ to the classical *distance function* of the set Ω defined by

$$\operatorname{dist}(x; \Omega) := \inf_{y \in \Omega} \|y - x\|, \quad x \in \Omega, \tag{5}$$

which corresponds to (2) when $F = I\!\!B$, the closed unit ball in X. In fact, the vast majority of methods and results developed in the study of the minimal time function (4) under the interiority requirement $0 \in \operatorname{int} F$ are inspired by their counterparts for the distance function (5); see more details and discussions in the reference above. In the absence of the latter requirement the minimal time function may be quite different from the distance one; e.g., for $F = [0, 1] \subset I\!\!R$ and $\Omega = (-\infty, 0]$ we have the expression

$$T_\Omega^F(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

It is worth noting that functions of type (2) arise not only in the control framework and have not only the "minimal time" interpretation. Their importance has been well recognized in approximation theory; see, e.g., [7, 10]. Furthermore, functions of type (2) belong to a broader class of the so-called *marginal functions*

$$\mu(x) := \inf_{w \in \Omega(x)} \varphi(x, \omega), \quad x \in \Omega, \tag{6}$$

describing, in particular, *optimal values* in general problems of parametric optimization and playing a significant role in sensitivity, stability, and other aspects of variational analysis and its applications; see, e.g., [2, 11, 12, 16, 18, 21] and the references therein. However, the *special structure* of the cost function/Minkowski gauge in (4) is *crucial* for the most interesting results obtained for the minimal time and distance functions and cannot be deduced from those known for more general classes of marginal functions (6).

A characteristic feature of the minimal time function (2) is its *intrinsic nonsmoothness*, which requires the usage of appropriate tools of generalized differentiation. A

number of results for evaluating various subdifferentials of (2) were given in [5, 6, 8, 14, 22] under the underlying assumption $0 \in \text{int}\, F$, which ensures that the Lipschitz continuous function $T_\Omega^F(x)$ behaves similarly to the distance function (5) from the viewpoint of generalized differentiation. It is definitely not the case when the assumption $0 \in \text{int}\, F$ is violated.

To the best of our knowledge, the first effort in dealing with the minimal time functions of type (2) in the absence of the interiority condition $0 \in \text{int}\, F$ was made in [9], where certain formulas for evaluating their proximal and Fréchet subdifferentials were obtained. However, the major results in the out-of-set case $\bar{x} \notin \Omega$ were derived in [9] under the *calmness* property [18] of $T_\Omega^F(\cdot)$ at $\bar{x}$ meaning that

$$\left| T_\Omega^F(x) - T_\Omega^F(\bar{x}) \right| \leq \kappa \|x - \bar{x}\| \quad \text{for all } x \text{ near } \bar{x} \tag{7}$$

with some constant $\kappa > 0$, which is a "one-point" refinement of the classical Lipschitz continuity of the minimal time function discussed above.

The primary goal of this paper is to develop subdifferential properties of the minimal time function (2) with *no imposing either* the interiority condition $0 \in \text{int}\, F$ *or* the calmness condition (7); the latter is required only in the second part of Theorem 4.6. Besides the pure theoretical interest of clarifying what is possible to get without the aforementioned requirements, the major motivation for our study comes from the application to the *generalized Fermat-Torricelli problem* of finding a point at which the sum of its distances to the given closed sets is minimal. This problem is comprehensively studied in the associated paper [15] from both qualitative and quantitative/numerical viewpoints.

We pay the main attention to the two robust limiting constructions by Mordukhovich: the *basic/limiting* and *singular* subdifferentials for minimal time functions. The first of them was studied in our recent paper [14] in the case of $0 \in \text{int}\, F$ while the second one, being trivial for Lipschitzian functions, was not considered in [14] or anybody else in the literature on minimal time functions. As a preliminary technical step (but of its own interest) we evaluate ε-subdifferentials of the Fréchet type for (2). The latter construction reduces to the usual Fréchet subdifferential studied in [9], while we need its ε-enlargements in the general Banach space setting. Note that some results obtained here for Fréchet subgradients of (2) recover those from [9], while the most of them are new in the settings under consideration, even in the case of convex data with no calmness assumption.

The rest of the paper is organized as follows. Section 2 contains preliminaries from generalized differentiation used in the formulations and proofs of the main results.

Section 3 concerns general (non-subdifferential) properties of minimal time functions important for their own sake and useful for the subsequent study of subdifferentials.

Section 4 deals with ε-subdifferentials of (2) at $\bar{x} \in X$ considering both in-set $\bar{x} \in \Omega$ (easier) and out-of-set $\bar{x} \notin \Omega$ (more difficult) cases. The crucial result in the latter case is representing ε-subgradients of the minimal time function via appropriate normals at perturbed projections on the target with proofs based on variational arguments.

In Sections 5–7 we present the main results of the paper related to evaluating basic

and singular subgradients of minimal time functions in both convex and nonconvex settings. Most of the results obtained in these lines are new even for the case of $0 \in \text{int}\, F$ and are illustrated by numerical examples.

Section 5 is particularly devoted to upper estimates and precise representations of the basic and singular subdifferentials of (2) at in-set points $\bar{x} \in \Omega$ of general non-convex target sets. It contains upper estimates and equalities for evaluating basic and singular subgradients of the minimal time function T_Ω^F via the limiting normals to the target Ω and appropriate characteristics of the dynamics F.

Section 6 concerns upper estimates and equalities for the basic and singular subdifferentials of T_Ω^F and their one-sided counterparts at out-of-set points $\bar{x} \notin \Omega$ of the general target set Ω. We derive two types of results in this direction: those expressed via limiting normals to Ω at projection points and those involving the limiting normal cone to the corresponding enlargements of the target.

Section 7 is devoted to the minimal time problem (1) with convex data. The exact calculations of the convex subdifferential of (2) obtained here recover some results of [9] but without the calmness condition and also provide new subdifferential formulas involving the Minkowski gauge (3) in the absence of the interiority condition $0 \in \text{int}\, F$. Besides computing the convex subdifferential of (2), we give the exact evaluation of the singular subdifferential of the convex minimal time function, which has never been considered in the minimal time literature. It is worth also mentioning that the singular subdifferential has not been systematically studied and applied in the general framework of convex analysis.

Out notation is basically standard in variational analysis and generalized differentiation; see, e.g., [11, 18]. Unless otherwise stated, the space X in question is arbitrary *Banach* with the norm $\| \cdot \|$, the closed unit ball $I\!B$, and the canonical pairing $\langle \cdot, \cdot \rangle$ between X and its topological dual X^*. As usual, the symbol $x_k \to \bar{x}$ stands for the norm convergence in X while $x_k^* \xrightarrow{w^*} x^*$ as $k \in I\!N := \{1, 2, \ldots\}$ signifies the sequential weak* convergence in the dual space X^*. Given a set-valued mapping $G \colon X \rightrightarrows X^*$, we denote

$$\mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} G(x) := \left\{ x^* \in X^* \;\middle|\; \exists \text{ sequences } x_k \to \bar{x},\; x_k^* \xrightarrow{w^*} x^* \text{ as } k \to \infty \right. $$
$$\left. \text{with } x_k^* \in G(x_k) \text{ for all } k \in I\!N \right\} \tag{8}$$

the *sequential* Painlevé-Kuratowski upper/outer limit of G as $x \to \bar{x}$. If no confusion arises, the symbol $x \xrightarrow{\Omega} x$ means that $x \to \bar{x}$ with $x \in \Omega$ for a set Ω, while $x \xrightarrow{\varphi} \bar{x}$ indicates that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ for an extended-real-valued function $\varphi \colon X \to \overline{I\!R} := (-\infty, \infty]$.

2. Preliminaries from Generalized Differentiation

Here we define the constructions of generalized differentiation in variational analysis used in this paper and review some of their properties. We mostly follow the book [11], where the reader can find comprehensive material in this direction with the vast

commentaries and references on these and related topics; cf. also [2, 12, 18, 19] for additional issues.

Given a set $\Omega \subset X$ with $\bar{x} \in \Omega$ and given $\varepsilon \geq 0$, the collection of ε-*normals* to Ω at $\bar{x}$ is

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \,\Big|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\}, \quad \bar{x} \in \Omega, \tag{9}$$

with $\widehat{N}_\varepsilon(\bar{x}; \Omega) = \emptyset$ if $\bar{x} \notin \Omega$ for convenience. When $\varepsilon = 0$ in (9), the set $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$ is a cone known as the *Fréchet/regular normal cone* to Ω at $\bar{x}$. For *convex* sets Ω we have

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) = \left\{ x^* \in X^* \,\big|\, \langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\| \text{ whenever } x \in \Omega \right\}, \quad \bar{x} \in \Omega, \tag{10}$$

i.e., $\widehat{N}(\bar{x}; \Omega)$ reduces to the normal cone of convex analysis, while for nonconvex sets Ω the cone $\widehat{N}(\bar{x}; \Omega)$ and its ε-enlargement (9) do not generally possess appropriate properties expected from natural notions of normals. In particular, $\widehat{N}(\bar{x}; \Omega)$ if often trivial $(= \{0\})$ for boundary points of closed sets; there is no robustness and good calculus for (9), etc.

The situation dramatically changes when we consider the robust sequential regularization (8) of the set-valued mapping (9) near $\bar{x}$ defined by

$$N(\bar{x}; \Omega) := \underset{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}}{\mathrm{Lim}\sup} \, \widehat{N}_\varepsilon(x; \Omega) \tag{11}$$

and known as the *basic/limiting/Mordukhovich normal cone* of Ω at $\bar{x}$. The latter cone enjoys a number of good properties in the general Banach space setting and perfect ones in Asplund spaces (including all reflexive) characterized as those Banach spaces, where every separable subspace has a separable dual; see [2, 11, 17] for more details. In this paper we do not need to impose the Asplund structure on X. Note that the normal cone (11) and the corresponding subdifferentials are usually *nonconvex* (in contrast to the majority of their known counterparts), while their important properties and applications are mainly based on *variational/extremal principles* of variational analysis.

In this paper we employ the following three subgradient constructions for extended-real-valued functions $\varphi \colon X \to \overline{\mathbb{R}}$ generated by normals (9) and (11) to their epigraphs $\mathrm{epi}\, \varphi := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}$. For convenience we present these constructions in the equivalent analytic forms. Given a function $\varphi \colon X \to \overline{\mathbb{R}}$ and a point $\bar{x}$ from its domain $\mathrm{dom}\, \varphi := \{x \in X \mid \varphi(x) < \infty\}$, the ε-*subdifferential* of the Fréchet type of φ at $\bar{x}$ is given by

$$\widehat{\partial}_\varepsilon \varphi(\bar{x}) := \left\{ x^* \in X^* \,\Big|\, \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}, \quad \varepsilon \geq 0, \tag{12}$$

with $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0 \varphi(\bar{x})$. For *convex* functions φ the ε-subdifferential (12) reduces to

$$\widehat{\partial}_\varepsilon \varphi(\bar{x}) = \left\{ x^* \in X^* \,\big|\, \langle x^*, x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon \|x - \bar{x}\| \text{ whenever } x \in X \right\}. \tag{13}$$

The *basic subdifferential* $\partial\varphi(\bar{x})$ and *singular subdifferential* $\partial^{\infty}\varphi(\bar{x})$ of Mordukhovich are generated, respectively, by "slant" and "horizontal" normals to epi φ at $(\bar{x}, \varphi(\bar{x}))$ in the sense of (11) and can be defined analytically as

$$\partial\varphi(\bar{x}) := \underset{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon \downarrow 0}}{\mathrm{Lim\,sup}}\; \widehat{\partial}_{\varepsilon}\varphi(x), \tag{14}$$

$$\partial^{\infty}\varphi(\bar{x}) := \underset{\substack{x \xrightarrow{\varphi} \bar{x} \\ \lambda \downarrow 0 \\ \varepsilon \downarrow 0}}{\mathrm{Lim\,sup}}\; \lambda\widehat{\partial}_{\varepsilon}\varphi(x). \tag{15}$$

It is worth observing (although it is not used in the paper) that we can equivalently put $\varepsilon = 0$ in (14) and (15) if X is Asplund and φ is lower semicontinuous (l.s.c.) around $\bar{x}$.

Recall that the Fréchet subdifferential $\widehat{\partial}\varphi(\bar{x})$ reduces to the classical Fréchet derivative of φ at $\bar{x}$ if φ is Fréchet differentiable at $\bar{x}$, while the basic subdifferential (14) reduces to the classical derivative $\partial\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$ if φ is strictly differentiable at $\bar{x}$ in the sense of

$$\lim_{\substack{x \to \bar{x} \\ u \to \bar{x}}} \frac{\varphi(x) - \varphi(u) - \langle\nabla\varphi(\bar{x}), x - u\rangle}{\|x - u\|} = 0,$$

which is automatic when φ is C^1 around $\bar{x}$. If φ is convex, both $\widehat{\partial}\varphi(\bar{x})$ and $\partial\varphi(\bar{x})$ agree with the subdifferential of convex analysis.

For the singular subdifferential (15) we have $\partial^{\infty}\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around $\bar{x}$ in arbitrary Banach spaces. In fact, the latter singular subdifferential condition is a full *characterization* of the local Lipschitzian property under some additional assumptions, which are automatic in finite dimensions; see [11, Theorem 3.52]. Thus the singular subdifferential carries nontrivial information only for non-Lipschitzian functions, which is not the case for the minimal time function (2) under the interiority condition $0 \in \mathrm{int}\,F$.

3.   General Properties of Minimal Time Functions

In this section we collect some properties of the minimal time function (2), which are not related to generalized differentiation. They are of their own interest while most of them are widely used in the subsequent sections for deriving subdifferential results of the paper. Note that, under our *standing assumptions* made in Section 1 and imposed in what follows, the minimal time function is merely extended-real-valued $T_{\Omega}^{F}: X \to \overline{\mathbb{R}}$ and does not share many common properties with the distance function (5) as in the case of $0 \in \mathrm{int}\,F$.

For the given target set Ω, consider the family of its *enlargements*

$$\Omega_r := \left\{x \in X \,\middle|\, T_{\Omega}^{F}(x) \leq r\right\}, \quad r > 0, \tag{16}$$

and establish the following relationship between $T_{\Omega_r}^{F}$ and T_{Ω}^{F}.

Proposition 3.1 (minimal time functions for enlargements of target sets).
Let $x \notin \Omega_r$ be such that $T_\Omega^F(x) < \infty$. Then

$$T_\Omega^F(x) = r + T_{\Omega_r}^F(x) \quad whenever \ r > 0. \tag{17}$$

Proof. Since $\Omega \subset \Omega_r$, we have $t_1 := T_{\Omega_r}^F(x) < \infty$. By the definition of $T_{\Omega_r}^F(x)$, for any $\varepsilon > 0$ there are $w_1 \in \Omega_r$ and $t_1 \le \gamma_1 < t_1 + \varepsilon$ satisfying

$$w_1 \in \Omega_r \cap (x + \gamma_1 F).$$

Then $T_\Omega^F(w_1) \le r$, and hence there are $w_2 \in \Omega$ and $\gamma_2 < r + \varepsilon$ such that

$$w_2 \in \Omega \cap (w_1 + \gamma_2 F).$$

Consequently we get $w_2 \in \Omega \cap (x + (\gamma_1 + \gamma_2)F)$ by the convexity of F. This gives

$$T_\Omega^F(x) \le \gamma_1 + \gamma_2 \le T_{\Omega_r}^F(x) + r + 2\varepsilon,$$

which imply in turn that $T_\Omega^F(x) \le T_{\Omega_r}^F(x) + r$ due to the arbitrary choice of $\varepsilon > 0$.

To justify the opposite inequality in (17), denote $t := T_\Omega^F(x) > r$ due to $x \notin \Omega_r$. Then for any $\varepsilon > 0$ there exist γ with $t \le \gamma < t + \varepsilon$ and $w \in X$ satisfying the relationship

$$w \in \Omega \cap (x + \gamma F).$$

The above element $w \in \Omega$ can be represented as $w = x + \gamma q$ with some $q \in F$. Define further $w_r := x + (\gamma - r)q$ and get $w_r \in \Omega_r$ by $w \in \Omega \cap (w_r + rF) \ne \emptyset$. Thus $w_r \in \Omega_r \cap (x + (\gamma - r)F)$, which implies the inequalities

$$T_{\Omega_r}^F(x) \le \gamma - r \le T_\Omega^F(x) + \varepsilon - r.$$

We therefore arrive at $T_{\Omega_r}^F(x) + r \le T_\Omega^F(x)$ and complete the proof of the proposition. $\qquad \square$

The next property is elementary while useful in what follows.

Proposition 3.2 (minimal time functions with shifted arguments). *For any $x \in \Omega_r$ with $r > 0$ and any $t \ge 0$ we have*

$$T_\Omega^F(x - tq) \le r + t \quad whenever \ q \in F.$$

Proof. Fix (x, r, t, q) in the formulation of the theorem and denote $\lambda := T_\Omega^F(x)$. Picking any $\varepsilon > 0$ and observing that $\lambda \le r$, find $\gamma > 0$ such that $\lambda \le \gamma < \lambda + \varepsilon$ and $w \in X$ satisfying

$$w \in \Omega \cap (x + \gamma F).$$

The latter directly implies the inclusions

$$w \in \Omega \cap (x - tq + tq + \gamma F) \subset \Omega \cap (x - tq + tF + \gamma F) \subset \Omega \cap (x - tq + (t + \gamma)F).$$

It follows then that $T_\Omega^F(x - tq) \le \gamma + t \le t + \lambda + \varepsilon \le t + r + \varepsilon$, and hence $T_\Omega^F(x - tq) \le r + t$ by the arbitrary choice of $\varepsilon > 0$. $\qquad \square$

Now we justify an important result ensuring the representation (4) of the minimal time function (2) via the Minkowski gauge (3) with no interiority requirement $0 \in \operatorname{int} F$.

Proposition 3.3 (relationship between minimal time and Minkowski functions). *Under the standing assumptions made we have the representation*

$$T_\Omega^F(x) = \inf_{w \in \Omega} \rho_F(w - x) \ \text{ for all } x \in X.$$

Proof. Let us first show that $T_\Omega^F(x) = \infty$ if and only if

$$\inf_{w \in \Omega} \rho_F(w - x) = \infty, \quad x \in X. \tag{18}$$

Indeed, it follows from definition (2) that $T_\Omega^F(x) = \infty$ for some fixed $x \in X$ if and only if $\Omega \cap (x + tF) = \emptyset$ whenever $t \geq 0$. The latter is equivalent to the fact that

$$\{t \geq 0 \mid w - x \in tF\} = \emptyset \ \text{ for any } w \in \Omega,$$

which is the same as $\rho_F(w - x) = \infty$ for all $w \in \Omega$, i.e., (18) holds.

Suppose now that $T_\Omega^F(x) < \infty$ and thus $\inf_{w \in \Omega} \rho_F(w - x) < \infty$ for some fixed $x \in X$. Then for any $t \geq 0$ with $\Omega \cap (x + tF) \neq \emptyset$ there is $w \in \Omega$ satisfying $w - x \in tF$, and hence $\rho_F(w - x) \leq t$. The latter implies that

$$\inf_{w \in \Omega} \rho_F(w - x) \leq t,$$

and so $\inf_{w \in \Omega} \rho_F(w - x) \leq T_\Omega^F(x)$. Put further $\gamma := \inf_{w \in \Omega} \rho_F(w - x) < \infty$ and, given $\varepsilon > 0$, find $w \in \Omega$ satisfying

$$\rho_F(w - x) < \gamma + \varepsilon.$$

Then there is $t \geq 0$ such that $t < \gamma + \varepsilon$ and $w - x \in tF$. This implies that

$$T_\Omega^F(x) \leq t \leq \gamma + \varepsilon,$$

and hence $T_\Omega^F(x) \leq \gamma = \inf_{w \in \Omega} \rho_F(w - x)$, which completes the proof. $\qquad \square$

Given $\bar{x} \in X$ with $T_\Omega^F(\bar{x}) < \infty$, consider the (generalized, minimal time) *projection* of $\bar{x}$ on the target set Ω defined by

$$\Pi_\Omega^F(\bar{x}) := \left(\bar{x} + T_\Omega^F(\bar{x})F\right) \cap \Omega. \tag{19}$$

It is not hard to check that if Ω is a compact set, the projection $\Pi_\Omega^F(\bar{x})$ is always nonempty with $T_\Omega^F(\bar{x}) = 0$ if and only if $\bar{x} \in \Omega$.

The next result reveals a kind of linearity of the minimal time functions with respect to projection points on arbitrary target sets.

Proposition 3.4 (minimal time linearity with respect to projections). *Let $\bar{x} \notin \Omega$, and let $\bar{w} \in \Pi_\Omega^F(\bar{x})$. Then for any $\lambda \in (0, 1)$ we have*

$$T_\Omega^F\left(\lambda \bar{w} + (1 - \lambda)\bar{x}\right) = (1 - \lambda)T_\Omega^F(\bar{x}). \tag{20}$$

Proof. It follows that $\bar{w} \in \bar{x} + tF$ for $t := T_\Omega^F(\bar{x}) < \infty$. Then

$$\lambda \bar{w} + (1 - \lambda)\bar{x} = \bar{w} + (1 - \lambda)(\bar{x} - \bar{w}) \in \bar{w} - (1 - \lambda)tF,$$

which implies the inclusion

$$\bar{w} \in \Omega \cap (\lambda \bar{w} + (1 - \lambda)\bar{x} + (1 - \lambda)tF), \quad 0 < \lambda < 1.$$

Hence $T_\Omega^F(\lambda \bar{w} + (1 - \lambda)\bar{x}) \leq (1 - \lambda)t = (1 - \lambda)T_\Omega^F(\bar{x})$ for such λ, which justifies the inequality "$\leq$" in (20). To prove the opposite inequality, denote $t_\lambda := T_\Omega^F(\lambda \bar{w} + (1 - \lambda)\bar{x}) < \infty$ and for any $\varepsilon > 0$ find $t_\lambda \leq \gamma < t_\lambda + \varepsilon$ with

$$\Omega \cap (\bar{x} + \lambda(\bar{w} - \bar{x}) + \gamma F) \neq \emptyset.$$

Thus we have that

$$\Omega \cap (\bar{x} + (\lambda t + \gamma)F) \neq \emptyset,$$

and so $T_\Omega^F(\bar{x}) \leq \lambda t + \gamma \leq \lambda T_\Omega^F(\bar{x}) + t_\lambda + \varepsilon$. It follows finally that

$$(1 - \lambda)T_\Omega^F(\bar{x}) \leq t_\lambda + \varepsilon,$$

which completes the proof by passing to the limit as $\varepsilon \downarrow 0$. $\qquad\square$

Let us now show that, not being Lipschitzian or calm under our assumptions, the minimal time function (2) enjoys the desired *lower semicontinuity property* provided some additional requirements needed for our subsequent applications. Recall that the lower semicontinuity of an extended-real-valued function $\varphi\colon X \to \overline{I\!R}$ is equivalent to the closedness of its *level sets* $\{x \in X \mid \varphi(x) \leq \alpha\}$ for all $\alpha \in I\!R$.

Proposition 3.5 (lower semicontinuity of minimal time functions). *In addition to our standing assumptions, suppose that the space X is either finite-dimensional, or it is reflexive and the target set Ω is convex. Then the minimal time function (2) is lower semicontinuous on its domain.*

Proof. Fix any real number $\alpha \geq 0$ and show that the level set

$$\mathcal{L}_\alpha := \{x \in X \mid T_\Omega^F(x) \leq \alpha\}$$

is closed under the assumptions made. Take an arbitrary sequence $\{x_k\} \subset \mathcal{L}_\alpha$ with $x_k \to \bar{x}$ as $k \to \infty$. Then we have from $T_\Omega^F(x_k) \leq \alpha$ and definition (2) that for every $k \in I\!N$ there is t_k such that $0 \leq t_k < \alpha + 1/k$ and

$$\Omega \cap (x_k + t_k F) \neq \emptyset, \quad k \in I\!N.$$

Fixing further $w_k \in \Omega$ with $w_k \in x_k + t_k F$, we find $q_k \in F$ satisfying $w_k = x_k + t_k q_k$ for all $k \in I\!N$. Observe that the sequences $\{t_k\}$ and $\{q_k\}$ are bounded due to the boundedness of F. If X is *finite-dimensional*, we get without loss of generality that $t_k \to \bar{t}$ and $q_k \to \bar{q}$ as $k \to \infty$ for some elements $\bar{t} \in [0, \alpha]$ and $\bar{q} \in F$. Then $w_k = x_k + t_k q_k \to \bar{x} + \bar{t}\bar{q} \in \Omega$, and thus $T_\Omega^F(\bar{x}) \leq \bar{t} \leq \alpha$.

If X is *reflexive*, we may assume that q_k converges weakly to some $\bar{q}$. It follows from the classical Mazur theorem that a convex combination of elements from the sequence $\{q_k\}$ converge to $\bar{q}$ strongly in X. By the closedness and convexity of F we conclude that $\bar{q} \in F$, and the same properties of Ω imply that $\bar{x} + \bar{t}\bar{q} \in \Omega$. Thus $T_\Omega^F(\bar{x}) \leq \bar{t} \leq \alpha$ in this case as well, which completes the proof of the proposition. $\qquad\square$

Next we characterize the *convexity* property of the minimal time function $T_\Omega^F(x)$.

Proposition 3.6 (convexity of minimal time functions).   *The minimal time function* (2) *is convex if and only if its target set Ω is convex.*

Proof. Suppose that the target set Ω is convex and show that in this case for any $x_1, x_2 \in X$ and for any $\lambda \in (0, 1)$ we have

$$T_\Omega^F(\lambda x_1 + (1 - \lambda)x_2) \le \lambda T_\Omega^F(x_1) + (1 - \lambda)T_\Omega^F(x_2). \tag{21}$$

Since (21) obviously holds if $T_\Omega^F(x_1) = \infty$ or $T_\Omega^F(x_2) = \infty$, assume in what follows that $t_1 := T_\Omega^F(x_1) < \infty$ and $t_2 := T_\Omega^F(x_2) < \infty$. Then for any $\varepsilon > 0$ there are numbers γ_i with

$$t_i \le \gamma_i < t_i + \varepsilon \quad \text{and} \quad \Omega \cap (x_i + \gamma_i F) \ne \emptyset, \quad i = 1, 2.$$

Take $w_i \in \Omega \cap (x_i + \gamma_i F)$ and by the convexity of Ω and F get $\lambda w_1 + (1 - \lambda)w_2 \in \Omega$ and

$$\begin{aligned}
\lambda w_1 + (1 - \lambda)w_2 &\in \lambda x_1 + (1 - \lambda)x_2 + \lambda \gamma_1 F + (1 - \lambda)\gamma_2 F \\
&\subset \lambda x_1 + (1 - \lambda)x_2 + (\lambda \gamma_1 + (1 - \lambda)\gamma_2)F.
\end{aligned}$$

The latter implies the inequalities

$$\begin{aligned}
T_\Omega^F(\lambda x_1 + (1 - \lambda)x_2) &\le \lambda \gamma_1 + (1 - \lambda)\gamma_2 \\
&\le \lambda T_\Omega^F(x_1) + (1 - \lambda)T_\Omega^F(x_2) + \varepsilon,
\end{aligned}$$

which in turn justify (21) by the arbitrary choice of $\varepsilon > 0$.

To prove the converse statement of the proposition, observe that

$$\Omega = \left\{ x \in X \,\middle|\, T_\Omega^F(x) \le 0 \right\},$$

and thus Ω is obviously convex provided that T_Ω^F has this property. $\square$

The last result of this section establishes sufficient conditions for *concavity* property of the minimal time function under consideration.

Proposition 3.7 (concavity of minimal time functions). *Assume that the complement $\Omega^c := X \setminus \Omega$ of the target is convex. Then the minimal time function* (2) *is concave on Ω^c provided that it is finite on this set.*

Proof. If T_Ω^F is not concave on Ω^c, then there are $x_1, x_2 \in \Omega^c$ and $0 < \lambda < 1$ such that

$$T_\Omega^F(\lambda x_1 + (1 - \lambda)x_2) < \lambda T_\Omega^F(x_1) + (1 - \lambda)T_\Omega^F(x_2) < \infty. \tag{22}$$

By definition (2), find $t < \lambda T_\Omega^F(x_1) + (1 - \lambda)T_\Omega^F(x_2)$ and $w \in \Omega$ satisfying

$$w - (\lambda x_1 + (1 - \lambda)x_2) = tq$$

for some $q \in F$. Consider the points

$$u_i := x_i + \frac{tq}{\lambda T_\Omega^F(x_1) + (1 - \lambda)T_\Omega^F(x_2)} T_\Omega^F(x_i), \quad i = 1, 2,$$

and observe that $u_1, u_2 \in \Omega^c$. Indeed, assuming for definiteness that $u_1 \in \Omega$ yields that

$$T_\Omega^F(x_1) \leq \frac{t T_\Omega^F(x_1)}{\lambda T_\Omega^F(x_1) + (1 - \lambda) T_\Omega^F(x_2)} < T_\Omega^F(x_1),$$

a contradiction. At the same time we have the inclusion $w = \lambda u_1 + (1 - \lambda) u_2 \in \Omega$, which is impossible due to the convexity of Ω^c. Combining all the above shows that condition (22) does not hold under the assumptions made, and thus T_Ω^F is concave on Ω^c. $\qquad \square$

4. ε-Subgradients of Minimal Time Functions

This section is devoted to evaluating ε-subgradients (12) of the minimal time function (2) as $\varepsilon \geq 0$ via corresponding characteristics of the target and dynamics sets therein at both in-set and out-of-set points of the target in the general Banach space setting. In particular, our results for $\varepsilon = 0$ provide evaluations of Fréchet subgradients of (2) with no interiority and/or calmness assumptions essentially used in previous methods and results for this case.

We first consider *in-set* points $\bar{x} \in \Omega$. Involving the *support function* of the dynamics

$$\sigma_F(x^*) := \sup_{x \in F} \langle x^*, x \rangle, \quad x^* \in X^*, \tag{23}$$

and the exact *dynamics bound*

$$\|F\| := \sup \{\|q\| \text{ over } q \in F\}, \tag{24}$$

define the following *support level set*:

$$C_\varepsilon^* := \left\{ x^* \in X^* \,\middle|\, \sigma_F(-x^*) \leq 1 + \varepsilon \|F\| \right\}, \quad \varepsilon \geq 0, \tag{25}$$

which is denoted by C^* if $\varepsilon = 0$. Let us begin with upper estimating the ε-subdifferential of (2) via the support set (25) of the dynamics and the set of ε-normals (9) to the target.

Proposition 4.1 (upper estimate of ε-subdifferentials of minimal time functions at in-set points). *Let $\bar{x} \in \Omega$. Then we have*

$$\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x}) \subset \widehat{N}_\varepsilon(\bar{x}; \Omega) \cap C_\varepsilon^* \text{ for any } \varepsilon \geq 0.$$

Proof. Fix an arbitrary subgradient $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$. By definition (12) of the ε-subdifferential, for any $\eta > 0$ find $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq T_\Omega^F(x) - T_\Omega^F(\bar{x}) + (\varepsilon + \eta)\|x - \bar{x}\|$$
$$\leq T_\Omega^F(x) + (\varepsilon + \eta)\|x - \bar{x}\|$$

whenever $x \in \bar{x} + \delta \mathbb{B}$; this takes into account that $T_\Omega^F(\bar{x}) = 0$ on Ω. It follows that

$$\langle x^*, x - \bar{x} \rangle \leq (\varepsilon + \eta)\|x - \bar{x}\| \text{ for all } x \in \Omega,$$

and thus $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$. Fix further any $q \in F$ and get

$$\langle x^*, -tq \rangle \leq T_\Omega^F(\bar{x} - tq) + (\varepsilon + \eta)\|tq\|$$
$$\leq t + t(\varepsilon + \eta)\|F\|$$

when $t > 0$ is sufficiently small. Since $\eta > 0$ is also arbitrarily small, the latter implies that $\sigma_F(-x^*) \leq 1 + \varepsilon\|F\|$ and completes the proof of the proposition. $\square$

The next result provides a certain opposite estimate to Proposition 4.1.

Proposition 4.2 (lower estimate of ε-subdifferentials of minimal time functions at in-set points). *Let $\bar{x} \in \Omega$, and let $\varepsilon \geq 0$. Then for any $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega) \cap C_\varepsilon^*$ we have*

$$x^* \in \widehat{\partial}_{\mu\varepsilon} T_\Omega^F(\bar{x}) \quad \text{with } \mu = \mu(x^*) := 1 + 2\|F\| \cdot \|x^*\|. \tag{26}$$

Proof. Arguing by contradiction, suppose that $x^* \notin \widehat{\partial}_{\mu\varepsilon} T_\Omega^F(\bar{x})$. Then

$$\liminf_{x \to \bar{x}} \frac{T_\Omega^F(x) - T_\Omega^F(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} < -\mu\varepsilon,$$

and thus there exist $\gamma > 0$ and a sequence $x_k \to \bar{x}$ such that

$$T_\Omega^F(x_k) - \langle x^*, x_k - \bar{x} \rangle \leq (-\mu\varepsilon - \gamma)\|x_k - \bar{x}\|, \quad k \in \mathbb{N}.$$

It follows that $x_k \notin \Omega$ for k sufficiently large, since otherwise it contradicts the fact that $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$ due to $\mu\varepsilon + \gamma > \varepsilon$. This also implies for such k that

$$0 < T_\Omega^F(x_k) \leq \|x^*\| \cdot \|x_k - \bar{x}\|,$$

and hence $T_\Omega^F(x_k) \to 0$ as $k \to \infty$. Since $\|x_k - \bar{x}\|^2 > 0$, for each k sufficiently large there are $t_k \geq 0$, $w_k \in \Omega$, and $q_k \in F$ satisfying

$$w_k = x_k + t_k q_k \quad \text{and} \quad T_\Omega^F(x_k) \leq t_k < T_\Omega^F(x_k) + \|x_k - \bar{x}\|^2.$$

Consequently we have the relationships

$$\begin{aligned} \langle x^*, w_k - \bar{x} \rangle &= \langle x^*, x_k - \bar{x} \rangle + t_k \langle x^*, q_k \rangle \\ &\geq \langle x^*, x_k - \bar{x} \rangle + t_k(-1 - \varepsilon\|F\|) \\ &\geq \langle x^*, x_k - \bar{x} \rangle + (T_\Omega^F(x_k) + \|x_k - \bar{x}\|^2)(-1 - \varepsilon\|F\|) \\ &= \langle x^*, x_k - \bar{x} \rangle - T_\Omega^F(x_k) - (1 + \varepsilon\|F\|)\|x_k - \bar{x}\|^2 - \varepsilon T_\Omega^F(x_k)\|F\| \\ &\geq (\mu\varepsilon + \gamma - \varepsilon\|x^*\| \cdot \|F\|)\|x_k - \bar{x}\| - (1 + \varepsilon\|F\|)\|x_k - \bar{x}\|^2. \end{aligned}$$

On the other hand, it follows from $w_k \xrightarrow{\Omega} \bar{x}$ and $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$ that

$$\langle x^*, w_k - \bar{x} \rangle \leq (\varepsilon + \nu)\|w_k - \bar{x}\|$$

for any $\nu > 0$ and k sufficiently large. Observe also that

$$\|w_k - \bar{x}\| \leq \|x_k - \bar{x}\| + t_k\|F\| \leq (1 + \|x^*\| \cdot \|F\|)\|x_k - \bar{x}\| + \|x_k - \bar{x}\|^2\|F\|.$$

Comparing these inequalities and letting $\nu \downarrow 0$ and $k \to \infty$, we get the estimate

$$\mu\varepsilon + \gamma - \varepsilon\|x^*\| \cdot \|F\| \leq \varepsilon(1 + \|x^*\| \cdot \|F\|)$$

Taking into account the definition of μ in (26), we arrive at a contradiction and thus complete the proof of the proposition. $\qquad\square$

Let us now turn to the *out-of-set* case of $\bar{x} \notin \Omega$. The following important result is an extension of [14, Theorem 3.5] established under the interiority assumption $0 \in \operatorname{int} F$. The proof is based on variational arguments involving the seminal *Ekeland variational principle*.

Theorem 4.3 (ε-subgradients of minimal time functions at out-of-set points via perturbed normals to target sets). *Let $\bar{x} \notin \Omega$ with $T_\Omega^F(\bar{x}) < \infty$. Then for every $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$, $\varepsilon \geq 0$, and $\eta > 0$ there is $\bar{w} \in \Omega$ satisfying the relationships*

$$x^* \in \widehat{N}_{\varepsilon+\eta}(\bar{w}; \Omega) \quad and \quad \|\bar{x} - \bar{w}\| \leq \|F\| T_\Omega^F(\bar{x}) + \eta. \tag{27}$$

Proof. Fix (x^*, ε, η) from the formulation of the theorem and, using the ε-sub-differential construction (12), find $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \leq T_\Omega^F(x) - T_\Omega^F(\bar{x}) + \left(\varepsilon + \frac{\eta}{2}\right)\|x - \bar{x}\| \quad \text{for all } x \in \bar{x} + \delta I\!\!B. \tag{28}$$

The minimal time definition (2) ensures the existence of $t \geq 0$, $\widetilde{w} \in \Omega$, and $q \in F$ satisfying

$$T_\Omega^F(\bar{x}) \leq t < T_\Omega^F(\bar{x}) + \widetilde{\eta}^2 \quad \text{and} \quad \widetilde{w} = \bar{x} + tq \quad \text{with } \widetilde{\eta} := \min\left\{\frac{\delta}{2}, \frac{\eta}{2 + \|F\|}, 1\right\}. \tag{29}$$

It follows from (28) and (29) that for any $w \in \Omega \cap (\widetilde{w} + \delta I\!\!B)$ we have the estimates

$$\langle x^*, w - \widetilde{w}\rangle \leq T_\Omega^F(w - \widetilde{w} + \bar{x}) - T_\Omega^F(\bar{x}) + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\|$$

$$\leq T_\Omega^F(w - tq) - T_\Omega^F(\bar{x}) + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\|$$

$$\leq t - T_\Omega^F(\bar{x}) + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\|$$

$$\leq \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| + \widetilde{\eta}^2.$$

Consider further the complete metric space $E := \Omega \cap (\widetilde{w} + \delta I\!\!B)$ and define a continuous function $\varphi \colon E \to I\!\!R$ on it by

$$\varphi(w) := -\langle x^*, w - \widetilde{w}\rangle + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| + \widetilde{\eta}^2, \quad w \in E. \tag{30}$$

We conclude from the constructions and estimates above that

$$\varphi(\widetilde{w}) \leq \inf_{w \in E} \varphi(w) + \widetilde{\eta}^2.$$

Applying the Ekeland variational principle to φ on E allows us to find $\bar{w} \in E$ such that

$$\|\widetilde{w} - \bar{w}\| < \widetilde{\eta} \quad \text{and} \quad \varphi(\bar{w}) \leq \varphi(w) + \widetilde{\eta}\|w - \bar{w}\| \quad \text{whenever } w \in E.$$

This means by the definition of φ in (30) that

$$- \langle x^*, \bar{w} - \widetilde{w} \rangle + \left(\varepsilon + \frac{\eta}{2} \right) \|\bar{w} - \widetilde{w}\| + \widetilde{\eta}^2$$

$$\leq - \langle x^*, w - \widetilde{w} \rangle + \left(\varepsilon + \frac{\eta}{2} \right) \|w - \widetilde{w}\| + \widetilde{\eta}^2 + \widetilde{\eta}\|w - \bar{w}\|$$

for all $w \in E$. Taking into account the construction of $\widetilde{\eta}$ in (29), we get

$$\langle x^*, w - \bar{w} \rangle \leq \left(\varepsilon + \frac{\eta}{2} + \widetilde{\eta} \right) \|w - \bar{w}\| \leq (\varepsilon + \eta)\|w - \bar{w}\|. \tag{31}$$

It follows furthermore that

$$\|w - \widetilde{w}\| \leq \|w - \bar{w}\| + \|\bar{w} - \widetilde{w}\| < 2\widetilde{\eta} < \delta \quad \text{for any } w \in \Omega \cap (\bar{w} + \widetilde{\eta}I\!\!B).$$

This ensures that $\Omega \cap (\bar{w} + \widetilde{\eta}I\!\!B) \subset E$ and hence $x^* \in \widehat{N}_{\varepsilon + \eta}(\bar{w}; \Omega)$ by (9) and (31). Employing finally the choice of $(t, q, \widetilde{w}, \widetilde{\eta})$ in (29), we get

$$\|\bar{x} - \bar{w}\| \leq \|\bar{x} - \widetilde{w}\| + \|\widetilde{w} - \bar{w}\| \leq t\|q\| + \widetilde{\eta}$$
$$\leq \|F\| \left(T_\Omega^F(\bar{x}) + \widetilde{\eta}^2 \right) + \widetilde{\eta} \leq \|F\| T_\Omega^F(\bar{x}) + \widetilde{\eta}(\|F\| + 1)$$
$$\leq \|F\| T_\Omega^F(\bar{x}) + \eta,$$

which justifies the remaining estimate in (27) and completes the proof of theorem. $\square$

Next result fully describes behavior of the support function (23) at ε-subgradients of the minimal time function (2) taken at $\bar{x} \notin \Omega$ via the dynamics bound (24).

Proposition 4.4 (relationship between dynamics and ε-subgradients of minimal time functions at out-of-set points). *Let $\bar{x} \notin \Omega$ and $T_\Omega^F(\bar{x}) < \infty$ for (2). Then for any $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$ we have the two-sided estimates*

$$1 - \varepsilon\|F\| \leq \sigma_F(-x^*) \leq 1 + \varepsilon\|F\|, \quad \varepsilon \geq 0. \tag{32}$$

Proof. Fix $\varepsilon \geq 0$ and $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$. Picking an arbitrary number $\gamma > 0$ and using the ε-subgradient definition (12), find $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq T_\Omega^F(x) - T_\Omega^F(\bar{x}) + (\varepsilon + \gamma)\|x - \bar{x}\| \quad \text{for all } x \in \bar{x} + \delta I\!\!B.$$

Let $r := T_\Omega^F(\bar{x})$, which ensures that $\bar{x}$ belongs to the enlargement Ω_r defined in (16). By Proposition 3.2 we have the estimate

$$T_\Omega^F(\bar{x} - tq) \leq r + t \quad \text{whenever } q \in F \text{ and } t \geq 0.$$

Since $x := \bar{x} - tq \in \bar{x} + \delta I\!\!B$ when t is sufficiently small, it follows that

$$\langle x^*, -tq \rangle \leq T_\Omega^F(\bar{x} - tq) - T_\Omega^F(\bar{x}) + t(\varepsilon + \gamma)\|q\|$$
$$\leq t + t(\varepsilon + \gamma)\|F\|.$$

Letting $\gamma \downarrow 0$ yields that $\sigma_F(-x^*) \le 1 + \varepsilon\|F\|$, which is the upper estimate in (32).

To derive the lower estimate in (32), consider a sequence of $\nu_k \downarrow 0$ as $k \to \infty$ and for any $k \in \mathbb{N}$ find $t_k \ge 0$ such that

$$r \le t_k < r + \nu_k^2 \quad \text{and} \quad (\bar{x} + t_k F) \cap \Omega \ne \emptyset.$$

The latter implies there existence of $q_k \in F$ and $w_k \in \Omega$ satisfying

$$w_k = \bar{x} + t_k q_k = \bar{x} + \nu_k q_k + (t_k - \nu_k) q_k \quad \text{and} \quad T_\Omega^F(\bar{x} + \nu_k q_k) \le t_k - \nu_k.$$

Moreover, we have $x_k := \bar{x} + t_k \nu_k q_k \in \bar{x} + \delta \mathbb{B}$ when k is sufficiently large. This yields

$$\begin{aligned}
\langle x^*, \nu_k q_k \rangle &\le T_\Omega^F(\bar{x} + \nu_k q_k) - T_\Omega^F(\bar{x}) + (\varepsilon + \gamma)\nu_k \|q_k\| \\
&\le t_k - \nu_k - r + (\varepsilon + \gamma)\nu_k \|F\| \\
&\le \nu_k^2 - \nu_k + (\varepsilon + \gamma)\nu_k \|F\|
\end{aligned}$$

and justifies therefore that

$$1 - \nu_k - (\varepsilon + \gamma)\|F\| \le \langle -x^*, q_k \rangle \le \sigma_F(-x^*).$$

Thus we get $1 - \varepsilon\|F\| \le \sigma_F(-x^*)$ by letting $\nu_k \downarrow 0$ as $k \to \infty$ and taking into account that $\gamma > 0$ was chosen arbitrarily. This completes the proof of the proposition. $\square$

Next we obtain an upper estimate of the ε-subdifferentials of the minimal time function (2) at out-of-set points via the sets of ε-normals (9) to Ω at (generalized) *projection points* and the Minkowski gauge of the dynamics (3).

Proposition 4.5 (upper estimate of ε-subgradients of minimal time functions at out-of-set points via projections on targets). *Let $\bar{x} \notin \Omega$ with $T_\Omega^F(\bar{x}) < \infty$, and let $\Pi_\Omega^F(\bar{x}) \ne \emptyset$. Then for any $\bar{w} \in \Pi_\Omega^F(\bar{x})$ and $\varepsilon \ge 0$ we have the estimate*

$$\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x}) \subset -\widehat{\partial}_\varepsilon \rho_F(\bar{w} - \bar{x}) \cap \widehat{N}_\varepsilon(\bar{w}; \Omega). \tag{33}$$

Proof. Fix a number $\varepsilon \ge 0$ and an ε-subgradient $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$. Then picking any number $\eta > 0$ and employing (12), we find $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \le T_\Omega^F(x) - T_\Omega^F(\bar{x}) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{whenever } x \in \bar{x} + \delta \mathbb{B}. \tag{34}$$

Let us first show that, taking any projection point $\bar{w} \in \Pi_\Omega^F(\bar{x})$, we have the upper estimate

$$\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x}) \subset \widehat{N}_\varepsilon(\bar{w}; \Omega)$$

via ε-normals (9) to the target Ω. Indeed, fix $\bar{w} \in \Pi_\Omega^F(\bar{x})$ and observe that $\bar{w} \in \Omega \cap (\bar{x} + tF)$ with $t := T_\Omega^F(\bar{x}) > 0$. Hence $w \in \Omega \cap (w - \bar{w} + \bar{x} + tF)$ for any $w \in \Omega$. Specifying further $w \in \bar{w} + \delta \mathbb{B}$ with $\delta > 0$ from (34) and taking into account that $w - \bar{w} + \bar{x} \in \bar{x} + \delta \mathbb{B}$ and $T_\Omega^F(w - \bar{w} + \bar{x}) \le t = T_\Omega^F(\bar{x})$, we get by (34) that

$$\begin{aligned}
\langle x^*, w - \bar{w} \rangle &\le T_\Omega^F(w - \bar{w} + \bar{x}) - T_\Omega^F(\bar{x}) + (\varepsilon + \eta)\|w - \bar{w}\| \\
&\le (\varepsilon + \eta)\|w - \bar{w}\|.
\end{aligned}$$

This implies $x^* \in \widehat{N}_\varepsilon(\bar{w}; \Omega)$ by definition (9).

To continue the proof of estimate (33) by involving now the ε-subdifferential of the Minkowski gauge ρ_F, we set $\widetilde{x} = \bar{w} - \bar{x}$ and apply (34) with $\bar{x} - t(x - \widetilde{x})$ and $t > 0$ sufficiently small. Then (34), Proposition 3.3, and the convexity of ρ_F ensure the relationships

$$
\begin{aligned}
\langle x^*, -t(x - \widetilde{x}) \rangle &\leq T_\Omega^F(\bar{x} - t(x - \widetilde{x})) - T_\Omega^F(\bar{x}) + (\varepsilon + \eta)t\|x - \widetilde{x}\| \\
&\leq \rho_F(\bar{w} - \bar{x} + t(x - \widetilde{x})) - \rho_F(\bar{w} - \bar{x}) + (\varepsilon + \eta)t\|x - \widetilde{x}\| \\
&\leq \rho_F(tx + (1 - t)(\bar{w} - \bar{x})) - \rho_F(\bar{w} - \bar{x}) + (\varepsilon + \eta)t\|x - \widetilde{x}\| \\
&\leq t\rho_F(x) + (1 - t)\rho_F(\bar{w} - \bar{x}) - \rho_F(\bar{w} - \bar{x}) + (\varepsilon + \eta)t\|x - \widetilde{x}\| \\
&= t\left(\rho_F(x) - \rho_F(\widetilde{x})\right) + (\varepsilon + \eta)t\|x - \widetilde{x}\|.
\end{aligned}
$$

Thus $-x^* \in \widehat{\partial}_\varepsilon \rho_F(\bar{w} - \bar{x})$, and the proof is complete. $\qquad\square$

The last assertion of this section provides a two-sided estimate of ε-subgradients of the minimal time function (2) at out-of-set points $\bar{x} \in \Omega$ via the set of ε-normals to the target *enlargements* (16) and appropriate characteristics of the dynamics. The results obtained extend the ones from [14, Theorem 4.2] derived for the ε-subdifferential $\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$ under the interiority assumption $0 \in \text{int}\, F$ and those from [9, Theorem 4.2] given for the Fréchet subdifferential $\widehat{\partial} T_\Omega^F(\bar{x})$ under the calmness assumption (7).

In addition to (25), define the *two-sided support set*

$$
S_\varepsilon^* := \left\{ x^* \in X^* \,\middle|\, 1 - \varepsilon\|F\| \leq \sigma_F(-x^*) \leq 1 + \varepsilon\|F\| \right\}, \quad \varepsilon \geq 0, \tag{35}
$$

which reduces to $S^* := \{x^* \in X^* \mid \sigma_F(-x^*) = 1\}$ for $\varepsilon = 0$.

Theorem 4.6 (ε-subgradients of minimal time functions at out-of-set points via ε-normals to target enlargements). *Let $\bar{x} \notin \Omega$ with $r := T_\Omega^F(\bar{x}) < \infty$ under our standing assumptions. Then we have the upper estimate*

$$
\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x}) \subset \widehat{N}_\varepsilon(\bar{x}; \Omega_r) \cap S_\varepsilon^* \quad \text{for all } \varepsilon \geq 0. \tag{36}
$$

Conversely, suppose that the minimal time function T_Ω^F is calm at $\bar{x}$ with constant κ. Then for any $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r) \cap S_\varepsilon^*$ and $\varepsilon \geq 0$, we have the inclusion

$$
x^* \in \widehat{\partial}_{\ell\varepsilon} T_\Omega^F(\bar{x}) \quad \text{with } \ell = \ell(x^*) := 1 + 2\|x^*\| \cdot \|F\| + 2\kappa\|F\|. \tag{37}
$$

Proof. Fix $x^* \in \widehat{\partial}_\varepsilon T_\Omega^F(\bar{x})$ with $\varepsilon \geq 0$ and observe that the inclusion $x^* \in S_\varepsilon^*$ follows from Proposition 4.4. To justify $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r)$, pick $\eta > 0$ and find $\delta > 0$ such that inequality (34) is satisfied. Since $T_\Omega^F(x) \leq r = T_\Omega^F(\bar{x})$ for all $x \in \Omega_r$, we have

$$
T_\Omega^F(x) - T_\Omega^F(\bar{x}) \leq 0 \quad \text{whenever } x \in \Omega_r \cap (\bar{x} + \delta\mathbb{B}),
$$

which implies therefore that (34) reduces to

$$
\langle x^*, x - \bar{x} \rangle \leq (\varepsilon + \eta)\|x - \bar{x}\|
$$

for such x. Thus we get $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r)$ by (9) and justify the upper estimate (36).

To prove the converse inclusion (37) under the extra calmness assumption, pick any $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r) \cap S_\varepsilon^*$ with $\varepsilon \geq 0$ and, applying Proposition 4.2 and taking into account that $S_\varepsilon^* \subset C_\varepsilon^*$ and $\mu(x^*) \leq \ell(x^*)$ for $\mu(x^*)$ in (26) and $\ell = \ell(x^*)$ in (37), we get

$$x^* \in \widehat{\partial}_{\ell\varepsilon} T_{\Omega_r}^F(\bar{x}) \quad \text{with } r = T_\Omega^F(\bar{x}). \tag{38}$$

It follows from Proposition 3.1 that $T_\Omega^F(x) = T_{\Omega_r}^F(x) - r$ for any x with $T_\Omega^F(x) < \infty$ and $T_\Omega^F(x) \geq r$. This yields by (38) that

$$\begin{aligned}
&\liminf_{x \to \bar{x},\, T_\Omega^F(x) \geq r} \frac{T_\Omega^F(x) - T_\Omega^F(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \\
&\geq \liminf_{x \to \bar{x},\, T_\Omega^F(x) \geq r} \frac{T_{\Omega_r}^F(x) - T_{\Omega_r}^F(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\ell\varepsilon.
\end{aligned} \tag{39}$$

To justify (37), it remains to prove that

$$\liminf_{x \to \bar{x},\, T_\Omega^F(x) < r} \frac{T_\Omega^F(x) - T_\Omega^F(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\ell\varepsilon. \tag{40}$$

To proceed, take an arbitrary number $\gamma > 0$ and find $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq (\varepsilon + \gamma)\|x - \bar{x}\| \quad \text{whenever } x \in \Omega_r \cap (\bar{x} + \delta\mathbb{B}) \tag{41}$$

by $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r)$ and $|T_\Omega^F(x) - T_\Omega^F(\bar{x})| \leq \kappa\|x - \bar{x}\||$ for all $x \in \bar{x} + \delta\mathbb{B}$ by the calmness condition. Since $\sigma_F(-x^*) \geq 1 - \varepsilon\|F\|$, there is $q \in F$ such that $\langle -x^*, q \rangle \geq 1 - \varepsilon\|F\| - \gamma$. Fix further a point $x \in X$ such that $T_\Omega^F(x) < r$ and

$$x \in \bar{x} + \delta_1\mathbb{B} \quad \text{with } \delta_1 := \frac{\delta}{1 + \kappa\|F\|}. \tag{42}$$

Denoting $t := T_\Omega^F(x)$, we take a sequence of $\nu_k \downarrow 0$ as $k \to \infty$ and for any $k \in \mathbb{N}$ find $t_k \geq 0$, $w_k \in \Omega$, and $q_k \in F$ satisfying

$$t \leq t_k \leq t + \nu_k \quad \text{and} \quad w_k = x + t_k q_k.$$

It is easy to observe that

$$w_k = x - (r - t_k)q + (r - t_k)q + t_k q_k \subset x - (r - t_k)q + rF$$

when k is sufficiently large. Thus for such k we have

$$T_\Omega^F(x_k) \leq r \quad \text{with } x_k := x - (r - t_k)q$$

and, by using $r - t = T_\Omega^F(\bar{x}) - T_\Omega^F(x) \leq \kappa\|x - \bar{x}\|$ and the definition of δ_1 in (42), arrive subsequently at the upper estimates

$$\begin{aligned}
\|x_k - \bar{x}\| &\leq \|x - \bar{x}\| + (r - t_k)\|q\| \leq \|x - \bar{x}\| + (r - t)\|F\| \\
&\leq \|x - \bar{x}\| + \kappa\|x - \bar{x}\| \cdot \|F\| \leq (1 + \kappa\|F\|)\delta_1 < \delta,
\end{aligned} \tag{43}$$

and thus $x_k \in \bar{x} + \delta I\!B$ for all k sufficiently large. Plugging now $x := x_k$ into (41) and employing the middle estimate in (43), we get

$$\langle x^*, x - \bar{x} \rangle - (r - t_k)\langle x^*, q \rangle \leq (\varepsilon + \gamma)\|x_k - \bar{x}\|$$
$$\leq (\varepsilon + \gamma)(1 + \kappa\|F\|)\|x - \bar{x}\|$$

for the point x fixed above. The latter gives by letting $k \to \infty$ that

$$\langle x^*, x - \bar{x} \rangle \leq (r - t)\langle x^*, q \rangle + (\varepsilon + \gamma)(1 + \kappa\|F\|)\|x - \bar{x}\|$$
$$\leq t - r + (\varepsilon\|F\| + \gamma)(r - t) + (\varepsilon + \gamma)(1 + \kappa\|F\|)\|x - \bar{x}\|$$
$$\leq T_\Omega^F(x) - T_\Omega^F(\bar{x}) + \left[\kappa(\varepsilon\|F\| + \gamma) + (\varepsilon + \gamma)(1 + \kappa\|F\|)\right]\|x - \bar{x}\|,$$

which in turn implies that

$$\liminf_{\substack{x \to \bar{x} \\ T_\Omega^F(x) < r}} \frac{T_\Omega^F(x) - T_\Omega^F(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -(1 + 2\kappa\|F\|)\varepsilon \geq -\ell\varepsilon,$$

since $\gamma > 0$ was chosen arbitrarily. Thus we get (40) and, unifying it with (39), justify (37) and complete the proof of the theorem. $\qquad\square$

5.   Evaluating Basic and Singular Subdifferentials of Minimal Time Functions at In-set Points of General Targets

In this section we obtain various formulas of inclusion and equality types for efficient evaluations of both *basic* (14) and *singular* (15) subdifferentials of minimal time functions at *in-set* points $\bar{x} \in \Omega$ of general nonconvex target sets Ω.

Recall that a function $\varphi \colon X^* \to \overline{I\!R}$ is *sequentially weak* continuous at* x^* if for any sequence $x_k^* \xrightarrow{w^*} x^*$ we have $\varphi(x_k^*) \to \varphi(x^*)$ as $k \to \infty$. The function φ is sequentially weak* continuous *on* a subset $S \subset X^*$ if it has this property at each point of S.

In what follows we exploit the sequential weak* continuity of the dynamics support function (23), which is automatic in finite dimensions due to the following simple observation. The next proposition is well known while we present it for the reader's convenience.

Proposition 5.1 (Lipschitz continuity of support functions). *Let F be a bounded subset of a normed space X, and let σ_F be the associated support function* (23). *Then*

$$|\sigma_F(x_1^*) - \sigma_F(x_2^*)| \leq \|F\| \cdot \|x_1^* - x_2^*\| \quad \text{for any } x_1^*, x_2^* \in X^*, \tag{44}$$

i.e., σ_F is globally Lipschitz continuous with constant $\|F\|$ in the norm topology of X^*.

Proof. Fix $x_1^*, x_2^* \in X^*$ and for any $\eta > 0$ find by (23) such $q \in F$ that $\sigma_F(x_1^*) - \eta \leq \langle x_1^*, q \rangle$. Then we immediately have the estimates

$$\sigma_F(x_1^*) - \sigma_F(x_2^*) \leq \langle x_1^*, q \rangle - \sigma_F(x_2^*) + \eta$$
$$\leq \langle x_1^*, q \rangle - \langle x_2^*, q \rangle + \eta$$
$$\leq \|x_1^* - x_2^*\| \cdot \|F\| + \eta,$$

which imply in turn that $\sigma_F(x_1^*) - \sigma_F(x_2^*) \leq \|F\| \cdot \|x_1^* - x_2^*\|$, since $\eta > 0$ was chosen arbitrarily. Interchanging the role of x_1^* and x_2^* in the latter estimate gives us (44). $\square$

Let us now establish two-sided relationships between the basic subdifferential of (2) and the basic normal to the target in the *in-set* setting. The following theorem is new even for the case of $0 \in \text{int } F$ in finite dimensions; cf. [14, Theorem 3.6].

Theorem 5.2 (basic subgradients of minimal time functions and basic normals to targets at in-set points). *Let $\bar{x} \in \Omega$ for the minimal time function (2), and let C^* be defined in (25) as $\varepsilon = 0$. Then we have the upper estimate*

$$\partial T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega) \cap C^*, \tag{45}$$

which holds as equality when the dynamics support function (23) is sequentially weak* continuous at each point of the set $-[N(\bar{x}; \Omega) \cap C^*]$; in particular, when $\dim X < \infty$. If in addition $0 \in F$, then we have the normal cone representation

$$N(\bar{x}; \Omega) = \bigcup_{\lambda > 0} \lambda \partial T_\Omega^F(\bar{x}). \tag{46}$$

Proof. To justify the upper estimate (45), fix an arbitrary basic subgradient $x^* \in \partial T_\Omega^F(\bar{x})$ and by definition (14) find sequences $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, $T_\Omega^F(x_k) \to T_\Omega^F(\bar{x}) = 0$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k)$ for all $k \in \mathbb{N}$. If there is a subsequence of $\{x_k\}$ (with no relabeling) that belongs to Ω, then we get $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ and

$$\sigma_F(-x_k^*) \leq 1 + \varepsilon_k \|F\| \tag{47}$$

by Proposition 4.1. Passing there to the limit as $k \to \infty$ and employing definition (11) of the basic normal cone give us $x^* \in N(\bar{x}; \Omega)$. Since furthermore

$$\langle -x_k^*, v \rangle \leq 1 + \varepsilon_k \|F\| \quad \text{for all } v \in F,$$

it follows from (47) as $k \to \infty$ that $\langle -x^*, v \rangle \leq 1$, which justifies (45) when $\{x_k\} \subset \Omega$.

Consider now the other case when $x_k \notin \Omega$ for all $k \in \mathbb{N}$ sufficiently large and find by Theorem 4.3 a sequence $\{w_k\} \subset \Omega$ satisfying

$$x_k^* \in \widehat{N}_{\varepsilon_k + 1/k}(w_k; \Omega) \quad \text{and} \quad \|x_k - w_k\| \leq \|F\| T_\Omega^F(x_k) + 1/k, \quad k \in \mathbb{N}. \tag{48}$$

Since $T_\Omega^F(x_k) \to 0$, it follows from the inequalities in (48) that $w_k \to \bar{x}$ as $k \to \infty$, and thus $x^* \in N(\bar{x}; \Omega)$ by passing to the limit in the inclusions of (48). We also get from Proposition 4.4 that $\sigma_F(-x_k^*) \leq 1 + \varepsilon_k \|F\|$ in this case, which yields that $\sigma_F(-x^*) \leq 1$ as $k \to \infty$ and completes the proof of the upper estimate (45).

Let us next justify the opposite inclusion in (45) under the additional assumption made. Pick any $x^* \in N(\bar{x}; \Omega) \cap C^*$ and by definition (11) find sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ and $\sigma_F(-x^*) \leq 1$ for all $k \in \mathbb{N}$. Invoking the assumed sequential weak* continuity of σ_F on $-[N(\bar{x}; \Omega) \cap C^*]$, we get the convergence $\sigma_F(-x_k^*) \to \sigma_F(-x^*)$ as $k \to \infty$. If $\sigma_F(-x^*) < 1$, then $\sigma_F(-x_k^*) < 1$

for all large k. Proposition 4.2 gives us a sequence $\widetilde{\varepsilon}_k \downarrow 0$ such that $x_k^* \in \widehat{\partial}_{\widetilde{\varepsilon}_k} T_\Omega^F(x_k)$; hence $x^* \in \partial T_\Omega^F(\bar{x})$.

In the other case of $\sigma_F(-x^*) = 1$, denote $\gamma_k := \sigma_F(-x_k^*)$ and get by the assumed weak* continuity that $\gamma_k \to 1$ as $k \to \infty$. Then we have

$$\frac{x_k^*}{\gamma_k} \in \widehat{N}_{\varepsilon_k/\gamma_k}(x_k) \cap C^* \quad \text{and then} \quad \frac{x_k^*}{\gamma_k} \in \widehat{\partial}_{\widetilde{\varepsilon}_k} T_\Omega^F(x_k) \tag{49}$$

for some sequence $\widetilde{\varepsilon}_k \downarrow 0$, which exists by Proposition 4.2. Passing to the limit in (49) as $k \to \infty$ yields $x^* \in \partial T_\Omega^F(\bar{x})$ and completes the proof of equality in (45).

Let us finally justify representation (46). It immediately follows from the upper estimate (45) that the inclusion "$\subset$" holds in (46). It remains to show that under the additional assumption $0 \in F$ the opposite inclusion

$$N(\bar{x}; \Omega) \subset \bigcup_{\lambda > 0} \lambda \partial T_\Omega^F(\bar{x}), \quad \bar{x} \in \Omega$$

is satisfied. To proceed, fix any basic normal $x^* \in N(\bar{x}; \Omega)$ and find by (11) sequences $\varepsilon_k \downarrow 0$, $w_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(w_k; \Omega)$ for all $k \in \mathbb{N}$. Let

$$\lambda_k := \sigma_F(-x_k^*) + 1 = \sup_{v \in F} \langle -x_k^*, v \rangle + 1, \quad k \in \mathbb{N}.$$

and observe from $0 \in F$ that $\lambda_k \geq 1$ for every k. Moreover, the sequence $\{\lambda_k\}$ is bounded in $\mathbb{R}$ due to the boundedness of F in X and the boundedness of the weak* convergent sequence $\{x_k^*\}$ in X^* by the uniform boundedness principle. Without loss of generality, suppose that $\lambda_k \to \lambda > 0$ as $k \to \infty$. Then

$$\frac{x_k^*}{\lambda_k} \in \widehat{\partial}_{\alpha_k \varepsilon_k/\lambda_k} T_\Omega^F(w_k), \quad k \in \mathbb{N}, \tag{50}$$

with $\alpha_k := 2\|F\| \cdot \|x_k^*/\lambda_k\| + 1 \geq 1$ for all k. This implies that

$$x^* \in \lambda \partial T_\Omega^F(\bar{x})$$

by passing to the limit in (50), which completes the proof of the theorem. $\qquad\square$

The next theorem provides an upper estimate of the singular subdifferential of (non-Lipschitzian) minimal time functions at in-set points and also justifies a case of equality therein. As mentioned in the Introduction, the latter subdifferential has never been considered in the literature for minimal time functions while it is important for applications.

Theorem 5.3 (singular subgradients of minimal time functions via basic normals to targets at in-set points). *Define the positive dual cone of the dynamics in (2) by*

$$F_+^* := \left\{ x^* \in X^* \,\middle|\, \langle x^*, v \rangle \geq 0 \text{ for all } v \in F \right\}. \tag{51}$$

Then for any in-set point $\bar{x} \in \Omega$ we have the upper estimate

$$\partial^\infty T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega) \cap F_+^*. \tag{52}$$

Moreover, (52) holds as equality when $0 \in F$ and the support function σ_F in (23) is sequentially weak* continuous at each point of the set $-[N(\bar{x}; \Omega) \cap F_+^*]$.

Proof. To justify (52), fix any $x^* \in \partial^\infty T_\Omega^F(\bar{x})$ and by definition (15) find sequences $\lambda_k \downarrow 0$, $x_k \to \bar{x}$, $\varepsilon_k \downarrow 0$, and $x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k)$ such that $T_\Omega^F(x_k) \to T_\Omega^F(\bar{x}) = 0$ and

$$\lambda_k x_k^* \xrightarrow{w^*} x^* \quad \text{as } k \to \infty.$$

In the case of $x_k \in \Omega$ for a subsequence of $k \in \mathbb{N}$ (no relabeling) we have

$$x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega) \quad \text{and} \quad \sigma_F(-x_k^*) \le 1 + \varepsilon_k \|F\|, \quad k \in \mathbb{N},$$

which implies by construction (9) and Proposition 4.1 that $\lambda_k x_k^* \in \widehat{N}_{\lambda_k \varepsilon_k}(x_k; \Omega)$ and

$$\langle -\lambda_k x_k^*, v \rangle \le \lambda_k + \lambda_k \varepsilon_k \|F\| \quad \text{whenever } v \in F.$$

By passing to the limit in the latter relationships as $k \to \infty$, we get that $x^* \in N(\bar{x}; \Omega)$ and $\langle -x^*, v \rangle \le 0$ for all $v \in F$, respectively. This justifies (52) in the case under consideration.

In the other case of $x_k \notin \Omega$ for all large k, we proceed similarly to the above with using Theorem 4.3 and Proposition 4.4 for out-set points instead of Proposition 4.1 for $x_k \in \Omega$; cf. also the proof of Theorem 5.2. In this way we fully justify the upper estimate (52).

Let us finally prove the opposite inclusion in (52) under the additional assumptions made. Fix any $x^* \in N(\bar{x}; \Omega) \cap F_+^*$ and by definition (11) find sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$. We have furthermore that $\sigma(-x^*) = 0$ due to $0 \in F$ and $x^* \in F_+^*$. It follows from the assumed sequential weak* continuity of the support function σ_F that $0 \le \sigma_F(-x_k^*) \to \sigma_F(-x^*) = 0$. Set now

$$\lambda_k := \sigma_F(-x_k^*) + \sqrt[4]{\varepsilon_k} + 1/k, \quad k \in \mathbb{N},$$

and observe that $\lambda_k \downarrow 0$ as $k \to \infty$ and $x_k^*/\lambda_k \in \widehat{N}_{\varepsilon_k/\lambda_k}(x_k; \Omega) \cap C^*$. Since $\varepsilon_k/\lambda_k \downarrow 0$, by Proposition 4.2 we find a sequence $\widetilde{\varepsilon}_k \downarrow 0$ such that $\frac{x_k^*}{\lambda_k} \in \widehat{\partial}_{\widetilde{\varepsilon}_k} T_\Omega^F(x_k)$ for all $k \in \mathbb{N}$, and hence $x^* \in \partial^\infty T_\Omega^F(\bar{x})$ by passing to the limit as $k \to \infty$. This ensures the equality in (52) under all the assumptions made and thus complete the proof of the theorem. $\qquad \square$

Finally in this section, let us illustrate the results of Theorems 5.2 and 5.3, by the following example of a two-dimensional minimal time problem (1) with a nonconvex target set Ω and a convex dynamics set F of empty interior. In this case the minimal time function (2) is *non-Lipschitzian* and *nonconvex*.

Example 5.4 (basic and singular subgradients of nonconvex and non-Lipschitzian minimal time functions at in-set points). Consider the convex dynamics set $F := [-1, 1] \times \{0\} \subset \mathbb{R}^2$ with $\text{int } F = \emptyset$ and the nonconvex target set

$\Omega := I\!\!R^2 \setminus (-1,1) \times (-1,1)$ in the minimal time problem (1). Then the Minkowski gauge (3) and the minimal time function (2) are computed, respectively, by

$$\rho_F(x) = \begin{cases} |x_1| & \text{if } x \in I\!\!R \times \{0\}, \\ \infty & \text{otherwise}; \end{cases}$$

$$T_\Omega^F(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 - |x_1| & \text{if } x \notin \Omega. \end{cases} \tag{53}$$

We first verify Theorem 5.2 at the in-set point $\bar{x} = (1,0) \in \Omega$. It is easy to see that $\partial T_\Omega^F(\bar{x}) = [-1,0] \times \{0\}$ and that $\sigma_F(v) = |v_1|$ for any $v = (v_1, v_2) \in I\!\!R^2$. Then

$$N(\bar{x}; \Omega) \cap C^* = N(\bar{x}; \Omega) \cap \{v \in I\!\!R^2 \mid \sigma(-v) \le 1\} = [-1,0] \times \{0\},$$

and thus (45) holds as *equality* as well as that of (46). We can check further the fulfillment of (52) as equality in Theorem 5.3, which yields therefore that $\partial^\infty T_\Omega^F(\bar{x}) = \{0\}$. Due the result mentioned at the end of Section 2, the latter condition fully characterizes the local Lipschitzian property of T_Ω^F around $\bar{x}$, which can be seen directly from the explicit formula for the minimal time function given above.

Taking next another in-set point $\bar{y} = (0,1) \in \Omega$, we similarly check the fulfillment of (45) and (52) hold as *equalities* with $\partial T_\Omega^F(\bar{y}) = \{0\} \times I\!\!R^-$ and $\partial^\infty T_\Omega^F(\bar{y}) = \{0\} \times I\!\!R^-$. The latter confirms that T_Ω^F is non-Lipschitzian around $(0,1)$. We see from the precise formula (53) for T_Ω^F that this function is in fact discontinuous at $(0,1)$.

6.   Evaluating Basic and Singular Subdifferentials of Minimal Time Functions at Out-of-set Points of General Targets

This section is devoted to evaluating the basic and singular subdifferentials of the minimal time function (2) at *out-of-set* points $\bar{x} \notin \Omega$. We derive two types of results in this direction: via projection points to the target Ω and via enlargements Ω_r.

Focusing first on results of the projection type, we introduce and apply the following property of *well-posedness* for minimal time functions.

Definition 6.1 (well-posedness of minimal time functions). We say that the minimal time function (2) is *well posed* at $\bar{x} \notin \Omega$ with $T_\Omega^F(\bar{x}) < \infty$ if for any sequence $x_k \to \bar{x}$ as $k \to \infty$ with $T_\Omega^F(x_k) \to T_\Omega^F(\bar{x})$ there is a sequence of projection points $w_k \in \Pi_\Omega^F(x_k)$ containing a convergent subsequence.

The next proposition lists some conditions ensuring the well-posedness of (2). Recall that a norm on X is *Kadec* if the weak and strong (with respect to this norm) convergences agree on the boundary of the unit sphere of X.

Proposition 6.2 (sufficient conditions for well-posedness). *The minimal time function (2) is well posed at $\bar{x} \notin \Omega$ under one of the following conditions:*

(a)   *The target Ω is a compact subset of X;*
(b)   *The space X is finite-dimensional and Ω is a closed subset of X;*

(c) *X is reflexive, $\Omega \subset X$ is closed and convex, and the Minkowski gauge (3) generates an equivalent Kadec norm on X.*

Proof. The well-posedness of (2) under one of the conditions (a) and (b) is obvious. Let us justify it under condition (c). To proceed, fix a convergent sequence $x_k \to \bar{x}$ and observe that the property $T_\Omega^F(x_k) \to T_\Omega^F(\bar{x})$ is automatic when ρ_F generates a norm. It is well-known in this case that $\Pi_\Omega^F(x) \neq \emptyset$ for every $x \in X$ due to the convexity of Ω and the reflexivity of X. Pick any $w_k \in \Pi_\Omega^F(x_k)$ and observe that

$$T_\Omega^F(x_k) = \rho_F(x_k - w_k), \quad k \in \mathbb{N}. \tag{54}$$

It follows that the sequence $\{w_k\}$ is bounded in X, and hence—by the reflexivity of X—it contains a subsequence (with no relabeling) that weakly converges to some element $\bar{w}$. Since Ω is convex and closed in X, it is also weakly closed; this $\bar{w} \in \Omega$. By the lower semicontinuity of ρ_F in the weak topology of X and by (54) we have the relationships

$$\rho_F(\bar{x} - \bar{w}) \leq \liminf_{k\to\infty} \rho_F(x_k - w_k) = \liminf_{k\to\infty} T_\Omega^F(w_k - x_k) = T_\Omega^F(\bar{x}),$$

which imply that $\bar{w} \in \Pi_\Omega^F(\bar{x})$ and $T_\Omega^F(\bar{x} - \bar{w}) = \rho_F(\bar{x} - \bar{w})$. Since ρ_F generates a Kadec norm on X, it follows from $\rho_F(x_k - w_k) \to \rho_F(\bar{x} - \bar{w})$ and the weak convergence of $x_k - w_k$ to $\bar{x} - \bar{w}$ that in fact the sequence $x_k - w_k$ converges strongly in X, and hence $w_k \to \bar{w}$ as $k \to \infty$. This completes the proof of the proposition. $\square$

Now we use the well-posedness property of T_Ω^F to derive upper estimates of both basic and singular subdifferentials of the minimal time function at out-of-set points.

Theorem 6.3 (basic and singular subgradients of minimal time functions at out-of-set points via projections). *Let $\bar{x} \notin \Omega$ with $T_\Omega^F(\bar{x}) < \infty$, and let the minimal time function (2) be well posed at $\bar{x}$. Then we have the estimates*

$$\partial T_\Omega^F(\bar{x}) \subset \bigcup_{\bar{w} \in \Pi_\Omega^F(\bar{x})} \left[-\partial \rho_F(\bar{w} - \bar{x}) \cap N(\bar{w}; \Omega) \right], \tag{55}$$

$$\partial^\infty T_\Omega^F(\bar{x}) \subset \bigcup_{\bar{w} \in \Pi_\Omega^F(\bar{x})} \left[-\partial^\infty \rho_F(\bar{w} - \bar{x}) \cap N(\bar{w}; \Omega) \right]$$
$$\subset \bigcup_{\bar{w} \in \Pi_\Omega^F(\bar{x})} \left[N(\bar{w}; \Omega) \cap F_+^* \right] \tag{56}$$

with the positive dual cone F_+^* of the dynamics defined in (51).

Proof. Pick any basic subgradient $x^* \in \partial T_\Omega^F(\bar{x})$ and by definition (14) find sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{T_\Omega^F} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that

$$x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k) \quad \text{for all } k \in \mathbb{N}. \tag{57}$$

By the well-posedness property of (2) there is a sequence $w_k \in \Pi_\Omega^F(x_k)$, which contains a subsequence (no relabeling) converging to some $\bar{w}$. It follows from definitions (19)

of the generalized projection, the convergence $T_\Omega^F(x_k) \to T_\Omega^F(\bar{x})$, and the assumptions made that $\bar{w} \in \Pi_\Omega^F(\bar{x})$. Applying Proposition 4.5 to (57), we have

$$x_k^* \in -\widehat{\partial}_{\varepsilon_k} \rho_F(x_k - w_k) \cap \widehat{N}_{\varepsilon_k}(w_k; \Omega), \quad k \in I\!N,$$

which yields in turn the upper estimates (55) by passing to the limit as $k \to \infty$.

Let us now prove both inclusions in (56). Taking an arbitrary singular subgradient $x^* \in \partial^\infty T_\Omega^F(\bar{x})$, find by (15) sequences $\varepsilon_k \downarrow 0$, $\lambda_k \downarrow 0$, $x_k \xrightarrow{T_\Omega^F} \bar{x}$, and $x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k; \Omega)$ such that $\lambda_k x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ and

$$x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k) \quad \text{for all } k \in I\!N. \tag{58}$$

By the well-posedness property of (2) there is a sequence $w_k \in \Pi_\Omega^F(x_k; \Omega)$ that contains a subsequence (no relabeling) converging to some $\bar{w}$. As discussed above, we have $\bar{w} \in \Pi_\Omega^F(\bar{x})$. Applying Proposition 4.5 to (58) allows us to conclude that

$$-\lambda_k x_k^* \in \lambda_k \widehat{\partial}_{\varepsilon_k} \rho_F(x_k - w_k) \quad \text{and} \quad x_k^* \in \lambda_k \widehat{N}_{\varepsilon_k}(w_k; \Omega), \quad k \in I\!N. \tag{59}$$

Letting $k \to \infty$ in both inclusions of (59), we arrive at the first estimate in (56).

To justify the remaining inclusion in (56), observe by the arguments similar to the corresponding ones in Theorem 5.3 (cf. also the proof of Theorem 7.3 below for more details in the like setting) that we have the implication

$$-x_k^* \in \widehat{\partial}_{\varepsilon_k} \rho_F(x_k - w_k) \Longrightarrow \sigma_F(-x_k^*) \leq 1 + \varepsilon_k \|F\|, \quad k \in I\!N.$$

It yields by (59) that $x^* \in N(\bar{w}; \Omega) \cap F_+^*$ similarly to the proof of Theorem 5.3, which thus completes the proof of this theorem.   □

The following example illustrates some features of the results obtained in Theorem 6.3.

Example 6.4 (basic and singular subgradients of nonconvex and non-Lipschitzian minimal time functions at out-of-set points). Consider the setting of Example 5.4, where the minimal time function is computed by formula (53). Take the out-of-set point $\bar{z} = (1/2, 1/2) \notin \Omega$ and verify the conclusions of Theorem 6.3. The well-posedness property (54) holds by Proposition 6.2(*ii*). It is easy to check that $\Pi_\Omega^F(\bar{z}) = \{\bar{w}\}$ with $\bar{w} = (1, 1/2)$ for the *Euclidean norm* in the projection operator (19). Thus we arrive at the *equality*

$$\partial T_\Omega^F(\bar{z}) = \{(-1, 0)\} = -\partial \rho_F(\bar{w} - \bar{z}) \cap N(\bar{w}; \Omega)$$

in (55) and similarly get the equality in (56) with $\partial^\infty T_\Omega^F(\bar{z}) = \{0\}$, which is in accordance with the local Lipschitz property of T_Ω^F around this point that obviously follows from the explicit formula (53). Note that both inclusions (55) and (56) are *strict* in this example if the projection in (19) is taken with respect to the *maximum norm* on the plane.

Let us further address a natural question about getting counterparts of Theorems 5.2 and 5.3 on upper estimates for basic and singular subgradients of the minimal time function (2) at *out-of-set* points via basic normals to the enlargement Ω_r of the target set Ω. However, simple examples show the failure of such estimates. For instance, consider the minimal time problem (1) in $X = \mathbb{R}^2$ with $F = \mathbb{B}$ and $\Omega = \{x \in \mathbb{R}^2 | \|x\| \geq 1\}$. Then for $\bar{x} = 0$ and $r = T_\Omega^F(\bar{x}) = 1$ we have $N(\bar{x}; \Omega_r) = \{0\}$ while $\partial T_\Omega^F(\bar{x}) = \{x \in \mathbb{R}^2 | \|x\| = 1\}$.

It occurs that the appropriate analogs of the upper estimates in Theorem 5.2 and 5.3 hold at $\bar{x} \notin \Omega$ with the replacement of $\partial T_\Omega^F(\bar{x})$ and $\partial^\infty T_\Omega^F(\bar{x})$ therein by the *one-sided* modifications of these constructions for $\varphi = T_\Omega^F$ defined by

$$\partial_\geq \varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi^+} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(x), \tag{60}$$

$$\partial_\geq^\infty \varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi^+} \bar{x} \\ \varepsilon \downarrow 0 \\ \lambda \downarrow 0}} \lambda \widehat{\partial}_\varepsilon \varphi(x), \tag{61}$$

where the symbol $x \xrightarrow{\varphi^+} \bar{x}$ signifies that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ and $\varphi(x) \geq \varphi(\bar{x})$. Note that the basic one-sided construction (60) was introduced in [13] and applied therein to the study of distance function (see also [11, Sec.1.3.3] and [14]) while the singular one (61) appears here for the first time. Observe that we always have the inclusions

$$\widehat{\partial}\varphi(\bar{x}) \subset \partial_\geq \varphi(\bar{x}) \subset \partial \varphi(\bar{x}) \quad \text{and} \quad \partial_\geq^\infty \varphi(\bar{x}) \subset \partial^\infty \varphi(\bar{x})$$

which show, in particular, that $\partial_\geq \varphi(\bar{x}) = \partial \varphi(\bar{x})$ if φ is *subdifferentially regular* at $\bar{x}$, i.e., $\widehat{\partial}\varphi(\bar{x}) = \partial \varphi(\bar{x})$; the latter is always the case for convex function.

Now we are ready to establish the corresponding counterparts of Theorem 5.2 and 5.3 at out-of-set points by using the one-sided constructions (60) and (61).

Theorem 6.5 (one-sided basic and singular subgradients of minimal time functions at out-of-set points). *Let the minimal time function T_Ω^F be continuous around some point $\bar{x} \notin \Omega$, let $r = T_\Omega^F(\bar{x})$, and let the sets C^*, S^*, and F_+^* be defined in (25), (35), and (51), respectively. Then we have the upper estimates*

$$\partial_\geq T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega_r) \cap C^* \quad \text{and} \quad \partial_\geq^\infty T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega_r) \cap F_+^*, \tag{62}$$

where the first one can be replaced by the equality

$$\partial_\geq T_\Omega^F(\bar{x}) = N(\bar{x}; \Omega_r) \cap S^* \tag{63}$$

if the support function σ_F is sequentially weak* continuous on the set $-[N(\bar{x}; \Omega_r) \cap C^*]$ and if T_Ω^F is locally Lipschitzian around $\bar{x}$. Furthermore, the normal cone representation

$$N(\bar{x}; \Omega_r) = \bigcup_{\lambda \geq 0} \lambda \partial_\geq T_\Omega^F(\bar{x}) \tag{64}$$

takes place with the convention $0 \times \emptyset = 0$ provided that $0 \in \operatorname{int} F$.

Proof. We justify only the first inclusion in (62); the second one is proved similarly by taking into account the proof of Theorem 5.3. To proceed, pick any $x^* \in \partial_{\geq} T_\Omega^F(\bar{x})$ and by (60) find sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\varphi^+} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that

$$x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k) \text{ for all } k \in I\!N.$$

If $T_\Omega^F(x_k) = r$ for some subsequence of $\{k\}$, we have by the upper estimate (36) of Theorem 4.6 the relationships

$$x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_r) \quad \text{and} \quad 1 - \varepsilon_k \|F\| \leq \sigma_F(-x_k^*) \leq 1 + \varepsilon_k \|F\|$$

held along this subsequence. Passing there to the limit as $k \to \infty$ gives us the inclusions $x^* \in N(\bar{x}; \Omega_r)$ and $x^* \in C^*$, which justify the first estimate in (62) in this case even without the continuity assumption on the minimal time function.

In the other case of $T_\Omega^F(x_k) > r$ for all $k \in I\!N$ sufficiently large, the assumed continuity of T_Ω^F ensures that for such k we have that $T_\Omega^F(x) > r$ whenever x is near x_k. Employing then Proposition 3.1 ensures the equality

$$T_\Omega^F(x) = r + T_{\Omega_r}^F(x) \text{ for all } x \text{ near } x_k.$$

The latter implies by definition (12) that

$$x_k^* \in \widehat{\partial}_{\varepsilon_k} T_\Omega^F(x_k) = \widehat{\partial}_{\varepsilon_k} T_{\Omega_r}^F(x_k), \quad k \in I\!N.$$

The rest of the proof of the first inclusion in (62) follows the arguments in the proof of Theorem 5.2, which in turn are based on the variational result of Theorem 4.3.

Let us next justify equality (63) provided the fulfillment of the additional weak* continuity and Lipschitzian assumptions made in the theorem. It follows from the proof above that the latter assumption implies the inclusion "$\subset$" in (63). To justify the opposite inclusion "$\supset$" therein, fix any $x^* \in N(\bar{x}; \Omega_r) \cap S^*$ and find by (11) sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega_r} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ with $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_r)$, $k \in I\!N$. The sequential weak* continuity of σ_F at $-x^*$ ensures that

$$\gamma_k := \sigma_F(-x_k^*) \to \sigma_F(-x^*) = 1 \text{ as } k \to \infty.$$

By the definition of S^* in (35) we may assume with no loss of generality that

$$\frac{x_k^*}{\gamma_k} \in \widehat{N}_{\varepsilon_k/\gamma_k}(x_k; \Omega_r) \cap S^* \text{ for all } k \in I\!N. \tag{65}$$

It follows further that $T_\Omega^F(x_k) = r$ for large k, since the opposite assumption on $T_\Omega^F(x_k) < r$ implies by the continuity of T_Ω^F that $x_k \in \operatorname{int} \Omega_r$, which contradicts the condition $x^* \neq 0$ held by (65). Employing the second part of Theorem 4.6, find a sequence $\widetilde{\varepsilon}_k \downarrow 0$ such that

$$\frac{x_k^*}{\gamma_k} \in \widehat{\partial}_{\widetilde{\varepsilon}_k} T_\Omega^F(x_k) \text{ for all } k \in I\!N.$$

Passing there to the limit as $k \to \infty$ and using definition (60) justify equality (63).

Let us finally prove representation (64) correcting the corresponding arguments given in [14, Theorem 4.4]. Note that the inclusion "$\subset$" in (64) follows from the first inclusion (62) and the cone property of $N(\bar{x}; \Omega_r)$. To prove the opposite inclusion $\supset$" in (64), fix any $x^* \in N(\bar{x}; \Omega_r)$ and assume that $x^* \neq 0$, since otherwise x^* belongs to the right-hand side of (64) by our convention. In this case $\gamma := \sigma_F(-x^*) > 0$ due to $0 \in \operatorname{int} F$. By definition (11) of the basic normal cone, there are sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega_r} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ with $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega_r)$. By $0 \in \operatorname{int} F$ the minimal time function (2) is Lipschitz continuous and hence $T_\Omega^F(x_k) = r$ when k is sufficiently large. Indeed, if $T_\Omega^F(x_k) < r$ for a subsequence (without relabeling), then $x_k \in \operatorname{int} \Omega_r$, which implies that $\|x_k^*\| \leq \varepsilon_k$ and leads to a contradiction by $\|x^*\| \leq \liminf \|x_k^*\|$ as $k \to \infty$. Define further $\lambda_k := \sigma_F(-x_k^*)$ and observe by $x_k^* \xrightarrow{w^*} x^*$ that $\lambda_k \geq \gamma/2 > 0$ for all k sufficiently large. Moreover, λ_k is bounded, and hence we may assume that $\lambda_k \to \lambda \geq \gamma/2$ as $k \to \infty$. Then

$$\widetilde{x}_k^* := \frac{x_k^*}{\lambda_k} \in \widehat{N}_{\varepsilon_k/\lambda_k}(x_k) \quad \text{and} \quad \sigma_F(-\widetilde{x}_k^*) = 1,$$

which yields by Theorem 4.6 that $\widetilde{x}_k^* \in \widehat{\partial}_{\widetilde{\varepsilon}_k} T_\Omega^F(x_k)$ with $\widetilde{\varepsilon}_k \to 0$ as $k \to \infty$. The latter implies the inclusions

$$x^* \in \lambda \partial_\geq \lambda T_\Omega^F(\bar{x}) \subset \bigcup_{\lambda \geq 0} \lambda \partial_\geq T_\Omega^F(\bar{x}),$$

which justify (64) complete the proof of the theorem. $\qquad\square$

7. Computing Basic and Singular Subdifferentials of Convex Minimal Time Functions

The concluding section of the paper concerns the minimal time problem (1) with convex data, i.e., under the assumption that the target set Ω is a convex subset of an arbitrary Banach space X. By Proposition 3.6 this property is equivalent to the convexity of the minimal time function (2). In what follows we add the convexity of (2) to our standing assumptions formulated in Section 1 and refer to this setting as to the *convex minimal time problem* and/or the *convex minimal time function*.

Due to the representations of ε-normals to convex sets (10) and ε-subgradients of convex functions (13) we have specifications of the results obtained in Section 4 in the case of convex minimal time functions. The same can be said regarding the results of Sections 5 and 6 concerning the basic subdifferential and normal cone for convex functions and sets, which reduce to those in convex analysis. We can also specify to the case of convex minimal time functions the results derived above for the singular subdifferential; see [18, Proposition 8.12] for its various representations in the general framework of convex analysis.

In this section we show that, besides the aforementioned specifications, the convex case allows us to obtain *equalities* in the upper estimates of Sections 5 and 6 for the

basic and singular subdifferentials of (2) at both in-set and out-of-set points with no additional assumptions in general Banach spaces. Let us start with computing the basic subdifferential (12); cf. Theorem 5.2 and Theorem 6.5, where $\partial_{\geq} T_\Omega^F(\bar{x}) = \partial T_\Omega^F(\bar{x})$ in the convex case.

Theorem 7.1 (basic subgradients of convex minimal time functions). *Let the function T_Ω^F in (2) be convex. Then the following assertions hold:*

(i) *For any $\bar{x} \in \Omega$ we have the representation*

$$\partial T_\Omega^F(\bar{x}) = N(\bar{x}; \Omega) \cap C^*, \tag{66}$$

    where C^* is defined in (25).

(ii) *For any $\bar{x} \notin \Omega$ with $T_\Omega^F(\bar{x}) < \infty$ we have the representation*

$$\partial T_\Omega^F(\bar{x}) = N(\bar{x}; \Omega_r) \cap S^*, \tag{67}$$

    where $r = T_\Omega^F(\bar{x}) > 0$ and S^* is defined in (35).

Proof. Equality (66) in (i) follows directly from Propositions 4.1 and 4.2 with $\varepsilon = 0$ therein and the fact that $\widehat{\partial} T_\Omega^F(\bar{x}) = \partial T_\Omega^F(\bar{x})$ for convex functions.

To justify representation (67) in the out-of set case (ii), observe first that the inclusion "$\subset$" follows from the first part of Theorem 4.6. It remains to prove the converse inclusion "$\supset$". Fix $x^* \in N(\bar{x}; \Omega_r)$ with $\sigma_F(-x^*) = 1$ and show that

$$\langle x^*, x - \bar{x} \rangle \leq T_\Omega^F(x) - T_\Omega^F(\bar{x}) \quad \text{for all } x \in X. \tag{68}$$

Indeed, we get from $x^* \in N(\bar{x}; \Omega_r)$ and the normal cone construction for convex sets that

$$\langle x^*, x - \bar{x} \rangle \leq 0 \quad \text{whenever } x \in \Omega_r.$$

It follows from (66) that $x^* \in \partial T_{\Omega_r}^F(\bar{x})$ and hence

$$\langle x^*, x - \bar{x} \rangle \leq T_{\Omega_r}^F(x) \quad \text{for any } x \in X.$$

It is clear that (68) holds when $x \notin \Omega_r$, since in this case $T_{\Omega_r}^F(x) = T_\Omega^F(x) - r$ by Proposition 3.1. In the other case of $t = T_\Omega^F(x) \leq r$, for any $\varepsilon > 0$ sufficiently small pick $q \in F$ with $\langle x^*, -q \rangle \geq 1 - \varepsilon$ and get $T_\Omega^F(x - (r - t)q) \leq r$ by Proposition 3.2. This gives

$$\langle x^*, x - \bar{x} \rangle \leq (r - t)\langle x^*, q \rangle \leq (t - r)(1 - \varepsilon),$$

By the arbitrary choice of $\varepsilon > 0$ the latter justifies (68) in this case. Thus we arrive at $x^* \in \partial T_{\Omega_r}^F(\bar{x})$ and complete the proof of theorem. $\square$

The next result provides precise representations for the singular subdifferential of the convex minimal time function (2) in both in-set and out-of-set cases; cf. Theorems 5.3 and Theorem 6.5, where $\partial_{\geq}^\infty T_\Omega^F(\bar{x}) = \partial^\infty T_\Omega^F(\bar{x})$ in the convex case.

Theorem 7.2 (singular subgradients of convex minimal time functions). *Let the function T_Ω^F in (2) be convex and lower semicontinuous around $\bar{x}$, and let F_+^* be defined in (51). The following assertions hold:*

(*i*)   *If $\bar{x} \in \Omega$, then we have*

$$\partial^\infty T_\Omega^F(\bar{x}) = N(\bar{x}; \Omega) \cap F_+^*. \tag{69}$$

(*ii*)   *If $\bar{x} \notin \Omega$ and $T_\Omega^F(\bar{x}) < \infty$, then*

$$\partial^\infty T_\Omega^F(\bar{x}) = N(\bar{x}; \Omega_r) \cap F_+^* \quad \text{with } r = T_\Omega^F(\bar{x}). \tag{70}$$

Proof. Taking into account that the subdifferential of convex analysis agrees with the Fréchet subdifferential for convex functions and following the proof of [11, Lemma 2.37] with replacing the fuzzy sum rule for Fréchet subgradients of l.s.c. functions in Asplund spaces by the exact sum rule (Moreau-Rockafellar theorem) in convex analysis in Banach spaces, we get the singular subdifferential representations under the assumptions made:

$$\partial^\infty T_\Omega^F(\bar{x}) = \underset{\substack{x \xrightarrow{T_\Omega^F} \bar{x} \\ \lambda \downarrow 0}}{\mathrm{Lim\,sup}} \lambda \partial T_\Omega^F(x) = \left\{ x^* \in X^* \middle| (x^*, 0) \in N\left((\bar{x}, T_\Omega^F(\bar{x})); \mathrm{epi}\, T_\Omega^F\right) \right\}. \tag{71}$$

It is easy to check that

$$\left\{ x^* \in X^* \middle| (x^*, 0) \in N\left((\bar{x}, T_\Omega^F(\bar{x})); \mathrm{epi}\, T_\Omega^F\right) \right\} = N\left(\bar{x}; \mathrm{dom}\, T_\Omega^F\right),$$

and hence we have by the second representation in (71) and Theorem 5.3 that

$$\partial^\infty T_\Omega^F(\bar{x}) = N\left(\bar{x}; \mathrm{dom}\, T_\Omega^F\right) \subset N(\bar{x}; \Omega) \cap F_+^*. \tag{72}$$

Let us now justify the opposite inclusion in (72), i.e.,

$$N(\bar{x}; \Omega) \cap F_+^* \subset N\left(\bar{x}; \mathrm{dom}\, T_\Omega^F\right). \tag{73}$$

To proceed, pick arbitrary $x^* \in N(\bar{x}; \Omega) \cap F_+^*$ and $x \in \mathrm{dom}\, T_\Omega^F$ and then find by (1) a number $t \geq 0$ such that $(x + tF) \cap \Omega \neq \emptyset$. Fix further $q \in F$ and $w \in \Omega$ with $x + tq = w$ and obtain the relationships

$$\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \langle x^*, w - tq - \bar{x} \rangle \\
&= \langle x^*, w - \bar{x} \rangle - t\langle x^*, q \rangle \leq 0,
\end{aligned}$$

since $\langle x^*, w - \bar{x} \rangle \leq 0$ by $x^* \in N(\bar{x}; \Omega)$ and $\langle x^*, q \rangle \geq 0$ by $x^* \in F_+^*$. Thus we get (73) and arrive at the singular subdifferential representation (69) in the in-set case.

To justify further representation (70) in the out-of-set case $\bar{x} \in \Omega_r$ with $r = T_\Omega^F(\bar{x})$, observe from the equality in (72) that

$$\partial^\infty T_\Omega^F(\bar{x}) = N\left(\bar{x}; \mathrm{dom}\, T_\Omega^F\right) \subset N(\bar{x}; \Omega_r)$$

due to the obvious inclusions $\Omega_r \subset \mathrm{dom}\, T_\Omega^F$ and $N(\bar{x}; \Theta_2) \subset N(\bar{x}; \Theta_1)$ for any convex sets $\bar{x} \in \Theta_1 \subset \Theta_2$. Fix now $x^* \in \partial^\infty T_\Omega^F(\bar{x})$ and find by the first representation in (71) sequences $x_k \xrightarrow{T_\Omega^F} \bar{x}$, $x_k^* \in \partial T_\Omega^F(x_k)$, and $\lambda_k \downarrow 0$ such that

$$\lambda_k x_k^* \xrightarrow{w^*} x^* \quad \text{as } k \to \infty.$$

It follows from Theorem 7.1(ii) that $\sigma_F(-x_k^*) = 1$ whenever $k \in \mathbb{N}$ is sufficiently large. Hence picking any $q \in F$, we have $\langle -\lambda_k x_k^*, q \rangle \leq \lambda_k$ for all such k. This yields $\langle x^*, q \rangle \geq 0$ by letting $k \to \infty$. Thus it gives $x^* \in F_+^*$ justifying the inclusion

$$\partial^\infty T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega_r) \cap F_+^*.$$

To get (70), it remains to prove the converse inclusion

$$N(\bar{x}; \Omega_r) \cap F_+^* \subset N\left(\bar{x}; \operatorname{dom} T_\Omega^F\right).$$

Fix $x^* \in N(\bar{x}; \Omega_r) \cap F_+^*$ and pick any $x \in \operatorname{dom} T_\Omega^F$, which ensures the existence of $t \geq 0$ such that $(x + tF) \cap \Omega \neq \emptyset$. Take $q \in F$ and $w \in \Omega$ satisfying $x + tq = w$. Then

$$\begin{aligned}\langle x^*, x - \bar{x} \rangle &= \langle x^*, w - tq - \bar{x} \rangle \\ &= \langle x^*, w - \bar{x} \rangle - t\langle x^*, q \rangle \leq 0\end{aligned}$$

by $w \in \Omega \subset \Omega_r$ and $\bar{x} \in \Omega_r$, which completes the proof of the theorem. □

The last result of this section establishes representations of the convex subdifferential of T_Ω^F via that of the Minkowski gauge; in particular, it justifies the equality in the upper estimate of $\partial T_\Omega^F(\bar{x})$ from Theorem 6.3 at out-of-set points. Note that even the upper estimate (55) itself is new with no well-posedness assumption in general Banach spaces.

Theorem 7.3 (precise relationships between convex subdifferentials of minimal time and Minkowski functions in out-of-set points). *Let the function T_Ω^F in (2) be convex, and let $\bar{x} \notin \Omega$ be such that $\Pi_\Omega^F(\bar{x}) \neq \emptyset$ with $r = T_\Omega^F(\bar{x}) < \infty$. Then for any $\bar{w} \in \Pi_\Omega^F(\bar{x})$ we have the relationships*

$$\begin{aligned}\partial T_\Omega^F(\bar{x}) &= N(\bar{x}; \Omega_r) \cap \left[-\partial\rho_F(\bar{w} - \bar{x})\right] \\ &\subset N(\bar{w}; \Omega) \cap \left[-\partial\rho_F(\bar{w} - \bar{x})\right].\end{aligned} \tag{74}$$

If in addition $0 \in F$, then the inclusion in (74) holds as equality and thus

$$\partial T_\Omega^F(\bar{x}) = N(\bar{w}; \Omega) \cap \left[-\partial\rho_F(\bar{w} - \bar{x})\right].$$

Proof. It follows from Theorem 7.1(ii) that $\partial T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega_r)$. Furthermore

$$\partial T_\Omega^F(\bar{x}) \subset -\partial\rho_F(\bar{x} - \bar{w})$$

by Proposition 4.5 as $\varepsilon = 0$, and thus

$$\partial T_\Omega^F(\bar{x}) \subset N(\bar{x}; \Omega_r) \cap \left[-\partial\rho_F(\bar{w} - \bar{x})\right]. \tag{75}$$

To prove the opposite inclusion "⊃" to (75), fix any $x^* \in N(\bar{x}; \Omega_r) \cap \left[-\partial\rho_F(\bar{w} - \bar{x})\right]$. By Theorem 7.1$(ii)$ it suffices to show that

$$x^* \in S^*, \quad \text{i.e.,} \quad \sigma_F(-x^*) = 1. \tag{76}$$

To this end, observe that $T_{\{0\}}^F(x) = \rho_F(-x)$, which implies that

$$-\partial\rho_F(x) = \partial T_{\{0\}}^F(-x) \quad \text{and hence} \quad -\partial\rho_F(\bar{w} - \bar{x}) = \partial T_{\{0\}}^F(\bar{x} - \bar{w}), \quad x \in X.$$

Since $\bar{x} - \bar{w} \notin \{0\}$, we get (76) from Theorem 7.1(ii) and thus justify the equality in (74).

Further, it is not hard to check that $\partial T_\Omega^F(\bar{x}) \subset N(\bar{w}; \Omega)$ and hence

$$\partial T_\Omega^F(\bar{x}) \subset N(\bar{w}; \Omega) \cap \left[-\partial\rho_F(\bar{w} - \bar{x}) \right],$$

which implies the inclusion in (74).

To finish the proof, it remains to show that

$$N(\bar{w}; \Omega) \cap \left[-\partial\rho_F(\bar{w} - \bar{x}) \right] \subset N(\bar{x}; \Omega_r) \cap \left[-\partial\rho_F(\bar{w} - \bar{x}) \right] \qquad (77)$$

under the additional assumption that $0 \in F$ in which case we have $\rho_F(0) = 0$. It suffices to verify that for each $x^* \in N(\bar{w}; \Omega) \cap \left[-\partial\rho_F(\bar{w} - \bar{x}) \right]$ we have $x^* \in N(\bar{x}; \Omega_r)$.

To proceed, pick any $x \in \Omega_r$ and for an arbitrary small $\varepsilon > 0$ find $t < r + \varepsilon$, $q \in F$, and $w \in \Omega$ with $w = x + tq$. Then $\langle -x^*, q \rangle \le \sigma_F(-x^*) \le 1$ and

$$
\begin{aligned}
\langle x^*, x - \bar{x} \rangle &= \langle x^*, w - tq - \bar{x} \rangle \\
&= t\langle -x^*, q \rangle + \langle x^*, w - \bar{w} \rangle + \langle x^*, \bar{w} - \bar{x} \rangle \\
&\le t + \langle x^*, w - \bar{w} \rangle + \langle x^*, \bar{w} - \bar{x} \rangle \\
&\le T_\Omega^F(\bar{x}) + \varepsilon + \langle x^*, w - \bar{w} \rangle + \langle x^*, \bar{w} - \bar{x} \rangle.
\end{aligned}
$$

We have $\langle x^*, w - \bar{w} \rangle \le 0$ due to $x^* \in N(\bar{w}; \Omega)$ and

$$\langle x^*, \bar{w} - \bar{x} \rangle = \langle -x^*, 0 - (\bar{w} - \bar{x}) \rangle \le \rho_F(0) - \rho_F(\bar{w} - \bar{x}) = -T_\Omega^F(\bar{x})$$

by $-x^* \in \partial\rho_F(\bar{w} - \bar{x})$. It follows therefore that $\langle x^*, x - \bar{x} \rangle \le \varepsilon$ for all $x \in \Omega_r$, and hence $x^* \in N(\bar{x}; \Omega_r)$ because $\varepsilon > 0$ was chosen arbitrary small. Thus we arrive at (77) and complete the proof of the theorem. $\qquad \square$

Finally, let us present an example that illustrates computing the basic and singular subdifferentials of convex minimal time functions at in-set and out-of-set points.

Example 7.4 (subdifferentiation of convex minimal time functions). In $\mathbb{R}^2$, consider the convex dynamics $F = [-1, 1] \times \{0\}$ of empty interior and the convex target $\Omega = [-1, 1] \times [-1, 1]$. In this case the Minkowski gauge (3) and the minimal time function (2) of $x = (x_1, x_2) \in \mathbb{R}^2$ are computed by, respectively,

$$\rho_F(x) = \begin{cases} |x_1| & \text{if } x \in \mathbb{R} \times \{0\}, \\ \infty & \text{otherwise;} \end{cases}$$

$$T_\Omega^F(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ |x_1| - 1 & \text{if } |x_2| \le 1 \text{ and } |x_1| > 1, \\ \infty & \text{otherwise.} \end{cases}$$

Taking first the in-set $\bar{x} = (1, 0) \in \Omega$, we can easily check that $\partial T_\Omega^F(\bar{x}) = [0, 1] \times \{0\}$ and that $\sigma_F(v) = |v_1|$ for $v = (v_1, v_2) \in I\!\!R^2$. It is also clear that

$$N(\bar{x}; \Omega) \cap C^* = N(\bar{x}; \Omega) \cap \{v \in I\!\!R^2 \big| \sigma(-v) \leq 1\} = [0, 1] \times \{0\},$$

and thus we verify equality (66) in Theorem 7.1(*i*). Furthermore, it is easy to verify that $\partial^\infty T_\Omega^F(\bar{x}) = \{0\}$ in accordance with Theorem 7.2(*i*) in the in-set case; this confirms that T_Ω^F is locally Lipschitzian around $\bar{x} = (1, 0)$.

Considering another in-set point $\bar{y} = (0, 1) \in \Omega$, we have

$$\partial T_\Omega^F(\bar{y}) = N(\bar{y}; \Omega) \cap C^* = \{0\} \times I\!\!R^+,$$

which verifies the conclusion of Theorem 7.1(*i*). It follows similarly that $\partial^\infty T_\Omega^F(\bar{y}) = \{0\} \times [0, \infty)$, which is in accordance with Theorem 7.2(*i*) and with the non-Lipschitzian behavior of the minimal time function around $\bar{y} = (0, 1)$.

Considering finally the out-of-set point $\bar{z} = (2, 1/2) \notin \Omega$, with the projection singleton $\Pi_\Omega^F(\bar{z}) = \{\bar{w}\}$ computed by $\bar{w} = (1, 1/2)$. Then we arrive at the equalities

$$\partial T_\Omega^F(\bar{z}) = \{(1, 0)\} = -\partial \rho_F(\bar{w} - \bar{z}) \cap N(\bar{w}; \Omega) \quad \text{and} \quad \partial^\infty T_\Omega^F(\bar{z}) = \{0\},$$

which verify the conclusions of Theorem 7.2(*ii*) and Theorem 7.3 and confirm, in particular, the local Lipschitzian property of T_Ω^F around $\bar{z} = (2, 1/2)$.

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